THE COLLINEATION GROUPS OF FREE PLANES

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Free projective planes π^n (generated by n points on a line and two points off that line) have been studied for some time. Until now, however, virtually nothing was known about the collineation groups of the planes π^n except that the groups possessed no central collineations. In this paper, the collineation groups will be studied. We shall show that the collineation group of π^2 has three generators and exhibit them. Then, the orbit under this group of those points which lie on generating quadrilaterals will be determined. Finally, a subgroup H_n of the collineation group of π^n ($n > 2$) will be defined and it will be shown that there exists an integer m such that for every $n > 2$, H_n is generated by at most m collineations. It is the author's conjecture that H_n is in fact the full collineation group of π^n, but the methods of the present paper do not seem to generalize to give this result.

1. **Preliminaries.** A set of points and lines and an incidence relation is said to form a projective plane if the following three axioms are satisfied:

I. Any two distinct points are incident with exactly one line.

II. Given any two distinct lines, there is exactly one point incident with both.

III. There exist four points, no three incident with one line.

If, on the other hand, the following two axioms are satisfied, the set is said to constitute a *partial plane*:

1. There exists at most one line through any two distinct points.
2. There is at most one point incident with any two distinct lines.

Given any partial plane π_0 which is not complete (projective), we can define new partial planes as follows: For any two points in π_0 not already connected by a line in π_0, we adjoin a new line. Let L_0 be the set of all new lines adjoined in this manner. Then, for any two lines in $\pi_0 \cup L_0$ not already intersecting in a point of π_0, we adjoin a new point as their point of intersection. Let π_1 be the union of π_0, L_0, and the new points just adjoined. Clearly, π_1 is a partial plane. One can define $\pi_2, \pi_3, \cdots$ in an analogous manner, and set

$$\pi = \bigcup_{i=0}^{\infty} \pi_i, \tag{1}$$

with the obvious incidences. Then π is easily seen to be a complete plane. Any extension of a partial plane π_0 formed by letting each new point be the intersection

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of exactly two old lines, and each new line contain only two old points, is called a *free extension* of π_0.

We next state some lemmas which can be found in Hall [1, §4].

LEMMA 1. *The free extension of a partial plane to a complete plane is unique.*

LEMMA 2. *If π_2 is a free extension of π_1 and π_1 is a free extension of π_0, then π_2 is a free extension of π_0. The free extension of π_1 to a complete plane is the free extension of π_0 to a complete plane.*

DEFINITION. If π_1, π_2 are two partial planes, and if $\pi_2 \subset \pi_1$, π_2 is said to be *complete* in π_1 if

(1) any line of π_1 containing two points of π_2 already is in π_2;

(2) any point of π_1 contained in two lines of π_2 already is in π_2.

LEMMA 3. *If $\pi_1 \supset \pi_2$ and π_2 is complete in π_1, then the free extension π_1' of π_1 contains as a subplane the free extension π_2' of π_2. If $\pi_2 \neq \pi_1$, then $\pi_2' \neq \pi_1'$.*

The free plane $\pi^n (n \geqq 2)$ is defined to be the free extension of the partial plane π_0^n consisting of n points on a line and two points not on that line to a complete plane.

LEMMA 4. *A π^n contains a π^{n+1} as a subplane.*

LEMMA 5. *If $n \neq m$, π^n and π^m are not isomorphic.*

A collineation ϕ of a projective plane π is a 1–1 point-to-point and line-to-line mapping of the plane onto itself which preserves incidences. In what follows we shall study the collineation groups of the free planes π^n.

2. **General results.** We begin this section by proving some rather elementary results which will allow us to get some insight into the structure of the collineation group of π^n. Let $A_1, A_2, \cdots, A_n, B_1, B_2$ be the points (of π_0^n) generating π^n, where the A_i all lie on one line, and no other incidences hold.

THEOREM 1. *Let ϕ be a collineation of π^n. Then ϕ is uniquely determined by its action on the points A_i; $i = 1, 2, \cdots, n$, and B_j; $j = 1, 2$.*

Proof. This result can be proved by induction on the partial plane π_k^n, a free extension of π_0^n. For, if $L \in \pi_k^n$, $L \notin \pi_{k-1}^n$, then L can be described as the unique line joining two points in π_{k-1}^n. If ϕ is already defined (by induction) on these two points, then $(L)\phi$ must be the unique line joining the images of the points. Similarly, the definition of ϕ is extended to points in π_k^n but not in π_{k-1}^n.

The next question to which we address ourselves concerns the possible images of π_0^n under a collineation, ϕ, of π^n. The following theorem is evident.

THEOREM 2. *The image of π_0^n under a collineation ϕ must be a configuration C of n points on one line and two points off that line. Furthermore, the subplane of π^n generated by C must be all of π^n.*

Proof. The first statement follows from the fact that ϕ must preserve incidences, and the second, from the fact that ϕ must be onto.

A final theorem is needed before we can begin to characterize better the collineation group of π^n. If π is a finite partial plane, let P be the number of points of π, L the number of lines, and I the number of incidences defined by the points and lines of π. Hall has shown [1] that the number $2(P+L)-I$ is invariant under free extension. We call the number $\gamma(\pi)=2(P+L)-I$ the rank of π and observe that the rank of a generating configuration of a π^n can be computed as $\gamma(\pi_0^n)=n+6$.

THEOREM 3. *Let π^n be a free plane (of rank $n+6$). Let σ_0 be a finite subpartial plane of π^n which is of rank $n+6$ and which generates all of π^n. Then σ_0 generates π^n freely.*

Proof. To σ_0, adjoin the lines which can be added and then the points. Call the union of σ_0, the new points and the new lines σ_1. Then σ_i is defined inductively. If a *new* point (line) is always added as the intersection of two lines (points), the extension of σ_0 to π^n will be free. If not, then in some σ_k, an intersection two lines (points) of σ_{k-1} will be defined which does not create a new point (line). Thus an incidence is created, with no element being created, and we have $\gamma(\sigma_k)<\gamma(\sigma_0)=n+6$. Now for some m, $\sigma_k\subset\pi_m^n$. But if we extend σ_k to π_m^n by building outward from π_0^n and adding those elements not in σ_k, the rank can never be increased since a new line (point) is always created as the intersection of at least two points (lines). Thus we would have $\gamma(\pi_m^n)\leqq\gamma(\sigma_k)<n+6$. But by Theorem 4.10 of Hall [1] we know that $\gamma(\pi_m^n)=n+6$, which implies that σ_0 must generate π^n freely as asserted.

Because of Theorems 1, 2 and 3, we can study in detail the collineation group of π^n if we can determine all configurations of n points on a line and two points off the line which generate all of π^n. For let $\{C_i\}$ be the set of all such configurations ($C_1=\pi_0^n$). Then we have

THEOREM 4. *If $n>2$, there is a group of collineations of π^n isomorphic with $S_n\times S_2$, and if $n=2$, the corresponding group is isomorphic with S_4. Call this group S^n. Then if G_n is the collineation group of π^n, we have*

$$G_n=\sum_i S^nT_i, \tag{2}$$

where T_i is a collineation of π^n which maps π_0^n onto the configuration C_i.

Proof. If $n>2$, S^n consists of those collineations generated by permuting the n points A_i among themselves in all possible ($n!$) ways, and by permuting B_1 and

B_2 or leaving them fixed; S^2 is generated by the (4!) permutations of the points A_1, A_2, B_1, B_2 among themselves. Clearly, S^n consists of all collineations of π^n mapping π_0^n onto itself. Let ϕ_1, ϕ_2 be two elements in the same left coset of S^n in G_n. Then $\phi_1 = \theta\phi_2$, $\theta \in S^n$. But this implies that ϕ_1 and ϕ_2 both map π_0^n onto the same configuration C_i. Thus, T_i is a coset representative of S^n in G_n.

To help determine the desired configurations, we have

THEOREM 5. *Let C be any configuration of n points on a line and two points off the line contained in a free extension of π_0^n, π. Then C generates π^n if and only if the minimal complete sub-partial plane of π generated by C is π itself.*

Proof. To C adjoin (within π) points and lines until it is not possible to add any more elements. The set of points and lines so generated, π', will be a sub-partial plane of π *which is complete in* π, by the definition of π'. By Lemma 3, if C is to generate π^n, we must have $\pi' = \pi$.

In concluding this section, some concepts will be introduced which will be of much help in what follows. If π_0 is a partial plane, let L_0 be the set of all lines which can be added to π_0 as in §1. Then if P_0 is the set of all new points which can be added to π_0 as the intersection of lines (two at a time) of $\pi_0 \cup L_0$ which do not intersect in π_0, we shall call P_0 the set of *diagonal points* of π_0. Then, $\pi_1 = \pi_0 \cup L_0 \cup P_0$. If $\pi_0 = \pi_0^n$, the generating configuration for a free plane, we shall call the group S^n the *collineations of rank* 0, and define as the *collineations of rank* 1 that set of collineations which map π_0^n into π_1^n. With these concepts, and Theorems 1, 2, and 3 at our disposal, we can proceed to an examination of the collineation groups of the planes π^n.

3. **The case** $n = 2$. If A_1, A_2, B_1, B_2 are the four points of π_0^2, we let L_1 be the line containing A_1 and A_2 ($L_1 = A_1 \cdot A_2$), $L_2 = B_1 \cdot B_2$, $L_3 = A_1 \cdot B_1$, $L_4 = A_2 \cdot B_2$, $L_5 = A_1 \cdot B_2$, $L_6 = A_2 \cdot B_1$. The configuration π_0^2 has three diagonal points: $a_1 = L_1 \cap L_2$, $a_2 = L_3 \cap L_4$, $a_3 = L_5 \cap L_6$. By inspection, one can check that there are exactly seven configurations of four points in π_1^2 which generate π^2. They are:

$$C_1 = \pi_0^2,\ C_2 = \{a_1,\ a_2,\ A_1,\ B_2\},\quad C_3 = \{a_1, a_2, B_1, A_2\},$$

$$C_4 = \{a_1, a_3, A_1, B_1\},\ C_5 = \{a_1, a_3, A_2, B_2\},\quad C_6 = \{a_2, a_3, A_1, A_2\},$$

$$C_7 = \{a_2, a_3, B_1, B_2\}(^1).$$

The investigations in this section will show that G_2 is generated by S^2 and any T_i, $i = 2, 3, \cdots, 7$.

Before proving our main results, one final concept is needed. If C is a configuration of four points generating π^2, the ordered set of configurations

(1) The remaining sets of four points either have three points on a line, or generate proper subplanes of π^2.

$\pi_0^2 = C_{i_1}, C_{i_2}, \cdots, C_{i_k} = C$ is called a *chain of configurations between* π_0^2 *and* C *of length* k if (1) every C_{i_j} is a configuration of four points generating π^2; and (2) $C_{i_{j+1}}$ is contained in the seven points consisting of C_{i_j} and the diagonal points of C_{i_j}. We can now state and prove

THEOREM 6. *Let* C *be a configuration of four points of* π^2 *generating* π^2. *If* $C \subset \pi_k^2$, *then there exists a chain of length* k *or less between* π_0^2 *and* C.

Proof. The proof is by induction on k. If $k = 1$, the statement is true by definition. Let the statement of the theorem be true for $k-1$, and we shall prove it for k. If $C \subset \pi_{k-1}^2$, the theorem is true, so assume that $C \not\subset \pi_{k-1}^2$. Let L_k be the set of lines which can be adjoined to π_{k-1}^2 in a free extension, and let $\pi_{k-1}(C) = \pi_{k-1}^2 \cup L_k \cup C$. Then $\pi_{k-1}(C)$ is a free extension of π_{k-1}^2, and also of π_0^2. By hypothesis, C generates all of π^2, hence, by Theorem 4, the sub-partial plane of $\pi_{k-1}(C)$ generated by C must be all of $\pi_{k-1}(C)$. For the sake of definiteness let C consist of the points P_1, P_2, P_3, P_4 The sub-partial plane generated by C can be determined by first adding all lines $P_i \cdot P_j$ which are in $\pi_{k-1}(C)$, then all points $L_i \cap L_j$, and so on until the process terminates. If we assume that $k \geqq 2$, then $\pi_{k-1}(C)$ must contain at least eight points (seven in π_1^2 and one in C). But for C to generate points other than P_ι, some of the six lines $L_{ij} = P_i \cdot P_j = P_j \cdot P_i$ must be in $\pi_{k-1}(C)$. Also, it is easy to see that at least two of the three diagonal points of C must be in $\pi_{k-1}(C)$ as intersections of the lines L_{ij} if we are to be able to generate more than five points. Observe now that if all three diagonal points of C are in $\pi_{k-1}(C)$ as intersections of the L_{ij}, all six lines L_{ij} must be in $\pi_{k-1}(C)$. But in this case, every point P_i of C will have at least three lines through it which would imply that $C \subset \pi_{k-1}^2$, a contradiction. Hence, exactly two diagonal points of C must be in $\pi_{k-1}(C)$ as intersections of the L_{ij}. Let us say that these diagonal points are $d_1 = L_{12} \cap L_{34}$, $d_2 = L_{14} \cap L_{23}$. So far, we have six points of $\pi_{k-1}(C)$. To construct another, we must have the line $d_1 \cdot d_2$ intersecting with the line L_{13} or L_{24} (but not both—see Figure 1).

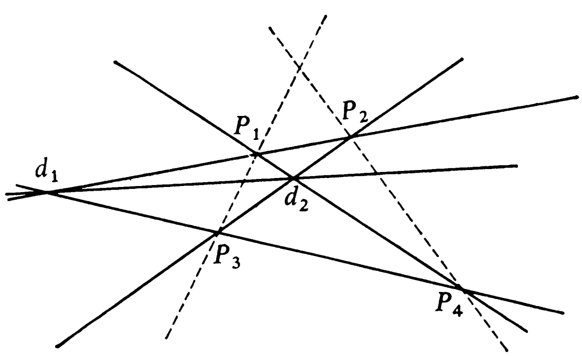

FIGURE 1

If, for example the line L_{13} is in $\pi_{k-1}(C)$, then the points P_1 and P_3 both have three lines through them and are therefore already in π^2_{k-1}. Also, the points d_1 and d_2 being in $\pi_{k-1}(C)$ and not in C must be in π^2_{k-1}. Thus the configuration $C' = \{P_1, P_3, d_1, d_2\}$ lies in π^2_{k-1}, and generates all of π^2, since it generates C which generates π^2 by hypothesis. Also by the induction hypothesis, there exists a chain of length $k-1$ from π^2_0 to C', and since all the points of C are contained among the points of C' and its diagonal points, the theorem is proved. Observe, incidentally, that the next line created must be either $P_4 \cdot [(d_1 \cdot d_2) \cap L_{13}]$ or $P_2 \cdot [(d \cdot d_2) \cap L_{13}]$. Thus either P_2 or P_4 has three lines through it, so exactly one point of the four was a newly created point.

We have, in Theorem 6, given a method for constructing any configuration generating π^2, and hence, any collineation. Our next task is to display explicitly the generators of the collineation group. Since a collineation is determined by its action on the points A_i, B_j, a collineation will be designated by

$$\begin{bmatrix} A_1 & \to & P_1 \\ A_2 & \to & P_2 \\ B_1 & \to & P_3 \\ B_2 & \to & P_4 \end{bmatrix}.$$

The group S^2 is isomorphic with S_4, and can be generated by the two collineations

$$(3) \qquad \theta_1 = \begin{bmatrix} A_1 & \to & A_2 \\ A_2 & \to & A_1 \\ B_1 & \to & B_1 \\ B_2 & \to & B_2 \end{bmatrix}, \qquad \theta_2 = \begin{bmatrix} A_1 & \to & A_2 \\ A_2 & \to & B_1 \\ B_1 & \to & B_2 \\ B_2 & \to & A_1 \end{bmatrix},$$

where $\theta_1^2 = \theta_2^4 = I$, the identity collineation. Let $a_1 = A_1 \cdot A_2 \cap B_1 \cdot B_2$, $a_2 = A_1 \cdot B_2 \cap A_2 \cdot B_1$, $a_3 = A_1 \cdot B_1 \cap A_2 \cdot B_2$ be the diagonal points of π^2_0. A typical collineation of rank one is

$$(4) \qquad \phi = \begin{bmatrix} A_1 & \to & A_1 \\ A_2 & \to & a_1 \\ B_1 & \to & B_1 \\ B_2 & \to & a_2 \end{bmatrix}.$$

(Note that $\phi^2 = I$, for $(a_1)\phi = (A_1)\phi \cdot (A_2)\phi \cap (B_1)\phi \cdot (B_2)\phi = A_1 \cdot a_1 \cap B_1 \cdot a_2 = A_2$, and $(a_2)\phi = (A_1)\phi \cdot (B_2)\phi \cap (A_2)\phi \cdot (B_1)\phi = A_1 \cdot a_2 \cap a_1 \cdot B_1 = B_2$.) We can now prove

THEOREM 7. *G_2 is generated by the collineations θ_1, θ_2, and ϕ.*

Proof. First, recall that there are exactly seven configurations of four points in π^2_1 which generate π^2. By Theorem 3, to find all rank 0 or 1 collineations we need only find one coset representative mapping π^2_0 onto each of these configurations. For example,

(5)

$$\theta_1\phi\theta_1 = \begin{bmatrix} A_1 \to a_1 \\ A_2 \to A_2 \\ B_1 \to B_1 \\ B_2 \to a_3 \end{bmatrix}, \quad \theta_1\phi\theta_1\theta_2^2 = \begin{bmatrix} A_1 \to a_1 \\ A_2 \to B_2 \\ B_1 \to A_1 \\ B_2 \to a_3 \end{bmatrix},$$

$$\theta_2^{-1}\phi\theta_2 = \begin{bmatrix} A_1 \to a_1 \\ A_2 \to A_2 \\ B_1 \to a_2 \\ B_2 \to B_2 \end{bmatrix}, \quad \theta_2^{-2}\phi\theta_2^2 = \begin{bmatrix} A_1 \to A_1 \\ A_2 \to a_2 \\ B_1 \to B_1 \\ B_2 \to a_1 \end{bmatrix},$$

$$\theta_1\phi\theta_1\theta_2 = \begin{bmatrix} A_1 \to a_2 \\ A_2 \to B_1 \\ B_1 \to B_2 \\ B_2 \to a_3 \end{bmatrix}, \quad \theta_1\phi\theta_1\theta_2^3 = \begin{bmatrix} A_1 \to a_2 \\ A_2 \to A_1 \\ B_1 \to A_2 \\ B_2 \to a_3 \end{bmatrix}.$$

These six collineations, together with the identity, give a set of coset representatives for the seven cosets which the rank 0 and 1 collineations constitute. The remainder of the proof proceeds by induction on the length of a chain between π_0^2 and the configuration giving rise to the collineation which we are trying to write in terms of θ_1, θ_2, and φ. Let C be the configuration into which this collineation, α, maps π_0^2:

(6) $$\alpha : \pi_0^2 \to C.$$

Then, since $C \subset \pi_0^n$ for some n, by Theorem 6 there exists a configuration $C' \subset \pi_0^{n-1}$ such that the points of C are contained among the points and diagonal points of C'. By the induction hypothesis, then, there is a collineation, β, generated by θ_1, θ_2, and ϕ with

(7) $$\beta : \pi_0^2 \to C'.$$

Now, the collineation $\gamma = \beta^{-1}\alpha$ maps C' onto C:

(8) $$\gamma : C' \to C.$$

But $\beta\gamma\beta^{-1}$, being a conjugate of γ (which maps C' into itself and its diagonal points), must map π_0^2 into π_1^2, and hence, $\beta\gamma\beta^{-1} = \eta$, which is in one of the seven cosets of S^2 given by the identity and (5). Thus,

(9) $$\beta^{-1}\eta\beta = \beta^{-1}\alpha,$$

where β and η are generated by ϕ, θ_1, and θ_2, which completes the proof of the theorem.

Indeed, it follows from (9), that α is a product of no more than n rank 0 or 1 collineations if α maps π_0^2 onto a configuration at the other end of a chain of length n.

Another question which can be asked about the collineation group G_2 concerns the orbits of the points of π^2 under G_2. Clearly the orbit containing the four original generators of π^2 consists of exactly those points which lie on some generating quadrilateral. The following theorem characterizes this set of points.

THEOREM 8. *Let P be a point of π^2 which is in π_k^2, but not in π_{k-1}^2. Thus $P = L_1 \cap L_2$, $L_i \in \pi_k^2$. Then P lies on a generating quadrilateral if and only if there is a generating quadrilateral π_{k-1}^2, two points of which are incident with L_1, and two with L_2.*

Proof. The first ("if") part of the theorem is obvious. To prove the converse, assume that P does lie on a generating quadrilateral $C \subset \pi_n^2$, and that P lies on no such generating quadrilateral of π_{n-1}^2. Clearly $n \geqq k$. We refer to Figure 1 and let $P = P_4$ where $C = \{P_1, P_2, P_3, P_4\}$. By the remarks following the proof of Theorem 5, exactly one of the P_i does not already belong to π_{n-1}^2.

Case 1. $P \in \pi_{n-1}^2$. By Theorem 6 there exists a generating configuration $C' \subset \pi_{n-1}^2$ consisting of two of the three P_i which are in π_{n-1}^2 and two of the diagonal points of C. If $P = P_4$ is one of the points of C', our choice of the configuration C as the generating quadrilateral containing P and of "least degree" is violated. Thus, we must have the situation in Figure 1, where $C' = \{d_1, d_2, P_1, P_3\}$ and P_2 is the point $\notin \pi_{n-1}^2$. But for C' to be a generating quadrilateral, the point $(d_1 \cdot d_2) \cap L_{13} = Q$ must be in π_{n-1}^2 (see the Proof of Theorem 6) which means that the generating configuration $C'' = \{Q,P,d_2,P_3\}$ is contained in π_{n-1}^2, again contradicting our choice of C.

Case 2. $P \notin \pi_{n-1}^2$. Thus, $k = n$, and by Theorem 6, there exists a generating quadrilateral contained in π_{n-1}^2 with P as one of its diagonal points. But clearly the only way P can be the diagonal point of a quadrilateral of π_{n-1}^2 is if this quadrilateral generates in π_n^2 the two lines which have created P and the proof is thus complete.

To see that there are points which lie on no generating quadrilateral, it suffices to choose four points of some π_k^2 no two of which lie on the same line of π_k^2. Then none of the three diagonal points of such a quadrilateral can lie on a generating quadrilateral. Four such points can be chosen when $k \geqq 2$.

4. $n > 2$. In this case, the group of collineations of π^n, H_n, generated by the rank 0 and rank 1 collineations, is somewhat more difficult to describe, and the following notation will be introduced to discuss the points and lines of π_1^n.

$$\text{lines:}\quad \begin{aligned} L_1 &= A_1 \cdots A_n, \\ L_2 &= B_1 \cdot B_2, \\ L_{ij} &= A_i \cdot B_j,\ i = 1,2,\cdots,n;\ j = 1,2. \end{aligned}$$

$$(10)\qquad \text{points:}\quad \begin{aligned} A_i, i &= 1,2,\cdots,n, \\ B_j, j &= 1,2, \\ A_{n+1} &= L_1 \cap L_2, \\ P_{ijkl} = P_{klij} &= L_{ij} \cap L_{kl}. \end{aligned}$$

Observe that there are $n+1$ points on L_1, three points on L_2, and $n+1$ points on L_{ij}. If $i=k$, $P_{ijkl}=A_i$, and if $j=l$, $P_{ijkl}=B_j$. Furthermore, P_{ijkl} and P_{abcd} are on a line of π_1^n if and only of $(i,j)=(a,b)$ or (c,d), or $(k,l)=(a,b)$ or (c,d). With this notation we can describe those configurations in π_1^n which generate π^n. For the moment, since the line L_2 introduces extra configurations when $n=3$, we shall assume $n>3$.

The lines which are candidates for being images of L_1 are L_1 and the L_{ij}. If L_1 is chosen, the set of n points can either be $A_1,\cdots,A_n$, or it can include A_{n+1}. In the former case, the two points can be B_1,B_2 (yielding π_0^n), or any two points P_{ijkl}, P_{abcd} not on a line of π_1^n. To see this, observe that if P_{ijkl},P_{abcd} are not on a line of π_1^n, neither can be a point B_i, so $j\neq l$, $b\neq d$. Let $j=b=1$, $l=d=2$. Then $B_1=L_{i1}\cap L_{a1}$, $B_2=L_{k2}\cap L_{c2}$. On the other hand, if two points ($\neq B_1,B_2$. are on a line of π_1^n, they must be on some L_{ij}, and hence collinear with some A_i and the whole configuration could not possibly generate a π^n. By a similar argument, it can be seen that if $A_1,\cdots,A_{n-1},A_{n+1}$ are taken to be the n points, the two points can either be two points collinear with A_n, or P_{i1k2},P_{a1c2} not on a line of π_1^n, as long as (a) neither $i=c=n$ nor $a=k=n$ is satisfied; (b) one of i, c,a,k is equal to n. For if neither $i=c=n$ nor $a=k=n$ holds, then either the pairs of lines $A_a\cdot P_{a1c2}$ and $A_i\cdot P_{i1k2}$ or the lines $A_c\cdot P_{a1c2}$ and $A_k\cdot P_{i1k2}$ can be defined and their intersection will yield B_1 or B_2, say $B_1=A_a\cdot P_{a1c2}\cap A_i\cdot P_{i1k2}$. But now $L_2=B_1\cdot A_{n+1}$, and $B_2=L_2\cap(A_c\cdot P_{a1c2})$ or $L_2\cap(A_k\cdot P_{i1k2})$, and finally if i,c,a, or k is n, a line through B_1 or B_2 and P_{a1c2} or P_{i1k2} will meet L_1 in A_n. If, on the other hand, $i=c=n$, only the lines $A_a\cdot P_{a1c2}$ and $A_k\cdot P_{i1k2}$ can be drawn from the set of $n+2$ points, and only their point of intersection P_{a1k2} will be added, proving that the configuration will not generate all of π^n. Finally, if i,c,a, and k are all distinct from n, the point A_n can never be generated by the given configuration

If some L_{ij}, say L_{11}, is the line which is the image of L_1 under the collineation, there are three possibilities for the set of n points:

(1) the set of points includes both A_1 and B_1;
(2) only A_1 is included;
(3) only B_1 is included.

We shall cover these cases separately and decide which points can be used as the two points in the various configurations. Under case (1), we can choose the points A_1,B_1,P_{11i2}, $i\neq 1,n$, as the n points. For the other two points, we can either choose any two points ($\neq P_{11n2}$) on the line L_{n2}, so long as one of them is B_2 or A_n; we can have any two points P_{i1k2}, P_{a1c2} not on a line of π_1^n so long as (a) $k,c\neq n$, and (b) $i=n$ or $a=n$; or we can have the point A_{n+1} together with a point on L_{n2}. To see this when the two points are on L_{n2} let P_{a1n2},P_{b1n2} be any two points ($\neq P_{11n2}$) on L_{n2}, and assume that $a,b\neq n$. Then the lines L_{n2},L_{a1},L_{b1} and the point P_{11n2} are generated, but the process goes no further. If $a=n$, however, the lines L_{n2}, $L_1 = A_1 \cdot A_n, L_{b1} = B_1 \cdot P_{b1n2}$, and the point $A_b = L_{b1} \cap L_1$ are generated. Then, $B_2 = L_{n2} \cap (A_b \cdot P_{11b2})$, and all of $\pi_1{}^n$ is easily generated. Similarly, when one of the points is B_2, π_1^n is generated readily. If, on the other hand, the two points are taken to be P_{i1k2}, P_{a1c2}, $i \neq a$, $k \neq c$, then if $c, k \neq n$, $L_{k2} = P_{11k2} \cdot P_{i1k2}$, $L_{c2} = P_{11c2} \cdot P_{a1c2}$, and $B_2 = L_{k2} \cap L_{c2}$. Then if $i = n$ or $a = n$, this case reduces to two points on L_{n2} as above. If neither $i = n$ nor $a = n$, the point A_n will never be generated, and the configuration cannot generate all of π_1^n. Furthermore, if, say, $k = n$, the only lines generated would be $L_{i1} = B_1 \cdot P_{i1k2}$, $L_{a1} = B_1 \cdot P_{a1c2}$, $L_{c2} = P_{11c2} \cdot P_{a1c2}$, and the only points generated would be $P_{i1c2} = L_{i1} \cdot L_{c2}$, thus again all of π_1^n would not be generated. Finally, if A_{n+1} is one of the points, the lines $L_1 = A_1 \cdot A_{n+1}$ and $L_2 = B_1 \cdot A_{n+1}$ are generated. Also, if P_{i1n2} is the point on L_{n2}, $A_i = L_1 \cap (B_1 \cdot P_{i1n2})$, and $B_2 = L_2 \cap (A_i \cdot P_{11i2})$, and we are in the same situation as previously.

In case (2), let the n points be A_1, P_{11i2}. As the two points, we can have any two points ($\neq B$) on a line ($\neq L_{11}$) through B_1, or some point A_i ($\neq A_{n+1}$) and any point P_{a1k2} ($k \neq i$). Finally, in case (3), if the n points are B_1, P_{11i2}, the two points can be any two points ($\neq A_1$) on L_1; or B_2 and any point ($\neq A_1$) on L_{12}; or two points P_{i1k2}, P_{a1c2} not on a line of π_1^n, such that $i = c$ and $a = k$ and such that $k, c \neq 1$; or A_{n+1} and any point not on L_{11} or L_{12}. The proofs in these cases are essentially the same and require a case analysis similar to the ones carried out above.

When $n = 3$, a further line, L_2, can be an image of L_1, and the other two points can be any two points P_{i1k2}, P_{a1c2} not on a line of π_1^3, as long as the equalities $i = c, k = a$ are not both satisfied.

The cases considered above allow one to explicitly write out a set of generators of H_n. They will not be listed here, but the list consists of three generators for $S_n \times S_2$, and representatives of the different types of collineations. We shall prove, however,

THEOREM 9. *There exists an integer m independent of n such that H_n is generated by m collineations for any $n \geqq 3$.*

Proof. Assume $n > 3$, for H_3 is certainly finitely generated. If $A_1, \cdots, A_n$ are the n points, then the two points (if they are not B_1, B_2) are of the form P_{i1k2}, P_{a1c2}, $i \neq a$, $k \neq c$, and are of three types:

(a) $1 \neq c$, $k \neq a$;

(b) $i = c$, $k \neq a$, or $i \neq c$, $k = a$;

(c) $i = c$, $k = a$.

Now, three collineations all fixing the A_i and one mapping B_1, B_2 into each of these types of pairs of points, P, P' will generate all the collineations of rank 1 having the set $\{A_i\}$ as the n points. To see this, for example, assume we wish to generate the collineation $A_i \to A_i$, $B_1 \to P_{x1y2}$, $B_2 \to P_{z1w2}$, $x \neq w$, $y \neq z$. To do this, conjugate the given collineation $\phi : A_j \to A_j$, all $j, B_1 \to P_{i1k2}$, $B_2 \to P_{a1c2}$ by the collineation $A_i \to A_x$, $A_k \to A_s$, $A_a \to A_z$, $A_c \to A_w$, in S^n. Pre-multiplication by elements of S^n will give all possible ways of mapping $\{A_i\}$ into $\{A_i\}$, and both ways of mapping B_j into $\{P_{x1y2}, P_{z1w2}\}$. A similar argument holds for each of the cases (1), (2), (3) above. For example, if a collineation ϕ is given which maps L_1 onto L_{11}, a collineation ϕ' mapping L_1 into L_{ij} can be obtained by conjugating ϕ by the collineation $A_1 \to A_i$, $B_1 \to B_j$. Thus, every collineation of rank 1 can be seen to be generated from a fixed number of collineations of rank 1, no matter which free plane is under consideration.

5. **Comments.** In §3, we have obtained generators for the collineation group of π^2. In §4, it was shown that the group of collineations of π^n, $n > 2$, generated by the rank 0 and rank 1 collineations always has less than a fixed number of generators. The generators were not explicitly written down, as it was felt that the space they would consume would not be justified inasmuch as §§2, 3, and 4 make evident the methods for obtaining them quite easily. A further omission was the relations holding between the generators of G_2. A finite set of relations (possibly not minimal) is known to the author, but the proof that they suffice to determine the group is too long and complicated for inclusion here. Finally, the large question left unanswered by this paper is the equality of H_n and G_n for $n > 2$. The author's conjecture is that they are equal, but the technique of Theorem 6 cannot be used in a proof for the general case.

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