PROJECTION CONSTANTS AND SPACES
OF CONTINUOUS FUNCTIONS

BY

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1. Theorems. A real Banach space X will be called *injective*[2] if for every Banach space Y and subspace Y_0, every linear operation[3] $T_0 : Y_0 \to X$ can be extended to a linear operation $T : Y \to X$. An equivalent condition is this: for every Banach space Z containing X as a subspace, there exists a projection from Z onto X. The class of injective spaces is usually denoted by $\mathfrak{P}$, and $\mathfrak{P}_s$ denotes the class of all spaces X for which such a projection can always be found with norm $\leqq s$. If $X \in \mathfrak{P}$, then $X \in \mathfrak{P}_s$ for some $s \geqq 1$; the infimum $p(X)$ of such numbers s is called the *projection constant* of X, and is called *exact* if the infimum is attained. Any Banach space Y can be isometrically embedded into a space Z belonging to $\mathfrak{P}_1$; and $Y \in \mathfrak{P}_s$ if and only if there exists a projection[4] $P : Z \to Y$ with $\| P \| \leqq s$ from this particular space, the best constant being independent of the choice of Z and of the embedding[5]. If $X \notin \mathfrak{P}$, we shall write $p(X) = \infty$.

The class $\mathfrak{P}_1$ is nicely characterized[6]: $X \in \mathfrak{P}_1$ if and only if X is isometric to a space $C(S)$ with S extremally disconnected and compact. The only known examples of spaces of $\mathfrak{P}$ are spaces isomorphic to these spaces of $\mathfrak{P}_1$[7].

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[2] We use the term *injective* to be consistent with the standard algebraic terminology. Two other terms are in use: *extension property* and *projection property* (note that this does not correspond to the algebraic term *projective*).

[3] In this paper, linear operations are continuous.

[4] The letter P will always denote projections and if $P : Z \to Y$ is a projection, we mean $P(Z) = Y$.

[5] Cf. Phillips [29], Goodner [13], Day [6, p. 94], Grünbaum [17], Nachbin [23], Sobczyk [37].

[6] Cf. Stone [38; 39], Nakano [24], Dilworth [7], Sikorski [35, p. 257; 36], Nachbin [22], Kelley [18], Aronszajn and Panitchpakdi [2], Grünbaum [16], Grothendieck [14; 15], Lindenstrauss [20; 21], Fullerton [11], and others. Let us mention the following characterization: $X \in \mathfrak{P}_1$ if and only if X is isometric to the quotient space of all bounded functions satisfying the condition of Baire (cf. [19, pp. 51–60, 306–311]) on a topological space T neglecting their values on sets of the first category.

[7] We do not know if there exists a space in $\mathfrak{P}$ which is not isomorphic to a space of $\mathfrak{P}_1$ and this problem seems to be difficult.

Two questions are considered in this paper. The first is the projection constants of spaces $C(S)$. The simplest example of a space $C(S)$ belonging to $\mathfrak{P}$ but not to $\mathfrak{P}_1$ is[8]

$$X = \{x \in C(\beta N) : x(t_1) = x(t_2)\},$$

where t_1 and t_2 are two points of $\beta N \setminus N$; in other words, S is obtained by identification of two nonisolated points of βN. We shall show that all the numbers $1, 2, 7/3, \cdots, 3 - 2/n, \cdots, 3$ are projection constants of spaces $C(S)$ (in our examples, they are exact), but we do not know what other numbers are possible.

The theorem below establishes some necessary and some sufficient conditions concerning the possible projection constants of spaces $C(S)$ in terms of the Gleason map of S. A. M. Gleason [12] has proved the following theorem: For each compact S there exists a unique extremally disconnected compact space G_S with a unique irreducible[9] map g_S from G_S onto S. We shall call G_S the *Gleason space* of S and g_S the *Gleason map* of S. Define the *order* $\iota(S)$ of the Gleason map as the least integer n such that all inverse sets $g_S^{-1}(t)$, $t \in S$, consist of at most n points, and $\iota(S) = \infty$ if no such integer exists; $\iota(S)$ has a simple topological characterization (see Lemma 7 below).

THEOREM 1([10]). *Let $p_C(S)$ be the projection constant of $C(S)$, S being a compact Hausdorff space. Then*

(i) *If $p_C(S) < 3$ then the Gleason map of S is of finite order; specifically*

$$\iota(S) \leqq 2[3 - p_C(S)]^{-1}.$$

(ii) *If $p_C(S) < 2$ then $p_C(S) = 1$.*

(iii) *If $\iota(S) = n$ and only a finite number of inverse sets $g_S^{-1}(t)$, $t \in S$, contain more than one point, then $p_C(S) = 3 - 2/n$.*

(iv) *There are spaces S such that $p_C(S) = 3$.*

This leaves the problem of topological characterization of spaces S for which $C(S)$ is injective entirely open. Neither of the conditions $p_C(S) < \infty$, $\iota(S) < \infty$, implies the other.

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(8) In the category of compact spaces and continuous mappings, extremally disconnected spaces are just those which are projective (cf. Gleason [12], Rainwater [30]), and in the category of Banach spaces, $\mathfrak{P}$ is just the class of injective spaces. If we consider $C(S)$ as a dual of S, then this example shows that the duals of projectives are injective but injectivity of the dual need not imply the projectivity of a compact space.

(9) A continuous map $f : A \rightarrow B$ will be called irreducible if $f(A) = B$ but $f(F) \neq B$ for any proper closed subset F of A. Uniqueness here means that for such G_S, g_S and G_S', g_S' there is a homeomorphism h from G_S' onto G_S such that $g_S' = g_S \cdot h$.

(10) While this paper was being typed, Professor Branko Grünbaum informed us of results similar to Theorem 1 obtained simultaneously by D. Amir. *Added in prooff.* Amir's paper has been published in Bull. Res. Council Israel **11** (1962).

The second topic considered in the paper is exactness of the projection constant of a Banach space. The question has been raised whether the projection constant of every space $X \in \mathfrak{P}$ is exact. The answer is *no* and it follows from the following theorem.

THEOREM 2. *Let S be a compact space containing infinitely many non-isolated points. Then there exists a Radon (= regular finite Borel signed) measure μ on S such that the hyperplane*

$$X = \left\{ x \in C(S) : \int x \, d\mu = 0 \right\}$$

does not admit any projection $P : C(S) \to X$ with norm $\| P \| \leq 2$. For spaces S containing only finitely many nonisolated points, there is no such measure[11].

Combine this theorem with the fact that any Banach space Z can be projected onto any of its hyperplanes by projections with norm $< 2 + \varepsilon$, for arbitrary $\varepsilon > 0$[12]. We conclude: *Every infinite dimensional space in $\mathfrak{P}_1$ contains a hyperplane having the nonexact projection constant 2.*

B. Grünbaum has asked whether the projection constant of every finite dimensional space is exact. The answer is *yes* and may be stated in more general form as follows.

THEOREM 3. *Let X be a subspace of a Banach space Z admitting a projection $P : Z \to X$ and let X be isometric to the conjugate space of a Banach space Y. Then there exists a projection $P_0 : Z \to X$ such that for any projection $P : Z \to X$ we have $\| P \| \geq \| P_0 \|$.*

In particular, if $X \in \mathfrak{P}$, then application of Theorem 3 to a space $Z \in \mathfrak{P}_1$ containing X yields the conclusion: *If $X \in \mathfrak{P}$ and X is a conjugate space, then the projection constant of X is exact.*

The paper is concluded with some remarks and problems about projection constants.

2. **Proofs.** Besides that in the preceding section, let us establish the following notation. S, S_0, $S_1, \cdots$ will denote always compact Hausdorff spaces and $C(S)$

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[11] Let Γ_α be the set of ordinals smaller than or equal to α, compact in the order topology. Γ_ω has just one limit point and Γ_{ω^2} has infinitely many of them, but $C(\Gamma_\omega)$ is isomorphic to $C(\Gamma_{\omega^2})$. Thus, the existence of projections of norm $\| P \| \leq 2$ onto each hyperplane is not an isomorphic invariant.

Dr. Joram Lindenstrauss recently informed us that he had proved the following: if the projection constant of a Banach space is equal to one, it is exact. At the same time, Dr. A. Pełczyński proved that for each number $s > 1$ there exists a space X such that $p(X)$ is equal to s and is not exact.

[12] This result of Bohnenblust [4] has been discussed by Grünbaum [16] who has proved the existence of a hyperplane H in l_1 which does not admit any projection $P : l_1 \leftarrow H$ with $\| P \| \leq 2$. Since $l_1 \notin \mathfrak{P}$, this example did not answer the question on exactness.

the space of real-valued continuous functions on S, $C(S \mid A) = \{x \in C(S) : x(t) = 0$ for $t \in A\}$. Further, $f : A \to B$ will always mean a *continuous* map from A into B. Similarly, if X, Y are Banach spaces, $T : X \to Y$ will mean a linear operation. χ_A will mean the characteristic function of A. Extremally disconnected may be abbreviated e.d. The m-product $(X_1 \times X_2 \times \cdots)_m$ of a sequence of Banach spaces means the space of all sequences $x = (x_1, x_2, \cdots)$ such that $x_n \in X_n$ and $\sup \| x_n \| < \infty$. Finally, $\mu(x) = \int x \, d\mu$ and $X_\mu = \{x \in C(S) : \mu(x) = 0\}$.

We begin with some lemmas which will be stated without proof.

LEMMA 1. *Let F be a closed nowhere dense subset of an e.d. compact space E and let S be the quotient space obtained by the identification of all points of F. Then S is compact, $E = G_S$, the identification map is the Gleason map of S. If F is infinite, $\iota(S) = \infty$.*

LEMMA 2. *Let F_1, F_2 be two nowhere dense disjoint closed subsets of an e.d. compact space E and let $h : F_1 \to F_2$ be a homeomorphism onto. Then S obtained by pairwise identification of the points t and $h(t)$ is compact, $E = G_S$, the identification map is the Gleason map of S and $\iota(S) = 2$.*

LEMMA 3. *Suppose that $0 \neq F = \bar{F} \subset S$ and that there exists a simultaneous extension of the continuous functions on F to continuous functions on S, i.e., a linear operation $T : C(F) \to C(S)$ such that $(Tx)(t) = x(t)$ for $t \in F$. Then:*

(i) *$P_1(x) = T(\mathrm{Rest}_F x)$ is a projection from $C(S)$ onto the subspace of extended functions and $\| P_1 \| = \| T \|$, Rest_F being the restriction to F.*

(ii) *$P_2(x) = x - P_1(x)$ is the complementary projection from $C(S)$ onto $C(S \mid F)$ and $\| P_2 \| = 1 + \| T \|$.*

(iii) *Given a point $t_0 \in F$, $P_3(x) = P_2(x) + x(t_0)$ is a projection from $C(S)$ onto the subspace of functions constant on F and $\| P_3 \| \leqq 2 + \| T \|$.*

LEMMA 4. *Let $S = F_1 \cup \cdots \cup F_m$ be a decomposition of an infinite space S into disjoint open sets and let $p_1 \in F_1, \cdots, p_m \in F_m$. Then*

$$P(x) = x + \sum_{k=1}^{m} x(p_k) \left(\frac{1}{m} - \chi_{F_k} \right)$$

is a projection from $C(S)$ onto $\{x \in C(S) : x(p_1) = \cdots = x(p_m)\}$ and $\| P \| = 3 - 2m^{-1}$.

LEMMA 5. *Given any sequence of Banach spaces $X_1, X_2, \cdots$, the equality $p[(X_1 \times X_2 \times \cdots)_m] = \sup_n p(X_n)$ holds (if one side is finite, so is the other). In particular, if S_n are mutually disjoint, then $p_C[\beta(\bigcup S_n)] = \sup p_C(S_n)$.*

LEMMA 6. *Any open nonvoid set in G_S contains at least one inverse set $g_S^{-1}(t)$, $t \in S$.*

Indeed, if it were not so, g_S would be reducible.

LEMMA 7. *Given a point $t_0 \in S$, the following conditions are equivalent:*

(a)   *There are n different points $u_1, \cdots, u_n$ in G_S such that $g_S(u_1) = \cdots = g_S(u_n) = t_0$.*

(b)   *There are n disjoint open subsets $H_1, \cdots, H_n$ of S such that $t_0 \in \bar{H}_1 \cap \cdots \cap \bar{H}_n$.*

Proof.   Suppose (a). Let $U_1, \cdots, U_n$ be disjoint open-closed subsets of G_S such that $U_1 \cup \cdots \cup U_n = G_S$ and $u_k \in U_k$ for $k = 1, \cdots, n$. Let $H_k = \{t \in S : g_S^{-1}(t) \subset U_k\}$ and $V_k = g_S^{-1}(H_k)$. Since $H_k = S \setminus \bigcup_{j \neq k} g_S(U_j)$, H_k and V_k are open in S and G_S, respectively, and V_k is dense in U_k (by Lemma 6). Obviously, $H_1, \cdots, H_n$ are disjoint and $t_0 = g_S(u_k) \in g_S(U_k) = \bar{H}_k$ for $k = 1, \cdots, n$.

Suppose (b). The sets $g_S^{-1}(H_k)$ are disjoint open sets in an e.d. space; hence their closures U_k are disjoint, $k = 1, \cdots, n$. Since $g_S(U_k)$ is a compact set containing H_k, there is at least one point $u_k \in U_k$ such that $g_S(u_k) = t_0$.

LEMMA   8.   *Let M be a Hausdorff space containing infinitely many non-isolated points. Then M has an infinite nowhere-dense subset.*

Proof.   If M does not contain any perfect[13] subset, the set of nonisolated points of M is nowhere dense. If M does contain a perfect Q, Q contains an infinite subset T which is discrete in the relative topology; so T is nowhere dense (cf. [19, p. 46]).

LEMMA   9.   *Let μ be a Radon measure on S such that $\|\mu\| = 1$ and let some isolated point t_0 of S have positive measure $\mu(t_0) = a_0 > 0$. Then there exists a projection $P : C(S) \to X_\mu$ of norm $\|P\| \leqq 2$.*

Proof.   If $a_0 = \|\mu\| = 1$, then there exists a projection of norm 1. Suppose $a_0 < 1$ and choose $y \in C(S)$ so that $\|y\| = 1$, $\mu(y) > a_0$, $\mu(y) > 1 - a_0$, $y(t_0) = 1$. Let $z(t) = y(t)$ for $t \neq t_0$ and $z(t_0) = 1 + (1 - \mu(y))a_0^{-1}$, and let $P(x) = x - \mu(x)z$; then $P : C(S) \to X_\mu$ as $\mu(z) = 1$. If $x \in C(S)$ and $\|x\| \leqq 1$, then $|x(t) - \mu(x)z(t)| \leqq 2$ for $t \neq t_0$ and $|x(t_0) - \mu(x)z(t_0)| \leqq |x(t_0)| \, |1 - a_0 z(t_0)| + z(t_0) \|x\| \mu(S \setminus t_0) \leqq 1 - a_0 z(t_0) + (1 - a_0) z(t_0) \leqq 1 - 2a_0 + z(t_0) < 2$. Thus, $\|P\| \leqq 2$.

Proof of Theorem 1.   Let $\iota(S) \geqq n$ and let P be any projection from $C(G_S)$ onto the subspace Z of $C(G_S)$ consisting of functions constant on the inverse sets $g_S^{-1}(t)$ (Z is the image of $C(S)$ in $C(G_S)$). We have to prove that $\|P\| \geqq 3 - 2n^{-1}$. By Lemma 7, there are open disjoint sets $H_1, \cdots, H_n$ in S and a point $t_0 \in S$ such that $t_0 \in \bar{H}_1 \cap \cdots \cap \bar{H}_n$, the closures U_k of $g_S^{-1}(H_k)$ are disjoint, and $G_S = U_1 \cup \cdots \cup U_n$. Let $u_k \in U_k \cap g_S^{-1}(t_0)$. Then $x(u_1) = \cdots = x(u_n)$ for $x \in Z$.

Let $\phi_k = P(\chi_{U_k})$ and $c_k = \phi_k(u_k) = \phi_k(u_j)$ for $k, j = 1, \cdots, n$. Then $\sum \phi_k = P(\sum \chi_{U_k}) = 1$ whence $\sum c_k = 1$. Choose i so that $c_i \leqq 1/n$. Let $f = 2\chi_{U_i} - 1$ and $q = Qf$ where $Q = I - P : C(G_S) \to \{x : Px = 0\}$. Then $q(u_i) = f(u_i) - (2c_i - 1) = 2 - 2c_i \geqq 2 - 2/n$. Given any $\varepsilon > 0$, there is $w_0 \in g_S^{-1}(H_i)$ such that $q(w_0) > 2 - 2n^{-1} - \varepsilon$. Let $h_0 : S \to [0,2]$ be such that $h_0(g_S(w_0)) = 2$ and

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[13] By a *perfect* set we mean a nonempty closed dense-in-itself set.

$h_0(t) = 0$ for $t \in S \setminus H_i$, and let $h = h_0 \cdot g_S$. Then $h \in Z$, $h(v) = 0$ for $v \in G_S \setminus U_i$, $|f(v) - h(v)| \leqq 1$ for $v \in U_i$ and $f(w_0) - h(w_0) = -1$, whence $\|h - f\| = 1$, $P(h - f) = h - f - Q(h) + Q(f) = h - f + q$ and $(h - f + q)(w_0) = 2 - 1 + q(w_0)$ $> 3 - 2n^{-1} - \varepsilon$. Thus, $\|P\| \geqq 3 - 2n^{-1}$ and (i) has been proved.

(ii) follows from (i), because if $p_C(S) < 2$, then g_S must be one-to-one. (iii) follows from Lemma 4 and (iv) follows from (iii) and Lemma 5. (iv) can be also proved in another way: Take any e.d. dense-in-itself compact space H (i.e., the Stone space of a complete atomless Boolean algebra). There exists βN_α containing H as a retract (cf. Rainwater [30]); obviously, $H \subset \beta N_\alpha \setminus N_\alpha$. Consider S obtained from βN_α by shrinking H into a point. By Lemma 3, there is a projection P_3 from $C(\beta N_\alpha)$ onto the subspace of functions constant on H, of norm less than or equal to 3. So $p_C(S) \leqq 3$; $p_C(S) < 3$ would contradict (i) as the identification map $\beta N_\alpha \to S$ is the Gleason map (by Lemma 1).

Proof of Theorem 2. Let S be compact, with infinitely many nonisolated points; let $\{t_k\}$ be an infinite nowhere dense set (Lemma 8) with a limit point t_0 such that $t_0 \neq t_k$ for $k = 1, 2, \cdots$. Define $a_0 = -1/2$; for $n = 1, 2, \cdots$, define a_n as $1/2^{n+1}$. Let μ be the atomic measure $\sum_{n=0}^{n=\infty} a_n \varepsilon_{t_n}$. The norm $\|\mu\|$ is 1; the support M of μ is the closure of $\{t_k\}$. Consider any $P : C(S) \to X_\mu$; necessarily $P(x) = x - \mu(x)u$ for some $u \in C(S)$ such that $\mu(u) = 1$. Since μ does not attain its norm, $\|u\| > 1$; we may write $\|u\| > 1/(1 - \varepsilon)$, $\varepsilon > 0$. There is an open set $V \subset S$, such that $V \cap M = \emptyset$ and either $u(t) > (1 - \varepsilon)^{-1}$ on V or $-u(t) > (1 - \varepsilon)^{-1}$ on V. Take x such that $\|x\| = 1$ and $\mu(x) > 1 - \varepsilon$; let y coincide with x on $(S \setminus V) \cup M$, but at some point v of V, let $y(v) = -\operatorname{sgn} u(v) \neq 0$. This can be done with $\|y\| = 1$. Since $y = x$ on M, $\mu(y) > 1 - \varepsilon$. Then $\|P(y)\| = \|y - \mu(y)u\| > 2$ as

$$\left| y(v) - \mu(y)\, u(v) \right| > \left| -1 - (1 - \varepsilon)(1 - \varepsilon)^{-1} \right| = 2.$$

Now, suppose that S has only finitely many nonisolated points. Given any Radon measure μ on S with $\mu(S) = 1$, we have to prove that there exists $P : C(S) \to X_\mu$ with $\|P\| \leqq 2$. If the measure is concentrated on nonisolated points, there exists $u \in C(S)$ such that $\|u\| = 1$ and $\mu(u) = 1$, whence $P(x) = x - \mu(x)u$ has the desired property; in the contrary case we apply Lemma 9.

Proof of Theorem 3. The relation $\psi(z, y) = (Tz)y$ establishes a one-to-one isometrically isomorphic correspondence from the space $B(Z, X)$ of linear operations from Z to X onto the space $H(Z, Y)$ of bilinear functionals on $Z \times Y$. Let $\{z_\alpha\}_{\alpha \in A}$ and $\{y_\beta\}_{\beta \in B}$ be normalized Hamel bases for Z and Y, respectively. The sets

$$U(T_0) = \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} \{ T \in B(Z, X) : \left| (Tz_{\alpha_i} - T_0 z_{\alpha_i}) y_{\beta_j} \right| < \varepsilon \}$$

with $\alpha_1, \cdots, \alpha_n \in A$, $\beta_1, \cdots, \beta_m \in B$, $\varepsilon > 0$, form a basis of neighborhoods for a

*-weak operator topology in $B(Z,X)$. The space H_0 of all (not necessarily continuous) bilinear forms on $Z \times Y$ is homeomorphic to a product of real lines, and

$$K_0 = \bigcap_{\alpha \in A} \bigcap_{\beta \in B} \{\psi \in H_0 : |\psi(z_\alpha, y_\beta)| \leq 1\}$$

is homeomorphic to a Tichonov cube. Since the unit cell $K = \{T \in B(Z,X) : \|T\| \leq 1\}$ is mapped onto a closed subset of K_0, K is compact.

Now, if $P_n : Z \to X$ are projections and $\|P_n\| \leq s + n^{-1}$, then any limit point P of P_n in this topology is a projection $P : Z \to X$ and $\|P\| \leq s$.

3. **Remarks and problems.** We begin by showing some counterexamples.

PROPOSITION 1. *There exists S such that $\iota(S) = 2$ and $p_C(S) = \infty$.*

Proof. Let N', N'' be two countable discrete spaces, $h : \beta N' \to \beta N''$ an identification homeomorphism, let $Q = \beta N' \cup \beta N''$ (as disjoint copies) and

$$X = \{x \in C(Q) : x(t) = x(h(t)) \text{ for all } t \in \beta N' \setminus N'\}.$$

Then X is isometric to a space $C(S)$ and $\iota(S) = 2$ by Lemma 2. On the other hand, let $(Tx)(t) = x(t)$ for $t \in \beta N'$ and $(Tx)(t) = x(t) - x(h^{-1}(t))$ for $t \in \beta N''$. Then T is an isomorphism from X onto $C(Q | \beta N'' \setminus N'')$ and the last space is obviously isomorphic to the Cartesian product of $C(\beta N)$ and $C(\beta N | \beta N \setminus N)$. In turn, $C(\beta N | \beta N \setminus N)$ is isometric to the space c_0 of sequences convergent to 0. Thus, $X \notin \mathfrak{P}$, because if X were injective, so would c_0 be, contradicting a well-known theorem of Phillips [29, p. 539].

PROPOSITION 2. *There exists S such that $\iota(S) = \infty$ and $p_C(S) = 3$.*

Such an example was considered in the proof of Theorem 1.

PROPOSITION 3. *There exists S such that $\iota(S) = 2$, $p_C(S) = 2$, but infinitely many inverse sets contain more than one point.*

Indeed, let S_0 be obtained by identification of two nonisolated points of βN and let S be β of the union of $\aleph_0$ disjoint copies of S_0.

Another construction of spaces with $p_C(S) < \infty$ can be obtained by the following generalization of Lemma 4.

PROPOSITION 4. *Suppose $A_1, \cdots, A_m$ are closed, nonvoid, nowhere dense subsets of S; $F_1, \cdots, F_m$ open disjoint sets such that $S = F_1 \cup \cdots \cup F_m$, $A_k \subset F_k$ and suppose $T_k : C(A_k) \to C(F_k)$ are simultaneous extensions $(k = 1, \cdots, m)$. Let h_{ij} be a homeomorphism of A_i onto A_j $(i, j = 1, \cdots, m)$ such that $h_{ij} = h_{kj} \cdot h_{ik}$. Given a function $f \in C(S)$, let*

$$g_k(t) = \sum_{j=1}^{m} \left(\frac{1}{m} - \delta_{jk}\right) f(h_{kj}(t))$$

*for $t \in A_k$ and $(Pf)(t) = f(t) + (T_k g_k)(t)$ for $t \in F_k$. Then $(Pf)(t) = 1/m \, \sum_j f(h_{ij}(t))$
for $t \in A_i$, whence*

$$P : C(S) \rightarrow \bigcap_{i,j} \bigcap_{t \in A_i} \{x \in C(S) : x(h_{ij}(t)) = x(t)\},$$

and $\|P\| \leqq 1 + (2 - 2n^{-1}) \sup \|T_k\|$.

LEMMA 10. *Let μ be any Radon measure on S, let V be an infinite open
subset of S and let $\varepsilon > 0$. There exists an open nonvoid set $W \subset V$ such that
$|\mu|(W) < \varepsilon$.*

PROPOSITION 5. *Let μ be a Radon measure on S, $\|\mu\| = 1$, and let M_1 and
M_2 be the supports of μ_+ and μ_-, respectively. Suppose that $M_1 \cup M_2$ does not
contain any of the isolated points of S and that $P : C(S) \rightarrow X_\mu$ is of the form
$P(x) = x - \mu(x)u$ with $u \in C(S)$ and $\mu(u) = 1$. Then $\|P\| \geqq 1 + \|u\| \geqq 2$. If
$M_1 \cap M_2 \neq \emptyset$, then $\|u\| > 1$ whence $\|P\| > 2$.*

We omit the proofs.

Many problems have been raised on the class $\mathfrak{P}$ and some of them seem to be
difficult. Besides those in [23; 25], and others, the following ones seem to be open.

Theorem 3 gives a sufficient condition for exactness of a projection constant,
but not necessary. There exist spaces in $\mathfrak{P}_1$ which are not isometric to any con-
jugate space (cf. Dixmier [9], Kelley [18], Wada [40]; the simplest example
is the space of bounded Baire functions on an interval modulo null functions,
but this space is isomorphic to $C(\beta N)$ (cf. [25, p. 223; 33])([14]).

PROBLEM 1. *Let $X \in \mathfrak{P}_1$. Is X isomorphic to a conjugate space?*

There are spaces in $\mathfrak{P}_1$ which are not isomorphic to any space $C(\beta N_\alpha)$, N_α
being a discrete space of power $\aleph_\alpha$ (cf. [25, p. 223]).

PROPOSITION 6. *Let $X \in \mathfrak{P}$, $Z \in \mathfrak{P}_1$ and $X \subset Z$; let $P : Z \rightarrow X$, $\|P\| = s > 1$.
Let $\| \ \|_0^*$ be the norm in X treated as the quotient Z/Z_0 where Z_0
$= (I - P)(Z) = \{z : Pz = 0\}$, i.e., let $\|x\|_0^* = \inf\{\|z\| : x = Pz\}$. Let $\| \ \|^*$ be
the Minkowski functional of the convex hull of the union of $\{x \in X : \|x\|_0^* \leqq 1\}$
and $\{x \in Z : \|z\| \leqq 1\}$. Then $\| \ \|^*$ is a norm in Z, $\|z\|^* \leqq \|z\| \leqq s\|z\|^*$ for
$z \in Z$, $\|x\|^* = \|x\|_0^*$ for $x \in X$ (cf. [1, p. 129; 37, p. 167]) and P treated as an
operation from $\langle Z, \| \ \|^* \rangle$ onto $\langle X, \| \ \|^* \rangle$ is a projection with $\|P\|^* = 1$.*

PROBLEM 2. *Does $\langle Z, \| \ \|^* \rangle$ belong to $\mathfrak{P}_1$?*

PROBLEM 3. *Characterize the compact spaces S such that $p_C(S) < \infty$. Estab-
lish all possible values of $p_C(S)$.*

The second part is related to the problem of estimation of the infimum of the
norms of simultaneous extensions of continuous functions from a given sub-

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[14] A. Pełczyński has remarked that if a space $C(S)$ is isomorphic to the conjugate space
of a Banach space, then $C(S) \in \mathfrak{P}$. Indeed, by a theorem of Dixmier [8, p. 1066], for every Banach
space X, there exists a projection of norm 1 from X^{***} onto the canonical image of X^* in X^{***}
and the second dual of any space $C(S)$ is in $\mathfrak{P}_1$.

space S_0 of a space S. There are some sufficient conditions for existence of such $T : C(S_0) \to C(S)$ with $\| T \| = 1$ (Borsuk [5], Kakutani, Dugundji [10] and others), but little is known about the case $\| T \| > 1$.

PROBLEM 4. *Suppose $p_C(S) < \infty$. Must this projection constant be exact? Must $C(S)$ be isomorphic to a space of $\mathfrak{P}_1$?*

Now, given a Banach space X, let X_s be the set of all functionals x^{**} of the bidual X^{**} which are *sequentially* continuous with respect to the weak topology $\sigma(X^*, X)$, i.e., such that the condition $x_n^*(x) \to 0$ for all $x \in X$ implies $x^{**}(x_n^*) \to 0$.

LEMMA 11. *If $T^{**} : X^{**} \to Y^{**}$ is the biconjugate of a linear operator $T : X \to Y$, then $T^{**}(X_s) \subset Y_s$.*

The proof is trivial. Let $\mathfrak{G}$ be the class of Banach spaces X such that $X_s = X^{**}$. A. Grothendieck [14, p. 168] has proved that $\mathfrak{P} \subset \mathfrak{G}$.

PROPOSITION 7. *If $T : X \to Y$ is onto and X belongs to $\mathfrak{G}$, so does Y. In particular, if $Z \subset X$ and $X \in \mathfrak{G}$, then $X/Z \in \mathfrak{G}$. If $C(S) \in \mathfrak{G}$ and $S_0 \subset S$, then $C(S_0) \in \mathfrak{G}$ as $C(S_0) = C(S)/C(S \mid S_0)$.*

PROPOSITION 8. *Let A be a σ-complete Boolean algebra and R an ideal in A. Let S be the Stone space of the algebra A/R. Then $C(S) \in \mathfrak{G}$.*

Proposition 7 follows from Lemma 11. Proposition 8 can be proved as follows. Let S_1 be the Stone space of A; then $C(S_1) \in \mathfrak{G}$ can be proved analogously to the original proof of Grothendieck, i.e., applying a theorem of Banach [3, p. 222] and reducing the question to a lemma of Phillips [29, p. 525]. Since the Stone space S of the quotient A/R is homeomorphic to a closed subset of S_1, $C(S) \in \mathfrak{G}$ follows from Proposition 7.

There are complete Boolean algebras A with A/R being not σ-complete. E.g., $\beta N \setminus N$ is the Stone space of the algebra of subsets of N modulo finite subsets, which is not σ-complete (Sierpiński [34]), but $C(\beta N \setminus N) = m/c_0 \in \mathfrak{G}$.

PROBLEM 5. *Do $C(\beta N_\alpha \setminus N_\alpha)$ belong to $\mathfrak{P}$?*

Pełczyński and Sudakov [28] have proved that if S is the Stone space of the algebra of all countable subsets of an uncountable set and their complements, then $C(S) \notin \mathfrak{P}$ though $C(S) \in \mathfrak{G}$([15]).

By a theorem of Banach [3, p. 124] and Dixmier [8, p. 1060], if X is separable, then X_s is just the canonical image of X in X^{**}. This is the way Grothendieck [14] has proved that any separable injective Banach space must be finite dimensional (for another proof, see [25, p. 222]).

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[15] J. Lindenstrauss has pointed out that a space X^* considered in §3 of [21] does not belong to $\mathfrak{P}$ though there exists a linear operation from $C(\beta N)$ onto X^* (whence $X^* \in \mathfrak{G}$). Another example is due to A. Pełczyński: If S contains a perfect subset, then the dual of $C(S)$ contains isomorphically the space L [27, p. 214] and L contains L^2 (spanned on Rademacher functions). Considering an adjoint operation, we get a linear operation from $C(S)$ *onto* L^2. Conversely, if $C(S)$ can be mapped onto L^2, then S contains a perfect set).

PROPOSITION 9. *Let $C(S) \in \mathfrak{G}$. Then no infinite closed subset of S is metrisable. In particular, the only G_δ-points of S are the isolated ones and the derivative of S is dense-in-itself.*

PROBLEM 6. *Characterize topologically those Borel subsets A of S such that the functional $x^{**}(\mu) = \mu(A)$ (defined for all Radon measures on S) belongs to X_s where $X = C(S)$, i.e., such that $\int x d\mu_n \to 0$ for all $x \in C(S)$ implies $\mu_n(A) \to 0$.*

There are some sufficient conditions; e.g., if A is closed and every G_δ-set containing A is a neighborhood of A, then A has this property (cf. [32]).

PROBLEM 7. *Let $X \in \mathfrak{P}_1$ and be infinite dimensional. Is X isomorphic to its Cartesian square?*

This problem is due to A. Pełczyński (cf. [26]). There exists a compact space S such that $C(S)$ is not isomorphic to its square (cf. [32]).

We conclude with two examples of very annoying problems which always arise in a field where so little is known as in this field.

PROBLEM 8. *Let $X \in \mathfrak{P}$ and be infinite dimensional. Is X isomorphic to any of its hyperplanes (equivalently, is X isomorphic to $X \times R$, R being the space of reals)?*

If $X \in \mathfrak{P}_1$, then X contains a subspace isometric to $m = C(\beta N)$ whence $X \sim m \times X/m \sim m \times m \times X/m \sim m \times X$ and similarly $X \sim X \times m \sim X \times m \times R \sim X \times R$, where $\sim$ means isomorphism (cf. [25]). On the other hand, if $X \in \mathfrak{P}$ and X_1 is a hyperplane in X, then $X_1 \in \mathfrak{P}$; suppose that $X \sim X^2$ and $X_1 \sim X_1^2$, then $X \sim X^2 \sim (X_1 \times R)^2 \sim X_1^2 \times R^2 \sim X_1 \times R^2 \sim X \times R$.

PROBLEM 9. *Does there exist a connected S such that $C(S) \in \mathfrak{G}$?*

Amir's results([10]) show that such $C(S)$ cannot be injective.

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