# A DECOMPOSITION THEORY FOR REPRESENTATIONS OF $C^*$-ALGEBRAS

BY

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**Introduction.** An important aspect of the representation theory of $C^*$-algebras is the decomposition of representations into direct integrals of representations. The original goal of this procedure was to reduce the classification of arbitrary representations to that of irreducible representations. A significant step in this direction was taken by G. W. Mackey (see [19]). Stated in the terminology of $C^*$-algebras, he considered the collection $\hat{\mathscr{A}}$ of unitary equivalence classes of irreducible representations of a separable $C^*$-algebra $\mathscr{A}$, defining on $\hat{\mathscr{A}}$ a natural Borel structure. Using direct integrals over $\hat{\mathscr{A}}$ he found a one-to-one correspondence between certain unitary equivalence classes of multiplicity-free representations of $\mathscr{A}$ and certain null-set equivalence classes of Borel measures on $\hat{\mathscr{A}}$. Due to a result of A. Guichardet [9], it is now known that all of the separable multiplicity-free representations occur in the correspondence, and the major obstacle remaining in the theory is to distinguish the measure classes of interest. It is known (see [19, p. 164]) that many Borel measure classes in $\hat{\mathscr{A}}$ do not give rise to multiplicity-free representations.

Recently John Ernest [4; 5] generalized Mackey's theory to the study of representations that are not of type I, i.e., which are not discrete sums of multiplicity-free representations. There does not seem to be any canonical reduction into irreducibles for such representations. Ernest considered instead decompositions over the Borel space $\hat{\mathscr{A}}$ of quasi-equivalence classes of separable factor representations. In particular, he found a one-to-one correspondence between the quasi-equivalence classes of separable representations and certain "canonical" measure classes on $\hat{\mathscr{A}}$. Again, no criterion was discovered for recognizing the latter.

In this paper we shall consider a third decomposition. Our "dual space" will be the primitive ideal space $\mathrm{pr}\mathscr{A}$, together with the hull-kernel topology and Borel structure. Measures on this space correspond to the decomposition of representations into "homogeneous" representations, i.e., representations all of whose nontrivial subrepresentations have a given primitive ideal kernel. These reductions are useful for relating the ideal structure of $\mathscr{A}$ to the geometry of

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its representations. If $\mathscr{A}$ is of type I, $\mathrm{pr}\mathscr{A}$ coincides with $\hat{\mathscr{A}}$ (see [6, Theorem 3.1; 8, Theorem 1]) and we obtain an ideal-theoretic characterization of quasi-equivalence. Although the decomposition is necessarily "coarse" when $\mathscr{A}$ is not of type I, this is partially compensated by the absence of "bad" measures on $\mathrm{pr}\mathscr{A}$. Arbitrary measures on $\mathrm{pr}\mathscr{A}$ induce canonical measures on $\hat{\mathscr{A}}$ and $\tilde{\mathscr{A}}$.

In §1 we prove that each representation of a $C^*$-algebra $\mathscr{A}$ determines a "Galois type" correspondence between certain ideals in $\mathscr{A}$ and certain invariant subspaces of the representation. The "ideal center" generated by the latter splits the representation into homogeneous representations with distinct kernels. In general, an abelian von Neumann subalgebra of the commutant of a separable representation of a separable $C^*$-algebra determines a decomposition into homogeneous representations if and only if it contains the ideal center.

In §2, we prove that the primitive ideal space of a separable $C^*$-algebra is a standard Borel space. This is accomplished by showing that in Fell's identification of ideals with pseudo-norms [7], the primitive ideals correspond to "extremal" pseudo-norms.

§3 is devoted to the study of direct integrals over the primitive ideal space. A one-to-one correspondence is set up between the measure classes on $\mathrm{pr}\mathscr{A}$ and certain equivalence classes of representations. The nature of the latter equivalence is then investigated.

In §4 we review Ernest's theory for $\tilde{\mathscr{A}}$, including a detail not mentioned in [4] or [5]. We then indicate the manner in which measures on $\mathrm{pr}\mathscr{A}$ induce canonical measures on $\tilde{\mathscr{A}}$.

Many of the ideas in this paper were implicit in my doctoral dissertation at Harvard University. I wish to express my gratitude to Professor George W. Mackey for introducing me to the theory of representations, and guiding my research in the subject. I am also indebted to Professors John Ernest, James Fell, and James Glimm for stimulating conversations on the material of this paper.

1. **The ideal center.** Let $\mathscr{A}$ be a $C^*$-algebra. By a representation $L$ of $\mathscr{A}$ we mean a linear, adjoint-preserving homomorphism $A \to L_A$ of $\mathscr{A}$ into the algebra of bounded operators on a Hilbert space $H(L)$. Such homomorphisms must be norm-decreasing (see [15, p. 92]). $L(\mathscr{A})$ denotes the range of $L$, and $L(\mathscr{A})'$, $L(\mathscr{A})''$, and $\overline{L(\mathscr{A})}$ denote the commutant, double commutant, and weak closure of $L(\mathscr{A})$, respectively. There exists a unique projection $E$ in $L(\mathscr{A})' \cap L(\mathscr{A})''$ with $\overline{L(\mathscr{A})} = L(\mathscr{A})''E$. The range of $E$ is the closure of the union of the ranges of the operators in $L(\mathscr{A})$. $L(\mathscr{A})''$ consists of operators of the form $T + \lambda I$, where $T$ is in $\overline{L(\mathscr{A})}$, $I$ is the identity operator, and $\lambda$ is complex (for all of these facts, see [1, pp. 43–44]). If $E = I$, we say that $L$ is *proper*. If $E = 0$, i.e., $L_A = 0$ for all $A$ in $\mathscr{A}$, we call $L$ a *zero representation*. Every nonproper representation may be uniquely decomposed into the direct sum of a proper and a zero representation.

If $E$ is a projection in $L(\mathscr{A})'$, the corresponding subrepresentation will be written $L^E$. A representation $L$ of $\mathscr{A}$ is *homogeneous* if for all nonzero projections $E$ in $L(\mathscr{A})'$,

$$\text{kernel } L^E = \text{kernel } L,$$

i.e., the map $L_A \to L_A E$ defined for $A$ in $\mathscr{A}$ is an isomorphism of the $C^*$-algebras $L(\mathscr{A})$ and $L(\mathscr{A})E$. As any isomorphism between two $C^*$-algebras is an isometry (see [22, Corollary 4.8.6]) a representation $L$ is homogeneous if and only if $\| L_A E \| = \| L_A \|$ for all nonzero projections $E$ in $L(\mathscr{A})'$ and $A$ in $\mathscr{A}$. A homogeneous representation is proper if and only if it is nonzero.

A representation $M$ is a *factor representation* if the von Neumann algebra $M(\mathscr{A})'$ is a factor, i.e., if $M(\mathscr{A})' \cap M(\mathscr{A})''$ consists of multiples of the identity. A factor representation $M$ must be homogeneous. For if $E$ is a non-zero projection in $M(\mathscr{A})'$, the central cover of $E$ is the identity, hence the map $M_A \to M_A E$ is an isomorphism (see [1, p. 19]). Homogeneous representations need not be factor representations. If $\mathscr{A}$ is a separable $C^*$-algebra that is not of type I, there exist unitarily inequivalent irreducible representations $L$ and $M$ of $\mathscr{A}$ with kernel $L=$ kernel $M$ (see [8, Theorem 1]). $L \oplus M$ is homogeneous, but is not a factor representation.

A representation $N$ is *irreducible* if $N(\mathscr{A})'$ consists of multiples of the identity, i.e., there are no closed invariant subspaces for $N$.

Suppose that $L$ is a representation of the $C^*$-algebra $\mathscr{A}$. For each closed ideal $\mathscr{I}$ in $\mathscr{A}$, there exists a unique projection $Q(\mathscr{I})$ in $L(\mathscr{A})' \cap L(\mathscr{A})''$ such that

$$\overline{L(\mathscr{I})} = L(\mathscr{A})'' \, [I - Q(\mathscr{I})].$$

This follows as $\overline{L(\mathscr{I})}$ is a weakly closed ideal in $L(\mathscr{A})''$. The range of $I - Q(\mathscr{I})$ is the closure of the union of the ranges of the operators in $L(\mathscr{I})$. $L^{Q(\mathscr{I})}$ defines a representation $L^{\mathscr{A}/\mathscr{I}}$ of the $C^*$-algebra $\mathscr{A}/\mathscr{I}$ by

$$L^{\mathscr{A}/\mathscr{I}}_{A+\mathscr{I}} = L^{Q(\mathscr{I})}_A.$$

If $\mathscr{I}$ and $\mathscr{J}$ are ideals in $\mathscr{A}$ with $\mathscr{I} \subseteq \mathscr{J}$, then $Q(\mathscr{I}) \geqq Q(\mathscr{J})$. $Q(\{0\}) = Q(\text{kernel } L) = I$, and $Q(\mathscr{A})$ is the complement of the projection $E$ described in the first paragraph of this section. In particular, $L$ is proper if and only if $Q(\mathscr{A}) = 0$. To avoid confusion, we shall at times write $Q_L(\mathscr{I})$ for $Q(\mathscr{I})$.

If $\mathscr{I}$ is a norm-closed ideal in $\mathscr{A}$, it is self-adjoint (see [22, Theorem 4.9.2]) and thus is itself a $C^*$-algebra. There exists an approximate identity $u_\alpha$ in $\mathscr{I}$, i.e., a net of positive elements $u_\alpha$ such that $\| u_\alpha \| \leqq 1$ and $u_\alpha A$ converges in norm to $A$ for each $A$ in $\mathscr{I}$ (see [22, Theorem 4.8.14]).

LEMMA 1.1. *With the above notations, the sequence of operators $L_{u_\alpha}$ converges strongly to $I - Q(\mathscr{I})$ on $H(L)$.*

**Proof.** For $A$ in $\mathscr{I}$ and $\phi$ in $H(L)$, the net $L_{u_\alpha} L_A \phi = L_{u_\alpha A} \phi$ converges in norm to $L_A\phi$. As the operators $L_{u_\alpha}$ are all bounded by 1, $L_{u_\alpha}[I - Q(\mathscr{I})]$ converges strongly to $I - Q(\mathscr{I})$. For any $\phi$ in $H(L)$,

$$L_{u_\alpha}Q(\mathscr{I})\phi = Q(\mathscr{I})L_{u_\alpha}\phi = 0,$$

hence $L_{u_\alpha} = L_{u_\alpha}[I - Q(\mathscr{I})]$ converges strongly to $I - Q(\mathscr{I})$.

If $E$ is a projection in $L(\mathscr{A})'$, let

$$R(E) = \operatorname{kernel} L^E.$$

If $E \geqq F$, then $R(E) \subseteq R(F)$. At times we shall write $R_L(E)$ for $R(E)$.

THEOREM 1.2. *Let L be a representation of the $C^*$-algebra $\mathscr{A}$. If $\mathscr{I}$ is any closed ideal in $\mathscr{A}$, then*

$$RQ(\mathscr{I}) \supseteq \mathscr{I}.$$

*If E is any projection in $L(\mathscr{A})'$, then*

$$QR(E) \geqq E.$$

**Proof.** Let $u_\alpha$ be an approximate identity for $\mathscr{I}$. Then from Lemma 1.1, if $A$ is in $\mathscr{I}$,

$$L_A = \lim L_{u_\alpha A} = \lim L_{u_\alpha}L_A = [I - Q(\mathscr{I})]L_A,$$

hence $L_AQ(\mathscr{I}) = 0$ and $A$ is in $RQ(\mathscr{I}) = \operatorname{kernel} L^{Q(\mathscr{I})}$.

Let $v_\alpha$ be an approximate identity for $R(E) = \operatorname{kernel} L^E$. Then from Lemma 1.1,

$$[I - QR(E)]E = \lim L_{v_\alpha}E = 0,$$

hence $E \leqq QR(E)$.

COROLLARY 1.3. *For any closed ideal $\mathscr{I}$ in $\mathscr{A}$,*

$$QRQ(\mathscr{I}) = Q(\mathscr{I})$$

*and $\mathscr{J} = RQ(\mathscr{I})$ is the largest ideal in $\mathscr{A}$ with*

$$\overline{L(\mathscr{J})} = \overline{L(\mathscr{I})}.$$

*For any projection E in $L(\mathscr{A})'$,*

$$RQR(E) = R(E),$$

*and $F = QR(E)$ is the largest projection in $L(\mathscr{A})'$ such that*

$$\operatorname{kernel} L^F = \operatorname{kernel} L^E .$$

**Proof.** As $RQ(\mathscr{I}) \supseteq \mathscr{I}$, $Q(RQ(\mathscr{I})) \leqq Q(\mathscr{I})$. But from Theorem 1.2, $QR(Q(\mathscr{I})) \geqq Q(\mathscr{I})$. If $\mathscr{K}$ is any ideal with $\overline{L(\mathscr{K})} = \overline{L(\mathscr{I})}$, then $Q(\mathscr{K}) = Q(\mathscr{I})$, and from Theorem 1.2,

$$\mathscr{K} \subseteq RQ(\mathscr{K}) = RQ(\mathscr{I}).$$

Similar arguments give the corresponding results for projections.

We call the ideals $R(E)$ with $E$ a projection in $L(\mathscr{A})'$ *projection ideals*, and the projections $Q(\mathscr{I})$ for closed ideals $\mathscr{I}$ in $\mathscr{A}$, *ideal projections*. From Corollary 1.3, the map $\mathscr{I} \to Q(\mathscr{I})$ defines an order-inverting, one-to-one correspondence between the projection ideals and the ideal projections, and $E \to R(E)$ is the inverse map. The ideal projections generate an abelian von Neumann subalgebra of the center of $L(\mathscr{A})'$ which we call the *ideal center* of $L$.

THEOREM 1.4. *A representation $L$ of a $C^*$-algebra $\mathscr{A}$ is homogeneous if and only if the ideal center of $L$ consists of multiples of the identity operator on $H(L)$.*

**Proof.** Suppose that the ideal center contains other projections than 0 and $I$. Then it contains an ideal projection $P$, $P \neq 0, I$. Noting that $I$ is an ideal projection as $I = Q(\{0\})$,

$$\operatorname{kernel} L^P = R(P) \neq R(I) = \operatorname{kernel} L,$$

hence $L$ is not homogeneous. Conversely if $L$ is not homogeneous, let $E$ be a nonzero projection in $L(\mathscr{A})'$ with

$$R(E) = \operatorname{kernel} L^E \neq \operatorname{kernel} L = R(I).$$

Then $0 \neq QR(E) \neq QR(I) = I$, and $QR(E)$ is a nontrivial projection in the ideal center of $L$.

An ideal $\mathscr{I}$ in a $C^*$-algebra $\mathscr{A}$ is *prime* if $\mathscr{I} \neq \mathscr{A}$, and there do not exist ideals $\mathscr{J}$, $\mathscr{K}$ in $\mathscr{A}$ with $\mathscr{J} \nsubseteq \mathscr{I}$, $\mathscr{K} \nsubseteq \mathscr{I}$, and $\mathscr{J}\mathscr{K} \subseteq \mathscr{I}$. A *primitive ideal* is the kernel of a nonzero irreducible representation. Equivalently, it is the kernel of an algebraically irreducible representation of $\mathscr{A}$ on a vector space (see [13, p. 233; 11]). Any primitive ideal is prime. Conversely, Dixmier has shown [2, p. 100] that if $\mathscr{A}$ is separable, then any prime ideal in $\mathscr{A}$ is primitive. The following corollary is a straightforward generalization of [13, Theorem 7.3].

COROLLARY 1.5. *The kernel of a nonzero homogeneous representation $L$ of a $C^*$-algebra $\mathscr{A}$ is prime.*

**Proof.** Suppose that $\mathscr{J}$, $\mathscr{K}$ are ideals in $\mathscr{A}$ with $\mathscr{J}\mathscr{K} \subseteq$ kernel $L$. Taking the norm closures, $\bar{\mathscr{J}}\bar{\mathscr{K}} \subseteq$ kernel $L$. As $L$ is homogeneous, either $Q(\bar{\mathscr{K}}) = 0$ or $Q(\bar{\mathscr{K}}) = I$. As the vectors $L_A\phi$ with $A$ in $\bar{\mathscr{K}}$ and $\phi$ in $H(L)$ are dense in the range of $I - Q(\bar{\mathscr{K}})$, $L(\bar{\mathscr{J}})$ annihilates that range. Thus if $Q(\bar{\mathscr{K}}) = 0$, $L(\bar{\mathscr{J}})$ annihilates $H(L)$ and $\bar{\mathscr{J}} \subseteq \operatorname{kernel} L$. If $Q(\bar{\mathscr{K}}) = I$,

$$L(\bar{\mathscr{K}}) \subseteq L(\mathscr{A})''[I - Q(\bar{\mathscr{K}})] = 0,$$

and $\bar{\mathscr{K}} \subseteq \operatorname{kernel} L$. As $L$ is nonzero, kernel $L \neq \mathscr{A}$.

COROLLARY 1.6. *The kernel of a nonzero homogeneous representation of a separable $C^*$-algebra is a primitive ideal.*

Let $\mathscr{A}$ be a separable $C^*$-algebra. For each cardinal $n = 1,2,\cdots, \aleph_0$ let $H_n$ be a fixed Hilbert space of dimension $n$. Let $\mathscr{A}_n^c$ and $\mathscr{A}_n^{cp}$ be the sets of all representations and of all proper representations on $H_n$, respectively. $\mathscr{A}_n^c$ and $\mathscr{A}_n^{cp}$ are given the *Mackey Borel structures*, i.e., the weakest Borel structures for which the functions $L \to L_A\phi \cdot \psi$ are Borel for all $A$ in $\mathscr{A}$ and $\phi,\psi$ in $H_n$. Let $\mathscr{A}^c = \bigcup_n \mathscr{A}_n^c$ and $\mathscr{A}^{cp} = \bigcup_n \mathscr{A}_n^{cp}$ have the "discrete union" Borel structures. $\mathscr{A}^c(\mathscr{A}^{cp})$ is called the *Borel space of (proper) concrete representations of* $\mathscr{A}$. Let $\mathscr{A}_n^h(\mathscr{A}_n^{hp})$ be the homogeneous representations in $\mathscr{A}_n^c$ $(\mathscr{A}^{cp})$ and $\mathscr{A}^h = \bigcup_{n=1}^{\infty} \mathscr{A}_n^h$ , $\mathscr{A}^{hp} = \bigcup_{n=1}^{\infty} \mathscr{A}_n^{hp}$, giving these the relative Borel structures.

THEOREM 1.7. *$\mathscr{A}^h$ and $\mathscr{A}^{hp}$ are standard Borel spaces.*

**Proof.** As Mackey has shown that $\mathscr{A}^c$ is standard and that $\mathscr{A}^{cp}$ is a Borel subset of $\mathscr{A}^c$ [19, Theorem 8.1 and p. 155], it suffices to prove that $\mathscr{A}_n^h$ is a Borel subset of $\mathscr{A}_n^c$. Let $A_1, A_2, \cdots$ be dense in $\mathscr{A}$, $\phi_1, \phi_2, \cdots$ vectors dense in $H_n$, and $E_1, E_2, \cdots$ projections weakly dense in the set of all projections on $H_n$ (see [1, p. 34]). For any positive integers $s,u,v,m$ and representation $L$ in $\mathscr{A}_n^c$, define $P_L(s,u,v,m)$ to be the set of all integers $p$ such that

$$|(L_{A_k}E_p - E_pL_{A_k})\phi_i \cdot \phi_j| < \frac{1}{m}$$

for all $i,j,k \leqq m$,

$$|L_{A_u}E_p\phi_i \cdot \phi_j| \leqq \left(\|L_{A_u}\| - \frac{1}{v}\right)\|\phi_i\| \; \|\phi_j\|$$

for all $i,j \leqq m$, and

$$\|E_p\phi_s\|^2 \geqq \left(1 - \frac{1}{3v\|L_{A_u}\|}\right) \|\phi_s\|^2 .$$

We assert that $\mathscr{A}_n^c - \mathscr{A}_n^h$ is the set of all $L$ in $\mathscr{A}_n^c$ such that there exist integers $s$, $u$, and $v$ with $P_L(s,u,v,m) \neq 0$ for all $m$.

Suppose that $L$ is not homogeneous. Then there exists a nonzero projection $E$ in $L(\mathscr{A})'$ such that $L_A \to L_AE$ is not an isometry on $L(\mathscr{A})$. As the operators $L_{A_k}$ are norm-dense in $L(\mathscr{A})$, there exist integers $u$ and $v$ such that $\|L_{A_u}E\| \leqq \|L_{A_u}\| - 2/v$. As $E$ is a nonzero projection, there exists an integer $s$ with

$$\|E\phi_s\|^2 \geqq \left(1 - \frac{1}{4v\|L_{A_u}\|}\right)\|\phi_s\|^2.$$

Let $p_w$ be a sequence such that $E_{p_w}$ converges weakly to $E$. Then for all $i,j,k$,

$$\lim |(L_{A_k}E_{p_w} - E_{p_w}L_{A_k})\phi_i \cdot \phi_j| = |(L_{A_k}E - EL_{A_k})\phi_i \cdot \phi_j| = 0,$$

$$\lim|L_{A_u}E_{p_w}\phi_i\cdot\phi_j| = |L_{A_u}E\phi_i\cdot\phi_j| \leqq \left(\|L_A\| - \frac{2}{v}\right)\|\phi_i\|\,\|\phi_j\|,$$

$$\lim\|E_{p_w}\phi_s\|^2 = \lim(E_{p_w}\phi_s\cdot\phi_s) = (E\phi_s\cdot\phi_s) \geqq \left(1 - \frac{1}{4v\|L_{A_u}\|}\right)\|\phi_s\|^2.$$

Thus for all $m$, $\{p_w\} \cap P_L(s,u,v,m) \neq 0$.

Conversely, suppose that $L$, $s$, $u$, and $v$ are such that $P_L(s,u,v,m) \neq 0$ for all $m$. For each $m$, let $p_m$ be an integer in $P_L(s,v,u,m)$. As the unit ball of the collection of all bounded linear operators in $H_n$ is weakly compact, there exists a subsequence $q_h$ of $p_m$ such that $E_{q_h}$ converges weakly to a positive operator $B$ (a weak limit of projections need not be a projection). Referring to the definition of $P_L(s,u,v,m)$, it is clear that $B$ is in $L(\mathscr{A})'$, $\|L_{A_u}B\| \leqq \|L_{A\mu}\| - 1/v$, and $B \geqq 1 - 1/3v\|L_{A_u}\|$. A simple application of spectral theory provides us with a nonzero projection $F$ doubly commuting with $B$, and thus in $L(\mathscr{A})'$, such that $cF \leqq B$, where $c = 1 - 1/2v\|L_{A_u}\|$. We have:

$$\|L_{A_u}F\|^2 = \|L_{A_u}FL_{A_u}^*\| \leqq c^{-1}\|L_{A_u}BL_{A_u}^*\| \leqq c^{-1}\|L_{A_u}B\|\,\|L_{A_u}\|$$

$$\leqq \|L_{A_u}\|^2\frac{\|L_{A_u}\| - \dfrac{1}{v}}{\|L_{A_u}\| - \dfrac{1}{2v}} \leqq \|L_A\|^2.$$

It follows that $L_A \to L_AF$ is not an isometry on $L(\mathscr{A})$, hence $L$ is not homogeneous

We have proved:

$$\mathscr{A}_n^c - \mathscr{A}_n^h = \bigcup_{s,u,v=1}^{\infty}\bigcap_{m=1}^{\infty}\{L : P_L(s,u,v,m) \neq 0\}.$$

For fixed $s$, $u$, $v$, and $m$,

$$\{L : P_L(s,u,v,m) \neq 0\} = \bigcap_{p\in S} R_1^p \cap R_2^p,$$

where

$$R_1^p = \{L : |(L_{A_k}E_p - E_pL_{A_k})\phi_i\cdot\phi_j| \leqq \frac{1}{m}\quad i,j,k = 1,\cdots,m\},$$

$$R_2^p = \{L : |L_{A_u}E_p\phi_i\cdot\phi_j| \leqq \left(\|L_A\| - \frac{1}{v}\right)\|\phi_i\|\,\|\phi_j\|\quad i,j,\cdots,m\},$$

$$S = \{p : \|E_p\phi_s\|^2 \geqq \left(1 - \frac{1}{3v\|L_{A_u}\|}\right)\|\phi_s\|^2\}.$$

As $R_1^p$ and $R_2^p$ are Borel subsets of $\mathscr{A}_n^c$, it follows that $\mathscr{A}_n^h$ is also Borel.

THEOREM 1.8. *Let $\mathscr{A}$ be a separable $C^*$-algebra, $(X,\mathscr{B},\mu)$ a standard measure space (see [19, p. 142]). Suppose that $L = \int L^x d\mu(x)$ where $x \to L^x$ is a measurable map of $(X,\mathscr{B},\mu)$ into $\mathscr{A}^c$. Then the representations $L^x$ are almost all homogeneous if and only if the abelian von Neumann subalgebra of $L(\mathscr{A})'$ corresponding to the direct integral contains the ideal center of $L$.*

**Proof.** Let $N$ be a Borel subset of $X$ with $\mu(N) = 0$, $X - N$ standard in the relative Borel structure, and $x \to L^x$ Borel on $X - N$. Let $S$ be the $x$ in $X - N$ for which $L^x$ is not homogeneous. From Theorem 1.7, $S$ is Borel. Suppose that $\mu(S) \neq 0$. Let $\mathscr{L}(H_n)$ be the bounded operators on $H_n$ with the Borel structure defined by the weak topology. Let $\mathscr{L} = \bigcup_{n=1}^{\infty} \mathscr{L}(H_n)$ have the discrete union Borel structure. The latter is standard (see [19, p. 150]). There exists a Borel subset $N_1 \subseteq S$ with $\mu(N_1) = 0$, and a Borel map $x \to E^x$ of $S - N_1$ into $\mathscr{L}$ such that $L^x_A \to L^x_A\, E^x$ is not an isometry and $E^x \neq 0, I$. To prove this we must employ some form of the "measurable axiom of choice." Using [19, Theorem 6.3], it suffices to show that

$$\mathscr{K} = \{(M,E)\colon M \in \mathscr{A}^c_n,\ E \in M(\mathscr{A})',\ 0 \neq E \neq I, \text{ and the map } M_A \to M_A E \text{ is not an isometry}\}$$

is a Borel subset of $\mathscr{A}^c_n \times \mathscr{L}(H_n)$. Let $A_1, A_2, \cdots$ be dense in $\mathscr{A}$, and $\phi_1, \phi_2, \cdots$ be dense in $H_n$. Then

$$\mathscr{K} = \{(M,E)\colon (M_{A_i}E - EM_{A_i})\phi_j \cdot \phi_k = 0 \qquad \text{for all } i,j,k,$$

and there exist positive integers $s,t,u,v$ with $\| M_{A_u}E\phi_i \cdot \phi_j\| < [\|M_{A_u}\| - 1/v]\|\phi_i\|\;\|\phi_j\|$ for all $i,j$, and

$$\|E\phi_s\|^2 \neq 0,\ \|(I-E)\phi_t\|^2 \neq 0\},$$

which is clearly Borel.

There exists a Borel subset $T$ of $S - N_1$, an element $A$ of $\mathscr{A}$, and a positive $\varepsilon$ such that $\mu(T) \neq 0$ and

$$\| L^x_A E^x\| \leqq \| L^x_A\| - \varepsilon$$

for all $x$ in $T$. Otherwise suppose that $A_1, A_2, \cdots$ are dense in $\mathscr{A}$, and that for each pair of integers $i,j > 0$ the Borel set

$$T_{i,j} = \left\{ x \in S - N_1 \colon \| L^x_{A_i}E^x\| \leqq \| L^x_{A_i}\| - \frac{1}{j}\right\}$$

has measure zero. Then $(S - N_1) - \bigcup_{i,j} T_{ij}$ is nonempty as it has positive measure. Let $x$ be a member of that set. For each $i$, $\| L^x_{A_i}E^x\| = \| L^x_{A_i}\|$ hence $L^x_A \to L^x_A E^x$ is an isometry. This contradicts the definition of $E^x$.

Extend the map $x \to E^x$ to all of $X$ by letting $E^x = 0$ outside of $S - N_1$, and set

$$E = \int_T E^x d\mu(x).$$

Let $B \to F(B)$ be the projection-valued measure defined on the Borel subsets $B$ of $X$ associated with the direct integral $L = \int L^x d\mu(x)$. We have

$$\begin{aligned} \| L_A^E \| &= \operatorname*{ess.\,sup}_{x \in T} \| L_A^x E^x \| \\ &\leqq \operatorname*{ess.\,sup}_{x \in T} \| L_A^x \| - \varepsilon \\ &= \| L_A^{F(T)} \| - \varepsilon. \end{aligned}$$

If the abelian von Neumann algebra of $L(\mathscr{A})'$ corresponding to the direct integral included the ideal center, there would be a Borel set $W$ with $F(W) = QR(E)$. As $QR(E) \geqq E$, we would have $W \supseteq T$ almost everywhere, i.e., $QR(E) \geqq F(T)$. This would contradict the inequality

$$\| L_A^{QR(E)} \| = \| L_A^E \| < \| L_A^{F(T)} \|.$$

Conversely suppose that the commutative von Neumann algebra associated with the direct integral does not contain the ideal center of $L$. Then there exists an ideal projection $E$ in $L(\mathscr{A})'$ not in the range of the projection-valued measure $F$. As $E$ is central, it is decomposable: there exists a measurable map $x \to E^x$ of $X$ into $\mathscr{L}$ (see above) such that $E = \int E^x d\mu(x)$. Let $u_n$ be an approximate identity in $\mathscr{I} =$ kernel $L^E$. From Lemma 1.1, $L_{u_n}$ converges strongly to $I - Q(\mathscr{I}) = I - QR(E) = I - E$. Taking a subsequence of the initial approximate identity, we may assume that $L^x_{u_n}$ converges strongly to $I^x - E^x$ for almost all $x$ (see [1, p. 162]). As $E$ is not in the range of $F$, we may find a Borel set $S$ in $X$ with $\mu(S) \neq 0$, and for all $x$ in $S$, $E^x \neq 0^x, I^x$ and $L^x_{u_n}$ converges strongly to $I^x - E^x$. From Lemma 1.1, $E^x = Q_{L^x}(\mathscr{I})$, i.e., $E^x$ is a nontrivial ideal projection. From Theorem 1.4, the representations $L^x$ with $x$ in $S$ are not homogeneous.

We say that two homogeneous representations $L$ and $M$ are *strongly disjoint* if kernel $L \neq$ kernel $M$. A more general definition will be given in §3.

LEMMA 1.9. *Suppose that $L$ is a separable homogeneous representation of a separable $C^*$-algebra $\mathscr{A}$, and that $L = \int L^x d\mu(x)$, where $x \to L^x$ is a measurable map of the standard measure space $(X, \mathscr{B}, \mu)$ into $\mathscr{A}^c$. Then for almost all* $x$, kernel $L^x =$ kernel $L$.

**Proof.** This is a trivial generalization of the corresponding result for factor representations [2, Remarque on p. 100].

THEOREM 1.10. *Suppose that $L$ is a separable representation of the separable $C^*$-algebra $\mathscr{A}$, and that $L = \int L^x d\mu(x)$, where $x \to L^x$ is a measurable map of* *the standard measure space* $(X,\mathscr{B},\mu)$ *into* $\mathscr{A}^c$. *Then the representations* $L^x$ *are almost all homogeneous and strongly disjoint if and only if the corresponding abelian von Neumann subalgebra of* $L(\mathscr{A})'$ *coincides with the ideal center of* $L$(2).

**Proof.** Say that almost all the $L^x$ are homogeneous and strongly disjoint. From Theorem 1.8, the commutative abelian subalgebra of $L(\mathscr{A})'$ determined by the decomposition contains the ideal center of $L$, i.e., the range of the projection-valued measure contains the projections of the ideal center. As in the proof of [19, Theorem 10.5], we obtain a standard quotient measure space $(Y, \tilde{\mathscr{B}},\tilde{\mu})$ of $(X,\mathscr{B},\mu)$ and decompositions:

$$\mu = \int_Y \mu_y d\tilde{\mu}(y), \tag{1}$$

$$L = \int_Y M^y d\tilde{\mu}(y), \tag{2}$$

where

$$M^y = \int_X L^x d\mu_y(x)$$

for almost all $y$, and the range of the projection-valued measure on $Y$ determining (2) is the set of all projections in the ideal center. From Theorem 1.8, (2) is another decomposition of $L$ into homogeneous representations almost everywhere. Let $N$ be a Borel subset of $X$ with $\mu(N) = 0$ and the $L^x$ homogeneous and strongly disjoint for all $x$ in $X - N$. Using (1), we select a Borel set $P$ in $Y$ with $\tilde{\mu}(P) = 0$, such that $\mu_y(N) = 0$ and $M^y$ is homogeneous for all y in $Y - P$. If $y$ is in $Y - P$, we have from Lemma 1.9 that there exists an $x_0$ in $X - N$ with kernel $L^x$ $=$ kernel $L^{x_0}$ for $\mu_y$-almost all $x$. It follows that

$$\mu_y(X - \{x_0\}) = \mu_y((X - N) - \{x_0\}) = 0$$

as for all $x$ in $(X - N) - \{x_0\}$, kernel $L^x \neq$ kernel $L^{x_0}$. Thus in the measure decomposition (1), $\tilde{\mu}$-almost all the measures $\mu_y$ are concentrated in points. Defining an inverse map almost everywhere on $Y$ into $X$ by sending $y$ into that point in which $\mu_y$ is concentrated, it may be verified that the quotient map of $X$ into $Y$ is a measure isomorphism. This implies that the ranges of the projection-valued measures on $X$ and $Y$ coincide, hence the abelian subalgebra of $L(\mathscr{A})'$ defined by $L = \int L^x d\mu(x)$ is the ideal center of $L$.

The converse is proved with a technique analogous to that of Guichardet [9]. Suppose that $L = \int L^x d\mu(x)$ is an ideal center decomposition, and let $S \to E(S)$ be the corresponding projection-valued measure. As the Boolean algebra of projections $\mathscr{P}$ in the ideal center is generated by the ideal projections, we may find a countable set of ideal projections $P_1, P_2, \cdots$ such that $\mathscr{P}$ is the smallest Boolean

(2) We shall call a direct integral with the latter property an *ideal center decomposition*.

$\sigma$-algebra containing the $P_i$. This follows as $\mathscr{P}$ has a countable generating family, and each member of the latter is contained in the $\sigma$-algebra generated by a countable number of ideal projections (see [10, Theorem D, p. 24]). Let $S_i$ be Borel sets in $X$ with $E(S_i) = P_i$. These define a measurable equivalence relation on $X$, i.e., the quotient measure space is countably separated, and thus standard (see [16, p. 124; 19, p. 143]). If $\pi$ is the quotient map, the quotient projection-valued measure $T \to E(\pi^{-1}(T))$ has the same range as $E$. The method for proving that isomorphisms of the ranges of projection-valued measures are implemented by measure isomorphisms of the underlying spaces now shows that $\pi$ is a measure isomorphism (see [18, pp. 90-91]). In particular, there is a Borel set $N$ in $X$ with $\mu(N) = 0$ such that on $X - N$, $\pi$ is a Borel isomorphism. In other words, the sets $T_i = S_i \cap (X - N)$ separate the points of $X - N$ in the sense that for any two distinct points in $X - N$, there exists a $T_i$ containing one but not the other.

Let $u_{ni}$ be an approximate identity for $\mathscr{I}_i = R(P_i) = \text{kernel}\, L^{P_i}$. By Lemma 1.1, $L^i_{u_n}$ converges strongly to $I - QR(P_i) = I - P_i$. Taking a subsequence of $u^i_n$ and enlarging $N$ by a null Borel set for each $i$, we may assume that $L^{x^i}_{u_n}$ converges strongly to $(I - P_i)^x = I^x - \chi_{T_i}(x)I^x$ for all $x$ in $X - N$ and all $i$ (see [1, p. 162]). From Lemma 1.1, $L^{x^i}_{u_n}$ also converges to $I^x - Q_{L^x}(\mathscr{I}_i)$, hence $x$ is in $T_i$ if and only if $Q_{L^x}(\mathscr{I}_i) = I^x$. But the latter condition is equivalent to $L^x(\mathscr{I}_i) = 0$, hence $x$ is in $T_i$ if and only if $\mathscr{I}_i \subseteq \text{kernel}\, L^x$. If $x$ and $y$ are in $X - N$ with kernel $L^x = \text{kernel}\, L^y$, we conclude that for all $i$, $x$ is in $T_i$ if and only if $y$ is in $T_i$, hence $x = y$.

2. **The primitive ideal space.** We denote the set of primitive ideals in a $C^*$-algebra $\mathscr{A}$ by $\text{pr}\mathscr{A}$. We give $\text{pr}\mathscr{A}$ the hull-kernel topology (see [22, pp. 76–80]). If $P$ is in $\text{pr}\mathscr{A}$ and $A$ is in $\mathscr{A}$ we denote the image of $A$ in $\mathscr{A}/P$ by $A(P)$.

Suppose that $\mathscr{A}$ is separable. We define a map $\pi : \mathscr{A}^{hp} \to \text{pr}\mathscr{A}$ by $\pi(L) = \text{kernel}\, L$. That $\pi$ is into follows from Corollary 1.6. If $P$ is a primitive ideal in $\mathscr{A}$, there exists an irreducible, hence homogeneous representation $L$ with $P = \text{kernel}\, L$ (see §1). $H(L)$ is separable, as if $p$ is a positive function on $\mathscr{A}$ corresponding to a vector in $H(L)$, $H(L)$ may be identified with the Hilbert space $H^p$ constructed from $\mathscr{A}$ in the usual way. If $A_1, A_2, \cdots$ are dense in $\mathscr{A}$, the corresponding elements will be dense in $H^p$. Thus $L$ is in $\mathscr{A}^h$, and $\pi$ is onto.

When $\mathscr{A}^{hp}_n$ is given the weak topology induced by the functions $L \to L_A\phi \cdot \psi$ for $A$ in $\mathscr{A}$, $\phi$ and $\psi$ in $H_n$, the map $\pi$ restricted to $\mathscr{A}^{hp}_n$ is continuous. For if $F \subseteq \text{pr}\mathscr{A}$ is the hull of the ideal $\mathscr{I}$ in $\mathscr{A}$ and $L^\alpha$ is a net in $\pi^{-1}(F) \cap \mathscr{A}^{hp}_n$ converging to a representation $L$ in $\mathscr{A}^{hp}_n$, then kernel $L^\alpha \supseteq \mathscr{I}$ for all $\alpha$ implies kernel $L \supseteq \mathscr{I}$. Thus $\pi$ is Borel when $\mathscr{A}^{hp}$ is given the Borel structure defined in §1 and $\text{pr}\mathscr{A}$ is given the Borel structure induced by the hull-kernel topology.

We wish to prove that if $\mathscr{A}$ is a separable $C^*$-algebra, $\text{pr}\mathscr{A}$ is standard in the hull-kernel Borel structure. We begin by using Fell's technique (see [7]) for imbedding pr $\mathscr{A}$ in a compact Hausdorff space, and we then characterize the image.

Let $\mathscr{A}$ be an arbitrary $C^*$-algebra. A *$C^*$-pseudo-norm* on $\mathscr{A}$ is a real-valued function $N$ on $\mathscr{A}$ such that for all $A,B$ in $\mathscr{A}$ and complex $\lambda$,

$$0 \leqq N(A) \leqq \|A\|,$$

$$\begin{aligned} N(A+B) &\leqq N(A) + N(B), \\ N(\lambda A) &= |\lambda|\, N(A), \\ N(AB) &\leqq N(A)N(B), \\ N(A^*A) &= N(A)^2. \end{aligned}$$

The collection $\mathscr{N}(\mathscr{A})$ of $C^*$-pseudo-norms on $\mathscr{A}$ is compact Hausdorff in the weak topology defined by $\mathscr{A}$. For each closed ideal $\mathscr{I}$ in $\mathscr{A}$ the map $A \to N_{\mathscr{I}}(A) = \|A(\mathscr{I})\|$, where $A(\mathscr{I})$ is the image of $A$ in the $C^*$-algebra $\mathscr{A}/\mathscr{I}$, is a $C^*$-pseudo-norm. As isomorphisms of $C^*$algebras are isometric (see [22, Corollary 4.8.6]), we have that for each $C^*$-pseudo-norm $N$ there is precisely one closed ideal $\mathscr{I}$ in $\mathscr{A}$ with $N = N_{\mathscr{I}}$.

If $\mathscr{I}$ and $\mathscr{J}$ are closed ideals in $\mathscr{A}$, then

$$N_{\mathscr{I} \cap \mathscr{J}} = N_{\mathscr{I}} \cup N_{\mathscr{J}}$$

where

$$(N_{\mathscr{I}} \cup N_{\mathscr{J}})(A) = \max\{N_{\mathscr{I}}(A), N_{\mathscr{J}}(A)\}.$$

For the natural map

$$\mathscr{A}/\mathscr{I} \cap \mathscr{J} \to \mathscr{A}/\mathscr{I} \oplus \mathscr{A}/\mathscr{J}$$

is an isomorphism into, hence an isometry. A $C^*$-pseudo-norm $N$ is *extremal* if there do not exist $C^*$-pseudo-norms $N_1$ and $N_2$ with $N_1 \ngeqq N$, $N_2 \ngeqq N$, and $N = N_1 \cup N_2$.

Theorem 2.1. *If $\mathscr{I}$ is a closed ideal in a $C^*$-algebra $\mathscr{A}$, then $\mathscr{I}$ is prime if and if only the corresponding $C^*$-pseudo-norm $N_{\mathscr{I}}$ is extremal and nonzero.*

**Proof.** The norm $N_{\mathscr{I}}$ is not extremal if and only if there exist closed ideals $\mathscr{J}$ and $\mathscr{K}$ with $\mathscr{J} \nsubseteq \mathscr{I}$, $\mathscr{K} \nsubseteq \mathscr{I}$, and $N_{\mathscr{I}} = N_{\mathscr{J}} \cup N_{\mathscr{K}}$, i.e., $\mathscr{I} = \mathscr{J} \cap \mathscr{K}$. Thus we must show that $\mathscr{I}$ is not prime if and only if there exist closed ideals $\mathscr{J}$, $\mathscr{K}$ in $\mathscr{A}$ with $\mathscr{J} \nsubseteq \mathscr{J}$, $\mathscr{K} \nsubseteq \mathscr{I}$ and $\mathscr{I} = \mathscr{J} \cap \mathscr{K}$.

First we note that if $\mathscr{P}$ and $\mathscr{Q}$ are closed ideals, then $\mathscr{P}\mathscr{Q} = \mathscr{P} \cap \mathscr{Q}$. Trivially, $\mathscr{P}\mathscr{Q} \subseteq \mathscr{P} \cap \mathscr{Q}$. Conversely if $A \geqq 0$ is in the $C^*$-algebra $\mathscr{P} \cap \mathscr{Q}$, so is $\sqrt{A}$, and $A = \sqrt{A}\sqrt{A}$ is in $\mathscr{P}\mathscr{Q}$. As any element in $\mathscr{P} \cap \mathscr{Q}$ is a linear combination of positive elements, $\mathscr{P} \cap \mathscr{Q} \subseteq \mathscr{P}\mathscr{Q}$.

If there exist ideals $\mathscr{J},\mathscr{K}$ with $\mathscr{J} \nsubseteq \mathscr{I}$, $\mathscr{K} \nsubseteq \mathscr{I}$, and $\mathscr{J} \cap \mathscr{K} = \mathscr{I}$, then $\mathscr{J}\mathscr{K} \subseteq \mathscr{I}$ and $\mathscr{I}$ is not prime. Conversely, suppose that there exist closed ideals $\mathscr{L}$, $\mathscr{M}$ with $\mathscr{L} \nsubseteq \mathscr{I}$, $\mathscr{M} \nsubseteq \mathscr{I}$ and $\mathscr{L}\mathscr{M} \subseteq \mathscr{I}$. Then $(\mathscr{L}+\mathscr{I})(\mathscr{M}+\mathscr{I}) \subseteq \mathscr{I}$ and letting $\mathscr{J} = \overline{\mathscr{L}+\mathscr{I}}, \mathscr{K} = \overline{\mathscr{M}+\mathscr{I}}$,

$$\mathscr{J} \cap \mathscr{K} = \mathscr{J}\mathscr{K} \subseteq \mathscr{I} \subseteq \mathscr{J} \cap \mathscr{K},$$

i.e., $\mathscr{I} = \mathscr{J} \cap \mathscr{K}$ where $\mathscr{J} \nsubseteq \mathscr{I}$ and $\mathscr{K} \nsubseteq \mathscr{I}$.

Corollary 2.2. *If $\mathscr{I}$ is a closed ideal in a separable $C^*$-algebra, then $\mathscr{I}$ is primitive if and only if the corresponding $C^*$-pseudo-norm $N_{\mathscr{I}}$ is extremal and nonzero.*

**Proof.** This is another application of [2, p. 100].

Lemma 2.3. *If $\mathscr{A}$ is an arbitrary $C^*$-algebra, the map $N : \mathrm{pr}\mathscr{A} \to \mathscr{N}(\mathscr{A})$: $P \to N_P$ is a Borel isomorphism.*

**Proof.** Let $F \subseteq \mathrm{pr}\mathscr{A}$ be the hull of a closed ideal $\mathscr{I}$ in $\mathscr{A}$. Then

$$N(F) = \{M \in \mathscr{N}(\mathscr{A}) \colon \; M(\mathscr{I}) = 0\} \cap N(\mathrm{pr}\mathscr{A}),$$

which is closed in $N(\mathrm{pr}\mathscr{A})$. As $N$ is one-to-one, $N$ maps Borel sets into Borel sets. Conversely, Kaplansky has shown (see [22, Theorem 4.9.17]) that for each $A$ in $\mathscr{A}$ the map $P \to \| A(P) \|$ is upper semi-continuous, i.e., for any real $\varepsilon$, the set $\{P : \| A(P) \| > \varepsilon\}$ is open. It follows that $P \to N_P(A) = \| A(P) \|$ is Borel for each $A$ in $\mathscr{A}$, hence $P \to N_P$ is Borel.

Theorem 2.4. *If $\mathscr{A}$ is a separable $C^*$-algebra, then* $\mathrm{pr}\mathscr{A}$ *is standard in the hull-kernel Borel structure.*

**Proof.** From Lemma 2.3 and Corollary 2.2, it suffices to show that the set $\mathscr{E}(\mathscr{A})$ of nonzero extremal $C^*$-pseudo-norms on $\mathscr{A}$ is standard. Let $A_1, A_2, \cdots$ be dense in $\mathscr{A}$. Then the metric

$$d(N,M) = \sum_{n=1}^{\infty} 2^{-n} |N(A_i) - M(A_i)|$$

defines the topology on $\mathscr{N}(\mathscr{A})$. As the latter is compact, the metric is complete. As any compact metric space is separable, the Borel structure on $\mathscr{N}(\mathscr{A})$ is standard.

As $\mathscr{A}^{hp}$ is standard and $N \circ \pi : \mathscr{A}^{hp} \to \mathscr{E}(\mathscr{A})$ is Borel and onto, $\mathscr{E}(\mathscr{A})$ is analytic. Thus it suffices to show that $\mathscr{E}(\mathscr{A})$ is also the complement of an analytic set in $\mathscr{N}(\mathscr{A})$ (see [14, p. 395]).

$$\mathscr{N}(\mathscr{A}) - \mathscr{E}(\mathscr{A}) = \{N : N \in \mathscr{N}(\mathscr{A}), \text{ and there exist } L,M \text{ in } \mathscr{N}(\mathscr{A}) \text{ with } L \ngeq N,\ M \ngeq N, \text{ and } N = L \cup M\} \cup \{0\}.$$

Letting $A_1, A_2, \cdots$ be dense in $\mathscr{A}$,

$$\mathscr{N}(\mathscr{A}) - \mathscr{E}(\mathscr{A}) = \mathrm{proj}_1(R) \cup \{0\},$$

where $\mathrm{proj}_1 : \mathscr{N}(\mathscr{A}) \times \mathscr{N}(\mathscr{A}) \times \mathscr{N}(\mathscr{A}) \to \mathscr{N}(\mathscr{A})$ is the projection on the first coordinate, and

$$R = \{(N,L,M) : L,M,N \in \mathscr{N}(\mathscr{A}),\ N(A_i) = \max[L(A_i),M(A_i)] \text{ for all } i,$$
$$\text{and there exist } j,k \text{ with } L(A_j) < N(A_j), \text{ and } M(A_k) < N(A_k)\}.$$

$\{0\}$ is closed and $R$ is Borel, hence $\mathscr{N}(\mathscr{A}) - \mathscr{E}(\mathscr{A})$ is analytic.

3. **Direct integrals on the primitive ideal space and the equivalence problem.** Each proper separable representation $L$ of a separable $C^*$-algebra $\mathscr{A}$ determines a certain measure class $\mathscr{M}(L)$ on pr$\mathscr{A}$. Let $(X,\mathscr{B},\mu)$ be a standard measure space and $x \to L^x$ a measurable map of $X$ into $\mathscr{A}^c$ such that $L = \int L^x d\mu(x)$ is an ideal center decomposition. Almost all the $L^x$ are proper as in general:

Lemma 3.1. *If $x \to M^x$ is a measurable map of the standard measure $(X,\mathscr{B},\mu)$ into $\mathscr{A}^c$, then $\int M^x d\mu(x)$ is proper if and only if almost all the $M^x$ are proper.*

**Proof.** If $\mathscr{J}$ is a closed ideal in $\mathscr{A}$, the map $x \to Q_{M^x}(\mathscr{J})$ is a measurable map of $X$ into $\mathscr{L}$ ($\mathscr{L}$ is defined in the proof of Theorem 1.8), and

$$Q_M(\mathscr{J}) = \int Q_{M^x}(\mathscr{J}) d\mu(x).$$

To prove this, note that as $Q_M(\mathscr{J})$ is in $M(\mathscr{A})' \cap M(\mathscr{A})''$, it is decomposable: there exists a measurable map $x \to Q^x$ with

$$Q_M(\mathscr{J}) = \int Q^x d\mu(x).$$

Letting $u_n$ be an approximate identity in $\mathscr{J}$, Lemma 1.1 implies that $M_{u_n}$ converges to $I - Q_M(\mathscr{J})$. Taking a subsequence of the approximate identity, we may assume that $M^x_{u_n}$ converges strongly to $I^x - Q^x$ for almost all $x$. From Lemma 1.1, $M^x_{u_n}$ converges strongly to $I^x - Q_{M^x}(\mathscr{J})$, hence $Q^x = Q_{M^x}(\mathscr{J})$ for almost all $x$.

In particular, we have

$$Q_M(\mathscr{A}) = \int Q_{M^x}(\mathscr{A}) d\mu(x),$$

hence $Q_M(\mathscr{A}) = 0$ if and only if $Q_{M^x}(\mathscr{A}) = 0^x$ for almost all $x$, i.e., $M$ is proper if and only if $M^x$ is proper for almost all $x$.

Using Theorem 1.10, we may thus find a Borel set $N$ in $X$ with $\mu(N) = 0$, $X - N$ standard, $x \to L^x$ a Borel map on $X - N$, $L^x$ nonzero and homogeneous for all $x$ in $X - N$, and kernel $L^x \neq$ kernel $L^y$ for distinct $x,y$ in $X - N$. $x \to$ kernel $L^x$

is a Borel isomorphism of $X - N$ with a Borel subset of pr $\mathscr{A}$ (see [19, Theorem 3.2]). Let $\tilde{\mu}$ be the Borel measure on pr$\mathscr{A}$ induced by $\mu$ and this map. Any ideal decomposition of $L$ will in this manner determine a measure on pr$\mathscr{A}$ equivalent to $\tilde{\mu}$ (see the analogous discussion in [18, pp. 105–106]). $\mathscr{M}(L)$ is defined to be the equivalence class of Borel measures containing $\tilde{\mu}$. The inverse of the map $x \to \text{kernel}\, L^x$ essentially determines a measurable cross-section $P \to L^P$ to the map $\pi: \mathscr{A}^{hp} \to \text{pr}\mathscr{A}: L \to \text{kernel } L$, i.e., $P \to L^P$ is such that kernel $L^P = P$ for almost all $P$. If $\nu$ is any measure in $\mathscr{M}(L)$, $L$ is unitarily equivalent to $\int L^P d\nu(P)$ (see [18. p. 77]).

Every Borel measure class on pr$\mathscr{A}$ is of the form $\mathscr{M}(L)$ for some proper separable representation $L$ of $\mathscr{A}$. For suppose that $\nu$ is a finite Borel measure on pr$\mathscr{A}$. From Theorem 2.4, (pr$\mathscr{A}$,$\mathscr{C}$,$\nu$) is a standard measure space, $\mathscr{C}$ denoting the hull-kernel Borel sets. Consequently, there exists a measurable cross-section $P \to L^P$ to the map $\pi : \mathscr{A}^{hp} \to \text{pr}\mathscr{A}$ (see [19, Theorem 6.3]). The direct integral $L = \int L^P d\nu(P)$ is ideal central (Theorem 1.10), $L$ is proper (Lemma 3.1), and $\mathscr{M}(L)$ is the measure class of $\nu$.

The equivalence problem is to characterize those proper representations $L$ and $M$ with $\mathscr{M}(L) = \mathscr{M}(M)$. The equivalence defined below provides a satisfactory criterion when pr$\mathscr{A}$ is sufficiently regular, as is the case if it is Hausdorff, or $\mathscr{A}$ is of type I.

Two representations $L$ and $M$ of an arbitrary $C^*$-algebra $\mathscr{A}$ are *ideal equivalent*, $L \sim_i M$, if they determine the same projection ideals in $\mathscr{A}$. Given a closed ideal $\mathscr{I}$ in $\mathscr{A}$, the smallest projection ideal determined by $L$ that contains $\mathscr{I}$ is $R_L Q_L(\mathscr{I})$ (Corollary 1.3), hence $L \sim_i M$ if and only if

$$R_L Q_L(\mathscr{I}) = R_M Q_M(\mathscr{I}),$$

i.e.,

$$\text{kernel } L^{\mathscr{A}/\mathscr{I}} = \text{kernel } M^{\mathscr{A}/\mathscr{I}}$$

for all closed ideals $\mathscr{I}$ in $\mathscr{A}$.

Two representations $L$ and $M$ of a $C^*$-algebra $\mathscr{A}$ are *strongly disjoint* if there do not exist projections $E$ in $L(\mathscr{A})'$ and $F$ in $M(\mathscr{A})'$ with $L^E \sim_i M^F$.

THEOREM 3.2. *Suppose that $\mathscr{A}$ is a separable $C^*$-algebra, and that $x \to L^x$, $x \to M^x$ are measurable maps of the standard measure space $(X,\mathscr{B},\mu)$ into $\mathscr{A}^c$. If $L^x \sim_i M^x$ for almost all $x$, then*

$$\int L^x d\mu(x) \sim_i \int M^x d\mu(x).$$

**Proof.** Let $L = \int L^x d\mu(x)$, $M = \int M^x d\mu(x)$. We must show that for all closed ideals $\mathscr{I}$ in $\mathscr{A}$,

$$\text{kernel } L^{Q_L(\mathscr{I})} = \text{kernel } M^{Q_M(\mathscr{I})}.$$

Let $N$ be a Borel subset of $X$ with $\mu(N)=0$ and $L^x \sim_i M^x$ for all $x$ in $X-N$. Let $u_n$ be an approximate identity for $\mathscr{I}$. From Lemma 1.1,

$$L_{u_n} \to I - Q_L(\mathscr{I}) \text{ strongly},$$

$$M_{u_n} \to I - Q_M(\mathscr{I}) \text{ strongly}.$$

Suppose that $A$ is in kernel $L^{Q_L(\mathscr{I})}$. Then

$$L_{Au_n} \to L_A(I - Q_L(\mathscr{I})) = L_A \text{ strongly}.$$

Enlarging $N$ by a Borel set of measure zero and taking a subsequence of the original approximate identity, we may assume that for all $x$ in $X-N$

$$L^x_{Au_n} \to L^x_A \text{ strongly}$$

(see [1, p. 162]). From Lemma 1.1,

$$L^x_A L^x_{u_n} \to L^x_A[I^x - Q_{L^x}(\mathscr{I})] \text{ strongly},$$

hence for $x$ in $X-N$,

$$A \in \text{kernel } L^{Q^x(\mathscr{I})} = \text{kernel } M^{Q_{M^x}(\mathscr{I})}$$

and again from Lemma 1.1,

$$M^x_{Au_n} \to M^x_A \text{ strongly}.$$

We conclude (see [1, p. 162]) that

$$M_{Au_n} \to M_A \text{ strongly},$$

and $A$ is in kernel $M^{Q_M(\mathscr{I})}$. The converse follows by symmetry.

COROLLARY 3.3. *If $L$ and $M$ are proper separable representations of a separable $C^*$-algebra $\mathscr{A}$ with $\mathscr{M}(L)=\mathscr{M}(M)$, then $L \sim_i M$.*

**Proof.** Suppose that $\mu$ is in $\mathscr{M}(L)$ and $\mathscr{M}(M)$. From the discussion at the beginning of this section, we may assume that

$$L = \int L^P d\mu(P),$$

$$M = \int M^P d\mu(P),$$

where $P \to L^P$ and $P \to M^P$ are measurable cross-sections to the map $\pi : \mathscr{A}^{hp} \to \mathrm{pr}\mathscr{A}$. As $L^P$ and $M^P$ are homogeneous and have the same kernel, $L^P \sim_i M^P$, and Theorem 3.2 is applicable.

As we shall see below, the converse of Corollary 3.3 is often false. Our next object is to determine when distinct measure classes on $\mathrm{pr}\mathscr{A}$ correspond to ideal equivalent representations.

If $\mathscr{A}$ is a separable $C^*$-algebra, then $\mathrm{pr}\mathscr{A}$ has a countable open basis. It is easily verified that if $A_i$ are dense in $\mathscr{A}$, the sets $\{P: \|A_i(P)\| > 1/n\}$ form such a basis. Given a finite Borel measure $\mu$ on a topological space $X$ with a countable open basis, let $\{V_\alpha\}$ be those open sets in $X$ such that $\mu(V_\alpha) = 0$. Then $\mu(\bigcup_\alpha V_\alpha) = 0$; for let $W_1, W_2, \cdots$ be those members of a fixed countable basis such that for each $i$, $W_i$ is contained in some $V_\alpha$. Then $\bigcup_\alpha V_\alpha = \bigcup_{i=1}^{\infty} W_i$, $\mu(W_i) = 0$, hence $\mu(\bigcup_\alpha V_\alpha) = 0$. We define the *support* of $\mu$, $\mathrm{supp}\,\mu$, to be $X - \bigcup_\alpha V_\alpha$. It is characterized by the fact that if $V$ is open in $X$, then $\mu(V) = 0$ if and only if $V \cap \mathrm{supp}\,\mu = 0$.

The *partial supports* of $\mu$ are those closed sets $K$ in $X$ for which if $V$ is open in $X$, then $\mu(V \cap K) = 0$ if and only if $V \cap K = 0$. If $B$ is a Borel set in $X$ and the measure $\mu_B$ is defined by $\mu_B(S) = \mu(B \cap S)$ for Borel $S$ in $X$, then $\mathrm{supp}\,\mu_B$ is a partial support of $\mu$. If $K$ is closed in $X$, then as $\mu_K(X - K) = 0$, $\mathrm{supp}\,\mu_K \subseteq K$. It follows that a closed set is of measure zero if and only if it contains no partial supports. If $K$ is a partial support of $\mu$, then $\mathrm{supp}\,\mu_K = K$. As equivalent measures have the same supports and partial supports we shall also speak of the *support and partial supports of a measure class.*

If $\mathscr{I}$ is an ideal in the $C^*$-algebra $\mathscr{A}$, we denote the hull of $\mathscr{I}$ in $\mathrm{pr}\mathscr{A}$, i.e., the set of primitive ideals containing $\mathscr{I}$, by $h(\mathscr{I})$. If $S$ is a subset of $\mathrm{pr}\mathscr{A}$, we denote the kernel of $S$, i.e., the intersection of the ideals in $S$, by $k(S)$.

LEMMA 3.4. *Suppose that $\mathscr{A}$ is a separable $C^*$-algebra, $\mu$ a finite Borel measure on $\mathrm{pr}\mathscr{A}$, and $P \to L^P$ a measurable cross-section for the map $\pi : \mathscr{A}^{hp} \to \mathrm{pr}\mathscr{A}$. Then*

$$\text{kernel} \int L^P d\mu(P) = \mathrm{k}(\mathrm{supp}\,\mu).$$

**Proof.** Let $L = \int L^P d\mu(P)$. If $A$ is in $k(\mathrm{supp}\,\mu)$, then $A(P) = 0$ for all $P$ in $\mathrm{supp}\,\mu$, hence

$$\|L_A\| = \text{ess.sup}\,\|L_A^P\| = \text{ess.sup}\,\|A(P)\| = 0.$$

Conversely, if $A$ is not in $k(\mathrm{supp}\,\mu)$, there exists a $P_0$ in $\mathrm{supp}\,\mu$ with $\|A(P_0)\| \neq 0$. As the set $\{P : \|A(P)\| \neq 0\}$ is open and intersects $\mathrm{supp}\,\mu$, it is of positive measure and

$$\|L_A\| = \text{ess.sup}\,\|A(P)\| \neq 0.$$

THEOREM 3.5. *Let $L$ be a proper separable representation of the separable $C^*$-algebra $\mathscr{A}$. The projection ideals determined by $L$ are just the kernels of the partial supports of $\mathscr{M}(L)$ in $\mathrm{pr}\mathscr{A}$.*

**Proof.** From the discussion at the beginning of this section, we may assume that $L = \int L^P d\mu(P)$, where $P \to L^P$ is a measurable cross-section for the map $\pi : \mathscr{A}^{hp} \to \mathrm{pr}\,\mathscr{A}$, and $\mu$ is in $\mathscr{M}(L)$. Let $S \to E(S)$ for Borel $S$ be the corresponding projection-valued measure.

If $S$ is a partial support for $\mu$, $S = \operatorname{supp} \mu_S$, hence from Lemma 3.4,

$$\begin{aligned} k(S) &= k(\operatorname{supp} \mu_S) = \text{kernel} \int L^P d\mu_S(P) \\ &= \text{kernel } L^{E(S)} = RE(S), \end{aligned}$$

i.e., $k(S)$ is a projection ideal.

Conversely suppose that $\mathscr{I}$ is a projection ideal. Then $\mathscr{I} = RQ(\mathscr{I})$, (Corollary 1.3). As $Q(\mathscr{I})$ is an ideal projection and the range of $E$ is the ideal center, there exists a Borel set $T$ in $\operatorname{pr}\mathscr{A}$ with $E(T) = Q(\mathscr{I})$. Thus

$$\begin{aligned} \mathscr{I} &= RQ(\mathscr{I}) = RE(T) = \text{kernel } L^{E(T)} \\ &= \text{kernel} \int L^P d\mu_T = k(\operatorname{supp} \mu_T). \end{aligned}$$

As we remarked above, $\operatorname{supp} \mu_T$ is a partial support for $\mu$.

COROLLARY 3.6. *If $L$ and $M$ are proper separable representations of a separable $C^*$-algebra $\mathscr{A}$, then $L \sim_i M$ if and only if the measure classes $\mathscr{M}(L)$ and $\mathscr{M}(M)$ have the same partial supports.*

For many topological spaces with countable open basis, measure classes that have the same partial supports must coincide. We say that such a space is *metrically regular.*

A subset of a topological space $X$ is *locally closed* if it is the intersection of an open and a closed set in $X$.

THEOREM 3.7. *Suppose that the topological space $X$ has a countable basis, and that it is a countable union of locally closed sets, each of which is Hausdorff and locally compact in the relative topology. Then $X$ is metrically regular.*

**Proof.** Let $\mu$ and $\nu$ be finite Borel measures on $X$ with the same partial supports. First we notice that if $Y$ is a locally closed subset of $X$, then the restrictions of $\mu$ and $\nu$ to $Y$ have the same partial supports in $Y$. To prove this, it suffices to show that if $\lambda$ is any Borel measure on $X$, then the partial supports of $\lambda_Y$ (considered as a measure on $Y$) are just those sets $F$ closed in $Y$ such that $\bar{F}$ is a partial support for $\lambda$ in $X$.

Say that $Y = V \cap K$ where $V$ is open and $K$ is closed in $X$. Suppose that $F$ is closed in $Y$ and $\bar{F}$ is not a partial support for $\lambda$ in $X$. Then there exists an open set $W$ in $X$ with $\lambda(W) = 0$ and $W \cap \bar{F} \neq 0$. It follows that $W \cap F \neq 0$, i.e., $(W \cap Y) \cap F \neq 0$, and $\lambda_Y(W \cap Y \cap F) = 0$. As $W \cap Y$ is relatively open in $Y$, $F$ is not a partial support for $\lambda_Y$ in $Y$. Conversely, if $F$ is closed in $Y$ but is not a partial support for $\lambda_Y$, let $W$ be an open set in $X$ with $\lambda_Y(W \cap F) = 0$ and $W \cap F \neq 0$. As $\bar{F} \subseteq K$ and $\bar{F} \cap Y = F$,

$$\begin{aligned} \lambda(W \cap V \cap \bar{F}) &= \lambda(W \cap V \cap K \cap \bar{F}) \\ &= \lambda(W \cap Y \cap \bar{F}) = \lambda_Y(W \cap F) = 0. \end{aligned}$$

But $W \cap V \cap F \supseteq W \cap F \cap F = W \cap F \neq 0$, hence $F$ is not a partial support for $\lambda$ in $X$.

Returning to the measures $\mu$ and $\nu$, let $X = \bigcup_{i=1}^{\infty} Y_i$ where $Y_i$ is locally closed, Hausdorff, and locally compact. It suffices to show that the restrictions $\mu_i$ and $\nu_i$ of $\mu$ and $\nu$ to $Y_i$ have the same null-sets. From above, $\mu_i$ and $\nu_i$ have the same partial supports on $Y_i$. As $X$ has a countable open basis, the same is true for $Y_i$. But any Borel measure on a locally compact Hausdorff space with countable open basis is "inner regular" (see [10, pp. 217–230]). For $\mu_i$ this means that if $S$ is Borel in $Y_i$,

$$\mu_i(S) = \sup \{\mu_i(K) : K \text{ compact in } Y_i,\ K \subseteq S\}.$$

The same being true for $\nu_i$, there exists a sequence of compact sets $K_n \subseteq S$ with $\mu_i(K_n)$ converging to $\mu_i(S)$ and $\nu_i(K_n)$ converging to $\nu_i(S)$. We have that $\mu_i(S) = 0$ if and only if $\mu_i(K_n) = 0$ for all $n$, and $\nu_i(S) = 0$ if and only if $\nu_i(K_n) = 0$ for all $n$. As the closed null-sets for a measure are simply the closed sets not containing a nonempty partial support, we have for each $n$ that $\mu_i(K_n) = 0$ if and only if $\nu_i(K_n) = 0$, hence $\mu(S) = 0$ if and only if $\nu(S) = 0$.

COROLLARY 3.8. *Let $\mathscr{A}$ be a separable $C^*$-algebra. If* pr$\mathscr{A}$ *is Hausdorff or $\mathscr{A}$ is of type* I, *then* pr$\mathscr{A}$ *is metrically regular.*

**Proof.** For any $C^*$-algebra $\mathscr{A}$, pr$\mathscr{A}$ is locally compact (see [6]). As $\mathscr{A}$ is separable, pr$\mathscr{A}$ has a countable open basis, hence if pr$\mathscr{A}$ is in addition separable, Theorem 3.7 is applicable.

If $\mathscr{A}$ is of type I, it is GCR (see [8, Theorem 1]), and there exists a collection $\{V_\alpha\}$ of distinct open sets in pr$\mathscr{A}$ indexed by ordinals $1 \leqq \alpha \leqq \alpha_0$, with $V_{\alpha+1} - V_\alpha$ Hausdorff for $\alpha < \alpha_0$, $V_\beta = \bigcup_{\alpha<\beta} V_\alpha$ for limit ordinals $\beta$, and $V_{\alpha_0} = \mathrm{pr}\,\mathscr{A}$. $\alpha_0$ is countable as pr$\mathscr{A}$ has a countable basis of open sets, hence the collection of locally closed Hausdorff spaces $\{V_{\alpha+1} - V_\alpha : \alpha < \alpha_0\}$ is countable. pr$\mathscr{A}$ is the union of these sets, and each is locally compact, as a locally closed subset of a locally compact space is locally compact in the relative topology. Thus we may again use Theorem 3.7.

THEOREM 3.9. *Let $L$ and $M$ be proper separable representations of a separable $C^*$-algebra $\mathscr{A}$. Suppose that* pr$\mathscr{A}$ *is metrically regular (see Corollary 3.8). Then $L$ and $M$ determine the same measure class on* pr$\mathscr{A}$ *if and only if they are ideal equivalent. $L$ and $M$ determine orthogonal measure classes if and only if they are strongly disjoint.*

**Proof.** $L$ and $M$ are ideal equivalent if and only if $\mathscr{M}(L)$ and $\mathscr{M}(M)$ have the same partial supports (Corollary 3.6). Thus as pr$\mathscr{A}$ is metrically regular, $L$ and $M$ are ideal equivalent if and only if $\mathscr{M}(L) = \mathscr{M}(M)$.

Suppose that $L$ and $M$ are not strongly disjoint. Let $\lambda$ and $\mu$ be elements of $\mathscr{M}(L)$ and $\mathscr{M}(M)$, respectively. From the discussion at the beginning of this section, we may assume that

$$L = \int L^P d\lambda(P),$$

$$M = \int M^P d\mu(P),$$

where $P \to L^P$, $P \to M^P$ are measurable cross-sections for $\pi : \mathscr{A}^{hp} \to \mathrm{pr}\mathscr{A}$. For Borel $S$ in $\mathrm{pr}\mathscr{A}$, let $S \to E(S)$, $S \to F(S)$ be the corresponding projection-valued measures. Let $G$ and $H$ be nonzero projections in $L(\mathscr{A})'$ and $M(\mathscr{A})'$ such that $L^G \sim_i M^H$. As the ideal center of $L$ is contained in $L(\mathscr{A})' \cap L(\mathscr{A})''$, $G$ is decomposible. There exists a Borel map $P \to G^P$ of $\mathrm{pr}\,\mathscr{A}$ into $\mathscr{L}$ (for this notation, see the proof of Theorem 1.8) with $G = \int G^P d\lambda(P)$. Similarly, we have $H = \int H^P d\mu(P)$. Let $S$ and $T$ be the Borel sets of those $P$ in $\mathrm{pr}\mathscr{A}$ for which $G^P \neq 0$ and $H^P \neq 0$, respectively. Then $\lambda(S) \neq 0$, $\mu(T) \neq 0$, and in the decompositions

$$L^G = \int (L^P)^{G^P} d\lambda_S(P),$$

$$M^H = \int (M^P)^{H^P} d\mu_T(P),$$

the representations $(L^P)^{G^P}$, respectively $(M^P)^{H^P}$, are homogeneous, proper (see Lemma 3.1), and strongly disjoint. From Theorem 1.10, we conclude that they are ideal center decompositions, hence $\lambda_S$ is in $\mathscr{M}(L^G)$ and $\mu_T$ is in $\mathscr{M}(M^H)$. As $L^G \sim_i M^H$, $\mathscr{M}(L^G)$ and $\mathscr{M}(M^H)$ have the same partial supports (Corollary 3.6), or since $\mathrm{pr}\mathscr{A}$ is metrically regular, $\mathscr{M}(L^G) = \mathscr{M}(M^H)$, $\lambda_{S \cap T} = \lambda_S$ and $\mu_T = \mu_{S \cap T}$ are equivalent, and $\lambda$ and $\mu$ are not orthogonal measures.

Conversely say that $\lambda$ and $\mu$ are nonorthogonal finite Borel measures on $\mathrm{pr}\mathscr{A}$. Then there exists a Borel set $S$ in $\mathrm{pr}\mathscr{A}$ with $\lambda(S) \neq 0$ and $\lambda_S$ equivalent to $\mu_S$. Defining $E(S)$ and $F(S)$ as above, we have from Theorem 3.2 and [18, p. 77] that $L^{E(S)} \sim_i M^{F(S)}$, hence $L$ and $M$ are not ideal disjoint.

Corollary 3.10. *If $\mathscr{A}$ is a separable $C^*$-algebra of type* I, *and $L$, $M$ are separable proper representations of $\mathscr{A}$, then $L$ and $M$ are quasi-equivalent or disjoint* (*see* [18, *Volume* I]) *if and only if they are ideal equivalent or strongly disjoint, respectively.*

**Proof.** Let $\mathscr{A}^{fp}$ be the proper factor representations in $\mathscr{A}^c$, and $\tilde{\mathscr{A}}^p$ the quasi-equivalence classes of representations in $\mathscr{A}^{fp}$. As $\mathscr{A}$ is separable of type I, any factor representation is of the form $nL$, where $L$ is irreducible, and irreducible representations are disjoint if and only if they have distinct kernels, i.e., they are strongly disjoint (see [8, Theorem 1]). If $M$ is a separable representation of $\mathscr{A}$ and $M = \int M^x d\mu(x)$ is its central decomposition, i.e., the decomposition with respect to $M(\mathscr{A})' \cap M(\mathscr{A})''$, then the representations $M^x$ are almost all disjoint factor representations (for arbitrary separable $C^*$-algebras see [4; 5]). As they are thus strongly disjoint homogeneous representations, $M(\mathscr{A})' \cap M(\mathscr{A})''$ coincides with the ideal center of $M$ (Theorem 1.10). In particular, $M$ is a factor representation if and only if it is homogeneous, and $\mathscr{A}^{fp} = \mathscr{A}^{hp}$. It is also known that $\tilde{\mathscr{A}}^p$ and pr$\mathscr{A}$ may be identified as Borel spaces (see [6; 8; 5]).

Ernest has proved [4; 5] that the central decompositions of two representations $L$ and $M$ will induce equivalent or orthogonal measures on $\tilde{\mathscr{A}}$ if and only if $L$ and $M$ are quasi-equivalent or disjoint, respectively. Since pr$\mathscr{A}$ is metrically regular (Corollary 3.8), proper representations $L$ and $M$ induce equivalent or orthogonal measures on pr$\mathscr{A}$ if and only if they are ideal equivalent or ideal disjoint, respectively. The corollary easily follows.

In order to prove that the converse of Corollary 3.3 is false, it suffices to find a separable $C^*$-algebra $\mathscr{A}$ for which pr $\mathscr{A}$ is not metrically regular. There will then exist inequivalent Borel measures $\mu$ and $\nu$ on pr$\mathscr{A}$ having the same partial supports. Letting $P \to L^P$ be a cross-section for the map $\pi : \mathscr{A}^{hp} \to \mathrm{pr}\mathscr{A}$, measurable relative to $\mu$ and $\nu$, the representations $\int L^P d\mu(P)$ and $\int L^P d\nu(P)$ will be ideal equivalent (Corollary 3.6).

Dixmier [2, pp. 101–104] has constructed a separable, primitive $C^*$-algebra $\mathscr{A}$ containing a sequence of distinct primitive ideals $P_1 \supseteq P_2 \supseteq \cdots$ with $P_i \neq \{0\}$ and $\bigcap_i P_i = \{0\}$ (this was pointed out to us by J. Glimm). Let $\mu$ and $\nu$ be the atomic measures concentrated in the $P_i$ and such that

$$\mu(\{P_i\}) = \frac{1}{2^i}, \quad i = 1, 2, \cdots, \quad \mu(\{P_0\}) = 0,$$

$$\nu(\{P_i\}) = \frac{1}{2^i}, \quad i = 1, 2, \cdots, \quad \nu(\{P_0\}) = 1,$$

where $P_0 = \{0\}$. Trivially any partial support for $\mu$ is a partial support for $\nu$. Suppose that $F$ is a partial support for $\nu$, but not for $\mu$. Then there exists an open set $V$ with $\mu(F \cap V) = 0$, $\nu(F \cap V) \neq 0$. We must have that

$$P_0 \in F \cap V, \quad P_i \in (\mathrm{pr}\mathscr{A} - F) \cup (\mathrm{pr}\mathscr{A} - V) \qquad \text{for } i > 0.$$

As $P_0 \in F$, $\overline{\{P_0\}} \subseteq F$, and as $P_i \supseteq P_0$, $P_i \in F$ for $i > 0$. Thus $P_i \in \mathrm{pr}\mathscr{A} - V$ for $i > 0$, and letting pr $\mathscr{A} - V$ be the hull of an ideal $\mathscr{I}$ in $\mathscr{A}$, $P_i \supseteq \mathscr{I}$. But then $P_0 \supseteq \mathscr{I}$ and $P_0 \in \mathrm{pr}\, \mathscr{A} - V$, a contradiction. We conclude that the inequivalent measures $\mu$ and $\nu$ have the same partial supports, hence pr$\mathscr{A}$ is not metrically regular.

4. **Canonical measure classes on the quasi-dual.** Let $\mathscr{A}$ be a separable $C^*$-algebra, and $\mathscr{A}^i$,$\mathscr{A}^f$ the irreducible and factor representations, respectively, in $\mathscr{A}^c$. $\mathscr{A}^i$ and $\mathscr{A}^f$ are Borel subsets of $\mathscr{A}^c$ (see [19; 5]) and are given the relative Borel

structures. The *dual* $\hat{\mathscr{A}}$ is the set of all unitary equivalence classes of representations in $\mathscr{A}^i$, and the *quasi-dual* $\tilde{\mathscr{A}}$ is the set of all quasi-equivalence classes in $\mathscr{A}^f$ (for the notion of quasi-equivalence, see [18, Volume I]). $\hat{\mathscr{A}}$ and $\tilde{\mathscr{A}}$ are given the quotient Borel structures.

In this section, we shall briefly consider certain measures on $\hat{\mathscr{A}}$ and $\tilde{\mathscr{A}}$ that are defined by measures on pr$\mathscr{A}$. As the arguments for $\hat{\mathscr{A}}$ and $\tilde{\mathscr{A}}$ are similar, we restrict our attention to the latter. We shall first review Ernest's results, and show how the analogy between his theory for $\tilde{\mathscr{A}}$ and Mackey's for $\hat{\mathscr{A}}$ may be completed.

Let $x \to L^x$ be a measurable map of a standard Borel space $(X,\mathscr{B},\mu)$ into $\mathscr{A}^c$. The direct integral $L = \int L^x d\mu(x)$ is said to be *central* if the corresponding decomposition algebra generated by the projection-valued measure coincides with $L(\mathscr{A})' \cap L(\mathscr{A})''$. In that case the $L^x$ are almost all factor representations and as Ernest and Naimark have proved [4; 5; 20], they are almost all disjoint. Thus the measure $\mu$ induces a Borel measure $\tilde{\mu}$ on $\tilde{\mathscr{A}}$ via the essentially one-to-one composition of $x \to L^x$ and the quotient map. Ernest showed [4; 5] that the measure class of $\mu$ depends only on the quasi-equivalence class $[L]$ of $L$. We denote the former by $\mathscr{C}([L])$. Following Ernest, we say that a Borel measure class on $\tilde{\mathscr{A}}$ is *canonical* if it is of the form $\mathscr{C}([L])$ for some separable representation $L$ of $\mathscr{A}$.

A Borel measure $\nu$ on $\tilde{\mathscr{A}}$ is *standard* if the corresponding measure space is standard. Canonical measures are always standard [5, Theorem 2]. As any measure equivalent to a standard measure is also standard, we shall say that a Borel measure class is standard if any one of its measures is standard. If $\nu$ is standard, there exists a measurable cross-section $x \to L^x$ for the quotient map of $\mathscr{A}^f$ onto $\tilde{\mathscr{A}}$. In order to show that the quasi-equivalence class of $\int L^x d\nu(x)$ does not depend on the particular cross-section used, we must generalize a result of Ernest [5, Proposition 5].

THEOREM 4.1. *Let $\mathscr{A}$ be a separable $C^*$-algebra and suppose that $x \to M^x$, $x \to N^x$ are measurable maps of a standard measure space $(X,\mathscr{B},\mu)$ into $\mathscr{A}^c$. Let $M = \int M^x d\mu(x)$, $N = \int N^x d\mu(x)$, and $\mathscr{M}, \mathscr{N}$ be the corresponding decomposition algebras. If $M^x$ and $N^x$ are quasi-equivalent for almost all $x$, there exists an ultraweakly continuous isomorphism of $M(\mathscr{A})'' \vee \mathscr{M}$ (the von Neumann algebra generated by $M(\mathscr{A})''$ and $\mathscr{M}$) onto $N(\mathscr{A})'' \vee \mathscr{N}$ carrying $M(\mathscr{A})''$ onto $N(\mathscr{A})''$ and $\mathscr{M}$ onto $\mathscr{N}$. In particular, $M$ and $N$ are quasi-equivalent.*

**Proof.** Let $H$ be a Hilbert space of dimension $\aleph_0$ and $I$ be the identity operator on $H$. For any representation $L$ of $\mathscr{A}$, the map sending $A$ onto the tensor product $L_A \otimes I$ defines a quasi-equivalent representation on $H(L) \otimes H$. The map $x \to L^x \otimes I$ is integrable and we have a natural unitary equivalence of $M \otimes I = (\int M^x d\mu(x)) \otimes I$ with $\int (M^x \otimes I) d\mu(x)$. The correspondence $S \to S \otimes I$ for $S$ in $M(\mathscr{A})''$ is an isomorphism. As $M_A + \lambda I \to M_A \otimes I + \lambda(I \otimes I)$ for $A$ in $\mathscr{A}$, and the map is ultraweakly continuous, the image of $M(\mathscr{A})''$ is $(M \otimes I(\mathscr{A})''$. Using the natural equivalence, it is seen that the image of $\mathscr{M}$ is the decomposition algebra for $\int M^x \otimes I d\mu(x)$. If $M^x$ and $N^x$ are quasi-equivalent, $M^x \otimes I$ and $N^x \otimes I$ are unitarily equivalent. As this is true for almost all $x$, there exists a unitary equivalence of $\int M^x \otimes I d\mu(x)$ and $\int N^x \otimes I d\mu(x)$ which preserves the decomposition algebras. (This is a trivial extension of [19, Theorem 10.1]). Composing these maps with the map $T \otimes I \rightarrow T$ for $T$ in $N(\mathscr{A})'' \vee \mathscr{N}$, we have the desired isomorphism. As the representations $M$ and $N$ of $\mathscr{A}$ are quasi-equivalent if and only if the map $M_A \rightarrow N_A$ is well-defined, one-to-one, and ultraweakly continuous (see [5, Lemma 2]), we have the last assertion.

Given a standard measure $\mu$ on $\tilde{\mathscr{A}}$, $[\mu]$ its equivalence class, we define $\mathscr{L}([\mu])$ to be the quasi-equivalence class of $\int L^x \, d\mu(x)$, where $x \rightarrow L^x$ is a measurable cross-section for the quotient map $\mathscr{A}^f \rightarrow \tilde{\mathscr{A}}$. The following is the analogue of [19, Theorem 10.6]:

Corollary 4.2. *If $\mu$ is a canonical measure on $\tilde{\mathscr{A}}$, the direct integral of each cross-section for the quotient map $\mathscr{A}^f \rightarrow \tilde{\mathscr{A}}$ is central, and $\mathscr{C}(\mathscr{L}([\mu])) = [\mu]$. If $L$ is a separable representation of $\mathscr{A}$, then $\mathscr{L}(\mathscr{C}([L])) = [L]$.*

**Proof.** Suppose that $M$ is a separable representation of $\mathscr{A}$ with $\mu$ in $\mathscr{C}([M])$. Then there exists a measurable cross-section $x \rightarrow M^x$ of $\tilde{\mathscr{A}}$ into $\mathscr{A}^f$ such that the decomposition algebra of $M = \int M^x d\mu(x)$ is $\mathscr{M} = M(\mathscr{A})' \cap M(\mathscr{A})''$, the center of $M(\mathscr{A})''$. By the result mentioned above, $\mu$ is standard. If $x \rightarrow N^x$ is any other measurable cross-section, with $\mathscr{N}$ the decomposition algebra for $N = \int N^x d\mu(x)$, then $M^x$ is quasi-equivalent to $N^x$ for all $x$, and we may apply Lemma 4.1. The resulting isomorphism carries the center $M(\mathscr{A})' \cap M(\mathscr{A})''$ of $M(\mathscr{A})''$ onto the center $N(\mathscr{A})' \cap N(\mathscr{A})''$ of $N(\mathscr{A})''$, and $\mathscr{M}$ onto $\mathscr{N}$. Thus $\mathscr{N} = N(\mathscr{A})' \cap N(\mathscr{A})''$, the decomposition $N = \int N^x d\mu(x)$ is central, and $\mu$ is in $\mathscr{C}([N]) = \mathscr{C}(\mathscr{L}([\mu]))$.

Suppose that $L = \int L^x d\mu(x)$ is a central decomposition where $x \rightarrow L^x$ is a measurable cross-section of $\tilde{\mathscr{A}}$ into $\mathscr{A}^f$. $\mu$ is standard, $\mathscr{L}(\mathscr{C}([L]))$ is defined, and $L$ is in $\mathscr{L}(\mathscr{C}([L]))$.

Given a Borel measure $\mu$ on pr$\mathscr{A}$, let $P \rightarrow L^P$ be a measurable cross-section for the natural map of $\mathscr{A}^{fp}$ onto pr$\mathscr{A}$. Composing with the quotient map of $\mathscr{A}^f$ onto $\tilde{\mathscr{A}}$, we obtain a Borel measure $\tilde{\mu}_L$ on the latter.

Theorem 4.3. *With the above notation, the measure $\tilde{\mu}_L$ on $\tilde{\mathscr{A}}$ is canonical.*

**Proof.** We must show that if $L = \int L^P d\mu(P)$ is a direct integral of strongly disjoint factor representations, then it is a central decomposition. From Theorem 1.10 the decomposition algebra coincides with the ideal center, and hence is a subalgebra of the center. On the other hand, it has been proved (see [12]) that the decomposition algebra of an integral of factor representations must contain the center.

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