DARBOUX PROPERTY
FOR FUNCTIONS OF SEVERAL VARIABLES

BY

C. J. NEUGEBAUER[1]

Introduction. Let E_n be the n-dimensional Euclidean space. A function f: $E_1 \rightarrow E_1$ is said to have the *Darboux* property if for $a < b$ and u between $f(a)$, $f(b)$ there is $t \in (a,b)$ satisfying $f(t) = u$. There are several important classes of functions which possess the Darboux property: approximately continuous functions and ordinary or approximate derivatives. The purpose of this paper is to extend these results to E_n by introducing a Darboux notion for E_n with the view of obtaining the following propositions: (1) approximately continuous functions on E_n, (2) derivatives of certain interval functions, and (3) partial derivatives (approximate or ordinary) of linearly continuous functions possess the Darboux property.

The Darboux property of a function on E_1 is equivalent with the notion of a connected mapping. The problem then is to define a class of connected sets in E_n, called Darboux class, relative to which the Darboux property of a function is defined. The Darboux class for E_n should contain the connected open subsets of E_n and has to be sufficiently large so as to reduce to the Darboux notion for $n = 1$, and sufficiently small so as to obtain the propositions (1), (2), and (3). With this in mind the extreme classes, i.e., the class of connected open sets and the class of all connected sets, do not form suitable Darboux classes. For the first class there would be no agreement for $n = 1$ [5] and, since an approximately continuous function on E_n ($n > 1$) need not be a connected mapping, the second class is too large. In view of the propositions (1), (2), (3) with which this paper is concerned, the Darboux class for E_n will depend upon the coordinate system in E_n.

1. **Definition and preliminaries.** By an interval I in E_n we mean the set of all $x = (x_1, \cdots, x_n)$ such that $-\infty < a_i \leqq x_i \leqq b_i < \infty, a_i < b_i, i = 1, \cdots, n$. For $A \subset E_n$, $|A|$ denotes the Lebesgue measure of A, $\delta(A)$ stands for the diameter of A, A^0 is the interior of A, and $\bar{A}$ is the closure of A.

Let $\{I\}$ be a collection of intervals in E_n. For $x \in E_n$, let $\{I\}_x = \{I : I \in \{I\}$ and $x \in I\}$. Let $\phi : \{I\} \rightarrow E_1$ be an interval function.

DEFINITION 1. ϕ is called *continuous* at $I_0 \in \{I\}$ if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that for each $I \in \{I\}$ with $|I - I_0| + |I_0 - I| < \delta$ there follows $|\phi(I) - \phi(I_0)| < \varepsilon$.

REMARK. This notion of continuity differs from the one given in [6]. Under some additional hypothesis one is able to select arbitrary small $I_x \in \{I\}_x$ so that $\phi(I_x)$ is continuous as a function of x, a property which is important later.

DEFINITION 2. ϕ is termed *balanced* if and only if for every $I_0 \in \{I\}$ the following holds. If $I \in \{I\}$, $I \subset I_0$, and $\phi(I) \geq r_0 \cdot |I|$ $[\phi(I) \leq r_0 \cdot |I|]$, then there exists $I' \in \{I\}$ such that $I' \subset I_0$, $\delta(I') \leq \delta(I)$, and $\phi(I') \leq r_0 \cdot |I'|$ $[\phi(I') \geq r_0 \cdot |I'|]$, where $r_0 = \phi(I_0)/|I_0|$.

EXAMPLES. Let $\{I\}$ be the collection of all intervals in E_n. Let $\phi(I) = |I|^2$ Then ϕ is continuous, but not balanced. Let $x_0 \in E_n$, and define $\phi(I) = |I|, x_0 \in I$, and $\phi(I) = 0, x_0 \notin I$. Then ϕ is balanced, but not continuous.

THEOREM 1. *If $\{I\}$ is the collection of all intervals of E_n, and if $\phi: \{I\} \to E_1$ is additive, then ϕ is balanced.*

Proof. Let $I_0 \in \{I\}$ and $I \subset I_0$ with $\phi(I) \leq r_0 \cdot |I|$, where $r_0 = \phi(I_0)/|I_0|$. Let $\{I_1, \cdots, I_K\}$ be a partition of I_0 with $I_1 = I$ and $\delta(I_j) \leq \delta(I_1)$, $j = 1, \cdots, K$. If $\phi(I_j) < r_0 \cdot |I_j|$, $j = 2, \cdots, k$, then $\phi(I_1) + \cdots + \phi(I_K) < r_0 |I_0| = \phi(I_0)$, contradicting the additivity of ϕ.

COROLLARY. *If $f: E_1 \to E_1$ and $\phi([a,b]) = f(b) - f(a)$, then ϕ is balanced.*

Let $\{I\}$ be a collection of intervals of E_n and let x be a point in E_n for which there is a sequence $\{I_j\} \subset \{I\}_x$ with $\delta(I_j) \to 0$.

DEFINITION 3. An interval function $\phi: \{I\} \to E_1$ has a *derivative* $\phi'(x)$ at x if and only if for $I \in \{I\}_x$ and $\delta(I) \to 0$,

$$\lim \frac{\phi(I)}{|I|} = \phi'(x).$$

Let us recall that we wish to study the connected sets that are carried by ϕ' into connected sets. Some further restrictions on ϕ are necessary as the following examples show. Let $\{I\}$ be the collection of all compact intervals of E_n. Let A be any nondegenerate connected subset of E_n and let $x_0 \in A$. Define $\phi(I) = |I|, x_0 \in I$, and $\phi(I) = 0$, $x_0 \notin I$. Then $\phi'(x) = 0$, $x \neq x_0$, and $\phi'(x_0) = 1$. Hence $\phi'(A)$ is not connected. Note that ϕ is balanced, but not continuous. If we let for $I \in \{I\}$, $d(I) = \text{dist}(I, x_0)$, then the interval function

$$\phi(I) = |I|^{1 + d(I)}$$

is continuous with $\phi'(x) = 0$, $x \neq x_0$, and $\phi'(x_0) = 1$. Again $\phi'(A)$ is not connected. We note that ϕ is not balanced.

In the sequel, the collection of intervals $\{I\}$ on which ϕ is defined has to be sufficiently large. We introduce the following definition.

DEFINITION 4. Let $A \subset E_n$. A collection of intervals $\{I\}$ of E_n is said to cover A *strongly* if and only if (1) for each $x \in A$ there is a sequence $\{I_j\} \subset \{I\}_x$ such that $\delta(I_j) \to 0$, (2) $\{I\}$ contains every interval I with $I^0 \subset A$.

2. **Darboux sets.** Let Q be the generic notation for a set in E_n such that $\bar{Q}$ is an interval and $(\bar{Q})^0 \subset Q \subset \bar{Q}$.

DEFINITION 1. A set $D \subset E_n$ will be called a *Darboux set* if and only if

(1)   for every $x \in D$ there is an interval I such that $x \in I$ and $I^0 \subset D$,

(2)   any two points $x, y \in D$ are *chained* in D, i.e., there are $Q_1, \cdots Q_j$ with $Q_i \subset D$, $i = 1, \cdots, j$, $x \in Q_1$, $y \in Q_j$, and $Q_i \cap Q_{i+1} \neq \emptyset$, $i = 1, \cdots, j - 1$.

It follows that every Darboux set is connected and that every connected open set is a Darboux set. The following example may help to clarify (2). Let $A = \{(x, y): (x, y) \in E_2$ and $y > |x|$ or $y \leq 0\}$. Then A is a connected set satisfying (1), but $(0, 0)$ and $(0, 1)$ are not chained in A.

DEFINITION 2. A function $f: A \to E_m$, $A \subset E_n$, is said to possess the *Darboux property* on A if and only if $f(D)$ is connected for every Darboux set $D \subset A$.

For the proof of Theorem 2 it is helpful to have the following terminology. Let $x_1, x_2 \in E_n$, $x_1 \neq x_2$, and let I_j be an interval containing x_j, $j = 1, 2$. We term I_1, I_2 *equivalent* if and only if there is a translation $\tau: E_n \to E_n$ with $\tau(I_1) = I_2$ and $\tau(x_1) = x_2$.

THEOREM 2. *Let $A \subset E_n$ and let $\{I\}$ cover A strongly. Let $\phi: \{I\} \to E_1$ be continuous, balanced, and differentiable in A. Then the derivative ϕ' has the Darboux property on A*(2).

Proof. Let $\bar{Q}$ be an interval, and let $(\bar{Q})^0 \subset Q \subset \bar{Q}$, $Q \subset A$. It suffices to show that $\phi'(Q)$ is connected. If we deny this, there is a point $x_0 \in E_1$ such that $\phi'(Q)$ is the union of two nonempty sets A', B' with $A' \subset \{x: x < x_0\}$, $B' \subset \{x: x > x_0\}$. Let $A = \{p: p \in Q$ and $\phi'(p) \in A'\}$, $B = \{p: p \in Q$ and $\phi'(p) \in B'\}$. Since $Q = A \cup B$, we may assume that $Q^0 \cap A \neq \emptyset$. Let $a \in Q^0 \cap A$, $b \in B$, and join a, b by a segment α.

Since $\phi'(a) < x_0$, $\phi'(b) > x_0$, there is an interval $I_b \in \{I\}_b$ such that $\delta(I_b) < 1$, $I_b \subset \bar{Q}$, $\phi(I_b) > x_0 \cdot |I_b|$, and if for $p \in \alpha - b$, I_p is the interval in $\{I\}_p$ equivalent with I_b, then $I_p \subset Q^0$ and $\phi(I_a) < x_0 \cdot |I_a|$. The function $\phi(I_p)$ is continuous in p on α. Hence there is a point $p_1 \in \alpha$ such that $\phi(I_1) = x_0 \cdot |I_1|$, where $I_1 = I_{p_1}$. We note that $I_1 \subset Q^0$ and $\delta(I_1) < 1$.

Let $a_1 \in I_1^0$. We may assume that $a_1 \varepsilon A$. There is $\frac{1}{2} > \varepsilon > 0$ such that if $I \in \{I\}_{a_1}$ and $\delta(I) < \varepsilon$, then $I \subset I_1$ and $\phi(I) < x_0 \cdot |I|$. Let then $I \in \{I\}_{a_1}$ with $\delta(I) < \varepsilon$. Since ϕ is balanced, there is an interval $I' \subset I_1$ such that $\delta(I') < \varepsilon$ and $\phi(I') \geq x_0 \cdot |I'|$. Let $p' \in I'$ and join p', a_1 by a line segment α_1. If we denote by I_p, $p \in \alpha_1$, the interval in $\{I\}_p$ equivalent with I', then $I_p \subset I_1$ and $\phi(I_{a_1}) < x_0 \cdot |I_{a_1}|$. By the continuity of ϕ, there is $p_2 \in \alpha_1$ such that $\phi(I_2) = x_0 \cdot |I_2|$, where $I_2 = I_{p_2}$. We note that $I_2 \subset I_1$ and $\delta(I_2) < \frac{1}{2}$.

Continuing in this manner, we obtain a sequence of intervals $\{I_j\} \subset \{I\}$ such that $Q^0 \supset I_1 \supset \cdots \supset I_j \supset \cdots$, $\delta(I_j) \to 0$, and $\phi(I_j) = x_0 \cdot |I_j|$. Let $p_0 \in \cap I_j$. Then $p_0 \in Q$ and $\phi'(p_0) = x_0$, a contradiction.

_____

(2) See also [9].

THEOREM 3. *Let C be a nondegenerate connected subset of E_n which is not a Darboux set. Then there exists a collection $\{I\}$ of intervals covering C strongly and a continuous, balanced, and differentiable interval function $\phi\colon \{I\} \to E_1$ such that its derivative ϕ' carries C into a disconnected set.*

Proof. There are two cases to be considered.

Case 1. There is a point $p_0 \in C$ such that for every interval I with $p_0 \in I$, $I^0 - C \neq \emptyset$. Let $\{I\}$ be the collection of all intervals I with $p_0 \notin I$ and the collection of all intervals I with $p_0 \in I^0$. Then $\{I\}$ covers C strongly. Define for $I \in \{I\}$, $\phi(I) = |I|$, $p_0 \in I^0$, and $\phi(I) = 0$, $p_0 \notin I$. Then ϕ is continuous, balanced, and $\phi'(p) = 0$, $p \neq p_0$, and $\phi'(p_0) = 1$. Hence $\phi'(C)$ is not connected.

Case 2. For every point $p \in C$ there is an interval I_p such that $p \in I_p$ and $I_p^0 \subset C$. Let $p_0 \in C$, and let

$$A = \{p\colon p \in C \text{ and } p, p_0 \text{ are chained in } C\},$$
$$B = \{p\colon p \in C \text{ and } p, p_0 \text{ are not chained in } C\}.$$

Since C is not Darboux, there follows that $C = A \cup B$ and $A \neq \emptyset$, $B \neq \emptyset$. We note that for each $p \in C$ the associated interval I_p has the property that either $I_p^0 \subset A$ or else $I_p^0 \subset B$.

Let $\{I\}$ be the collection of all intervals I with $I^0 \subset C$. Then $\{I\}$ covers C strongly. Define $\phi(I) = |I|$, if $I^0 \subset A$, and $\phi(I) = 0$, if $I^0 \subset B$. Then $\phi'(p) = 1$, $p \in A$, and $\phi'(p) = 0$, $p \in B$, and therefore $\phi'(C)$ is not connected. It is easy to see that ϕ is continuous and balanced.

Combining Theorems 2 and 3 we obtain the following characterization of Darboux sets.

THEOREM 4. *A nondegenerate connected set $D \subset E_n$ is a Darboux set if and only if for every collection of intervals $\{I\}$ covering D strongly and for every continuous, balanced, and differentiable $\phi\colon \{I\} \to E_1$, the derivative ϕ' carries D into a connected set.*

3. **Mean value property.** Let $\{I\}$ be the collection of all intervals in E_n and let $\phi\colon \{I\} \to E_1$ be a differentiable interval function with derivative ϕ'.

DEFINITION. ϕ is said to have the *mean value property* (simply MV-property) if and only if every $I \in \{I\}$ there is a point $p \in I^0$ such that

$$\frac{\phi(I)}{|I|} = \phi'(p).$$

The interval function $\phi(I) = |I|^2$ shows that continuity and differentiability are not enough for the MV-property. We note that ϕ is not balanced. The interval function defined by $(p_0 \in E_n)$ $\phi(I) = (1 - |I|) \cdot |I|$, $p_0 \in I$, and $\phi(I) = 0$, $p_0 \notin I$, is balanced, and $\phi'(p) = 0$, $p \neq p_0$, and $\phi'(p_0) = 1$. Thus ϕ does not possess the MV-property. It is clear that ϕ is not continuous. We have, however, the following theorem.

THEOREM 5. *Let $\{I\}$ be the collection of all intervals in E_n, and let $\phi\colon \{I\} \to E_1$ be continuous, balanced, and differentiable in E_n. Then ϕ has the MV-property([2]).*

Proof. The proof is similar to the proof of Theorem 2. Let $I_0 \in \{I\}$ and let $\phi(I_0)/|I_0| = x_0$. Let $p_0 \in I_0^0$, and assume that $\phi'(p_0) < x_0$. As in the proof of Theorem 2, there is an interval $I_1 \subset I_0^0$ such that $\delta(I_1) < 1$ and $\phi(I_1) = x_0 \cdot |I_1|$. Applying the same argument to I_1 and continuing in this way, we obtain a sequence $I_0^0 \supset I_1 \supset I_2 \supset \cdots, \delta(I_j) \to 0$, and $\phi(I_j) = x_0 \cdot |I_j|$. For $p \in \cap I_j$, we have $p \in I_0^0$ and $\phi'(p) = x_0$.

THEOREM 6. *Let $\{I\}$ be the collection of all compact intervals of E_n. Let $\phi\colon \{I\} \to E_1$ be continuous and differentiable. If ϕ possesses the MV-property, then its derivative ϕ' has the Darboux property.*

Proof. Let $\bar{Q}$ be an interval in E_n, and let $(\bar{Q})^0 \subset Q \subset \bar{Q}$. It suffices to show that $\phi'(Q)$ is connected. If this is not the case, we may assume that $Q = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$, and $\phi'(A) \subset \{x\colon x < x_0\}$, $\phi'(B) \subset \{x\colon x > x_0\}$ (see proof of Theorem 2). As in the proof of Theorem 2, there is an interval $I \subset Q$ such that $\phi(I) = x_0 \cdot |I|$. By the MV-property, there is $p_0 \in I^0$ such that $\phi'(p_0) = x_0$, a contradiction.

It is known that for functions of a real variable, the MV-property and Darboux property are equivalent (see e.g.[3]). The example $\phi(I) = |I|^2$ shows that this is false for interval functions. The next theorem contains some more information of this sort.

THEOREM 7. *For continuous, differentiable interval functions ϕ with derivative ϕ' the following holds: balanced $\to$ MV-property $\to$ Darboux property of ϕ'. None of the implications can be reversed.*

Proof. In view of the preceding theorems and remark we need only show that MV-property does not imply "balanced." Let $I_0 = [a_0, b_0]$ be an interval in E_1. Let $f\colon \{x\colon x \geqq 0\} \to \{x\colon x \geqq 0\}$ be continuous and have the property that $f(0) = 0$, $0 < f(x) < x/2$ for $x \in (0, |I_0|)$, and $f(x) = 0$ for $x \geqq |I_0|$. For I a compact interval in E_1, let $d(I)$ be the midpoint of I. Define

$$\phi(I) = [|d(I_0) - d(I)| + f(|I|)] \cdot |I|.$$

It is clear that ϕ is continuous and that $\phi'(x) = |d(I_0) - x|$ for $x \in E_1$. We will verify that ϕ has the MV-property. Let I be an interval. Since $0 \leqq f(|I|) < |I|/2$, there is a point $x \in I^0$ such that

$$|d(I_0) - x| = |d(I_0) - d(I)| + f(|I|).$$

Hence

$$\frac{\phi(I)}{|I|} = \phi'(x).$$

We show next that ϕ is not balanced. We have $\phi(I_0)=0$, and, since $0 < f(|I|)$ for $|I| < |I_0|$, for every proper subinterval I for I_0 we have $\phi(I) > 0$. This completes the proof.

4. **Applications.** In this section we will consider some classes of functions which possess the Darboux property.

A function $f: E_n \to E_1$ is called *approximately continuous* in E_n if and only if for each $p \in E_n$ there is a measureable set E such that p is a point of density one of E and $f|E$ is continuous at p (see [6]).

REMARK. We do not make a distinction between ordinary and strong metric density since the theorem below is true for either notion.

THEOREM 8. *If $f: E_n \to E_1$ is approximately continuous, then f has the Darboux property.*

Proof. We first assume that f is bounded. Then f is summable on every compact interval $I \subset E_n$. Let $\phi(I) = \int_I f$. It is clear that ϕ is continuous, and since ϕ is additive, we have by Theorem 1 that ϕ is balanced. By [6, p. 132], $\phi'(x) = f(x)$, $x \in E_n$. By Theorem 2, $\phi' = f$ has the Darboux property.

If f is unbounded, let $h: E_1 \to (0,1)$ be a homeomorphism. Then $g(x) = h[f(x)]$ is approximately continuous and bounded. Hence g and therefore f has the Darboux property.

REMARK. For further details concerning the class of connected sets that remain connected under approximately continuous transformations the reader should consult [4; 5].

THEOREM 9. *Let $f: E_n \to E_1$. If for each $(x_1^0, \cdots, x_n^0) \in E_n$, the functions $f(x_1^0, \cdots, x_{i-1}^0, x_i, x_{i+1}^0, \cdots, x_n^0)$, $i = 1, \cdots, n$, possess the Darboux property, then f possesses the Darboux property.*

Proof. For notational reasons we will restrict ourselves to $n = 2$. It suffices to show that for $\bar{Q}$ an interval and $(\bar{Q})^0 \subset Q \subset \bar{Q}$, $f(Q)$ is connected. By hypothesis, every horizontal and vertical line segment in Q is carried by f into a connected set.

Case 1. Every point in Q lies on a horizontal or vertical line segment which is contained in Q. In this case it follows from the above observation that $f(Q)$ is connected.

Case 2. There is a point $p_0 = (x_0, y_0)$ in Q such that p_0 does not lie on a horizontal or vertical line segment which is contained in Q. Then it follows that p_0 is a vertex of Q. Let H be a compact horizontal line segment such that $p_0 \in H$, $Q^0 \cup H \subset \bar{Q}$. Since $f(H)$ is connected, there is a sequence $p_n \in H - p_0$ such that $f(p_n) \to f(p_0)$. For each n, there is a vertical line segment V_n such that $p_n \in V_n$ and $V_n \subset \bar{Q}$. Since $f(V_n)$ is connected, there is a point $s_n \in V_n \cap Q^0$ such that $|f(s_n) - f(p_n)| < 1/n$. Then $f(s_n) \to f(p_0)$, and therefore $f(Q^0 \cup p_0)$ is connected This completes the proof.

REMARK.   It is easy to give examples showing that the converse of Theorem 8 is not true.

In the sequel we need the notion of an approximate derivative introduced by A. Denjoy [1]. The properties of approximate derivatives that will be used below can be found in [7; 8] with simpler proofs in [3]. In particular, if $f: E_1 \to E_1$ is an approximate derivative, then f has the MV-property, the Darboux property, and is a function of Baire type at most one.

We need the following known lemma, whose proof is short and will be given for completeness.

LEMMA.   *Let $I = [a, b]$ be an interval in E_1, and let A_1, A_2 be two nonempty measurable sets such that $I = A_1 \cup A_2$. Then A_i contains a point of positive upper density of A_j for some $i \neq j$, $i, j = 1, 2$.*

Proof.   Suppose this is not the case. Then $x \in A_1$ implies that x is a point of density one of A_i, $i = 1, 2$. Then the characteristic function c_1 of A_1 is approximately continuous. By Theorem 8, $c_1(I)$ should be connected, a contradiction since $c_1(A_1) = 1$, $c_1(A_2) = 0$.

DEFINITION.   A function $f: E_2 \to E_1$ will be called *linearly approximately continuous* if and only if for each $(x_0, y_0) \in E_2$, the functions $f(x_0, y)$, $f(x, y_0)$ are approximately continuous as a function of y, x, respectively.

THEOREM 10.   *Let $f: E_2 \to E_1$ be linearly approximately continuous, and assume that the approximate partial derivative with respect to x, $(f_x)_{ap}$, exists everywhere in E_2. Then $(f_x)_{ap}: E_2 \to E_1$ has the Darboux property.*

Proof.   Let $h = (f_x)_{ap}$. By [6, p. 299] h is measurable. Let $I = [a, b] \times [c, d]$. We will first prove that $h(I)$ is connected. If we deny this, there is a point $x_0 \in E_1$ and there are two sets $A \neq \emptyset$, $B \neq \emptyset$, such that $I = A \cup B$ and $h(A) \subset \{x: x < x_0\}$, $h(B) \subset \{x: x > x_0\}$. For each $y \in [c, d]$, let $H_y = \{(x, y): x \in [a, b]\}$. Since $h(H_y)$ is connected, $H_y \subset A$ or else $H_y \subset B$. Thus there are two sets A_1, A_2 such that $[c, d] = A_1 \cup A_2$, and $A = [a, b] \times A_1$, $B = [a, b] \times A_2$. Since h is measurable, A, B are measurable, and, therefore, A_1, A_2 are measurable subsets of $[c, d]$. By the lemma we may assume that there is $y_0 \in A_1$ which is also a point of positive upper density of A_2.

Since $f(a, y)$, $f(b, y)$ are approximately continuous, and since y_0 is a point of positive upper density of A_2, there is a sequence $\{y_n\}$ in A_2 such that $y_n \to y_0$ and $f(a, y_n) \to f(a, y_0)$, $f(b, y_n) \to f(b, y_0)$. For each n, $n = 0,1,2,\cdots$, there is a point $z_n \in (a, b)$ such that

$$f(b, y_n) - f(a, y_n) = h(z_n, y_n) \, (b - a).$$

Therefore, $h(x_n, y_n) \to h(z_0, y_0)$. Since $(z_n, y_n) \in B$, $h(z_n, y_n)$ is in $h(B)$. Therefore $h(z_0, y_0) > x_0$, contradicting $(z_0, y_0) \in A$. Thus $h(I)$ is connected.

To complete the proof, it suffices to show that for every set Q, satisfying $(\bar{Q})^0 \subset Q \subset \bar{Q}$, $\bar{Q}$ interval, $h(Q)$ is connected. If Q is open, Q can be written as an increasing sequence of compact intervals, and hence $h(Q)$ is connected. We may therefore assume that there is a point $p_0 \in Q - Q^0$. Let $I = [a, b] \times [c, d]$ be an interval in $\bar{Q}$ which has p_0 as a vertex, say $p_0 = (a, d)$. We only need show that $h(I^0 \cup p_0) = h(I^0) \cup h(p_0)$ is connected.

Since $h(x, d)$ possesses the Darboux property, there is a sequence $\{x_n\}$ in (a, b) such that $h(x_n, d) \to h(a, d)$. It suffices to show that for each n there is a point $p_n \varepsilon I^0$ such that $|h(x_n, d) - h(p_n)| < 1/n$. There is $z_n \in (a, b) - x_n$ such that

$$\left| h(x_n, d) - \frac{f(x_n, d) - f(z_n, d)}{x_n - z_n} \right| < \frac{1}{3n}.$$

By linear approximate continuity, there is $y_n \in (c, d)$ such that

$$\left| f(x_n, d) - f(x_n, y_n) \right| < \frac{1}{3n} \cdot \left| x_n - z_n \right|, \quad \left| f(z_n, d) - f(z_n, y_n) \right| < \frac{1}{3n} \cdot \left| x_n - z_n \right|.$$

An easy calculation reveals that

$$\left| h(x_n, d) - \frac{f(x_n, y_n) - f(z_n, y_n)}{x_n - z_n} \right| < \frac{1}{n}.$$

By the MV-property for approximate derivatives, there is a point τ_n between x_n, z_n such that

$$f(x_n, y_n) - f(z_n, y_n) = h(\tau_n, y_n)(x_n - z_n).$$

Consequently, $|h(x_n, d) - h(\tau_n, y_n)| < 1/n$, and the proof is complete.

BIBLIOGRAPHY

1. A. Denjoy, *Mémoire sur la totalization des nombres dérivés nonsommables*, Ann. Sci. Ecole Norm. Sup. **33** (1916), 127–222.

2. A. Gleyzal, *Interval functions*, Duke Math. J. **8** (1941), 223–230.

3. C. Goffman and C. J. Neugebauer, *On approximate derivatives*, Proc. Amer. Math. Soc. **11** (1960), 962–966.

4. C. Goffman, C. J. Neugebauer and T. Nishiura, *Density topology and approximate continuity*, Duke Math. J. **28** (1961), 497–505.

5. C. Goffman and D. Waterman, *Approximately continuous transformations*, Proc. Amer. Math. Soc. **12** (1961), 116–121.

6. S. Saks, *Theory of the integral*, Warsaw, 1937.

7. A. Khintshine, *Recherches sur la structure des fonctions mesurables*, Fund. Math. **9** (1927), 217–279.

8. G. Tolstoff, *Sur la dérivée approximative exacte*, Mat. Sb. (N. S.) **4** (1938), 499–504.

9. L. Mišik, *Der Mittelwertsatx für additive Intevallfunktionen*, Fund. Math. **45** (1958), 64–70.

PURDUE UNIVERSITY,
    LAFAYETTE, INDIANA