ABSTRACT ERGODIC THEOREMS(1)

BY

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Introduction. In this paper we prove certain maximal theorems and certain pointwise convergence theorems. Except for the Introduction and for the paragraph on notations and terminology, the material presented here is divided into two parts and an Appendix.

The main result of Part I is Theorem 1. This is a maximal theorem for certain operators on abstract L_E^p spaces, where $1 \leqq p < \infty$ and E is a Banach space. This theorem contains as immediate particular cases the classical maximal ergodic theorem and a maximal theorem for martingales. Hence this provides (see also the Appendix) a unified proof of the pointwise ergodic theorem and of the pointwise convergence theorem concerning the (decreasing) martingale. Theorem 2 is again a maximal theorem which contains as particular cases a maximal ergodic theorem and a maximal theorem for martingales. Another unified treatment of the ergodic theorem and the martingale theorem, quite different (in methods as well as results) from those presented here is given in [35](2).

The results of Part II are essentially generalizations of certain results concerning Banach space valued martingales. In particular, Theorem 4 shows that under certain conditions the strong pointwise convergence theorem for Banach space valued increasing martingales holds. In a weaker form this theorem was proved in [7; 12; 36].

The Appendix contains an almost everywhere convergence theorem which is used in Parts I and II.

The main results of this paper were announced in the Proceedings of the National Academy of Sciences, February 1962.

Notations and terminology. Let $(Z, \mathscr{E}, \mu)$ be a complete totally σ-finite measure space and E a Banach space. For each $1 \leqq p < \infty$ denote by $\mathscr{L}_E^p$ the vector space of all (Bochner) measurable mappings f of Z into E for which $z \to \|f(z)\|^p$ is μ-integrable(3); here $\mathscr{L}_E^p$ is endowed with the semi-norm $f \to \|f\|_p$

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(1) Research supported by the U. S. Army Research Office (Durham) under contract DA-ARO(D)-31-124-G218.

(2) The analogy between the maximal ergodic theorem (for operators corresponding to measure-preserving transformations) and the maximal theorem for martingales was noticed long ago (see for instance [29]).

(3) See [21]. In the same way we define the spaces $\mathscr{L}_E^p$ for not necessarily complete measure spaces.

$=(\int_Z \|f(z)\|^p d\mu(z))^{1/p}$. Denote by L_E^p the associated separated (Banach) space and by $f \to \tilde{f}$ the canonical mapping of $\mathscr{L}_E^p$ onto L_E^p. Let $\mathscr{S}_E$ be the vector space of all functions which are bounded and belong to $\mathscr{L}_E^1$; here $\mathscr{S}_E$ is endowed with the semi-norm $f \to \|f\|_\infty = \operatorname{ess\,sup}_{z \in Z} \|f(z)\|$. Denote by S_E the associated separated (normed) space and by $f \to \tilde{f}$ the canonical mapping of $\mathscr{S}_E$ onto S_E.

Let $\mathscr{D}$ be the set of all linear mappings T of S_E into S_E such that[4] $\|T\|_1 \leqq 1$ and $\|T\|_\infty \leqq 1$. Then $\|T\|_p \leqq 1$ for all $1 \leqq p < \infty$; hence T can be extended by continuity to L_E^p (we denote the extension by the same letter). For $T \in \mathscr{D}$ and $f \in \mathscr{V} = \bigcup_{1 \leqq p < \infty} \mathscr{L}_E^p$ we denote by Tf a (determined) representative of the class $T\tilde{f}$.

For each function $f \in \mathscr{V}$ and each $a > 0$ we write

$$G_f(a) = \{z \mid \|f(z)\| > a\}.$$

Let now $U_1, U_2, \cdots, U_n$ $(n \geqq 1)$ be n operators belonging to $\mathscr{D}$ and let $f \in \mathscr{L}_E^p \subset \mathscr{V}$ $(1 \leqq p < \infty)$. The definition of the functions $d_0, \cdots, d_n, f_0, \cdots, f_n$ which we shall introduce below and the relations which we shall establish were suggested by [5] and by a manuscript of J. Oxtoby. Define d_0 and f_0 by the equations

$$\begin{aligned} d_0(z) &= af(z)/\|f(z)\| && \text{if } a < \|f(z)\|, \\ d_0(z) &= f(z) && \text{if } a \geqq \|f(z)\|, \\ f_0 &= f - d_0. \end{aligned}$$

Since $\mu(G_f(a))$ is finite it follows that $d_0 \in \mathscr{L}_E^p$; hence $f_0 \in \mathscr{L}_E^p$ too. It is also obvious that $a - \|d_0(z)\| \geqq 0$ for all $z \in Z$. Suppose now that $k \in \{0, \cdots, n-1\}$ and that $d_0, \cdots, d_k, f_0, \cdots, f_k$ were defined, belong to $\mathscr{L}_E^p$ and

$$D_k(z) = a - \sum_{j=0}^k \|d_j(z)\| \geqq 0$$

for all $z \in Z$. We define then d_{k+1} and f_{k+1} by the equations

$$\begin{aligned} d_{k+1}(z) &= D_k(z) U_{k+1} f_k(z) / \|U_{k+1} f_k(z)\| && \text{if } D_k(z) < \|U_{k+1} f_k(z)\|, \\ d_{k+1}(z) &= U_{k+1} f_k(z) && \text{if } D_k(z) \geqq \|U_{k+1} f_k(z)\|, \\ f_{k+1} &= U_{k+1} f_k - d_{k+1}. \end{aligned}$$

It is easy to see that d_{k+1} and f_{k+1} belong to $\mathscr{L}_E^p$ and that we have $a - \sum_{j=0}^{k+1} \|d_j(z)\| \geqq 0$ for all $z \in Z$. By induction we define then the functions $d_0, \cdots, d_n, f_0, \cdots, f_n$; without difficulty we see that

(1) $d_0, \cdots, d_n, f_0, \cdots, f_n$ belong to $\mathscr{L}_E^p$;
(2) $\sum_{j=0}^k \|d_j(z)\| \leqq a$ for all $k \in \{0, \cdots, n\}$;
(3) $d_{k+1} + f_{k+1} = U_{k+1} f_k$ for all $k \in \{0, \cdots, n-1\}$;
(4) $\|d_0(z)\| + \|f_0(z)\| = \|f(z)\|$ for all $z \in Z$;
(5) $\|d_{k+1}(z)\| + \|f_{k+1}(z)\| = \|U_{k+1} f_k(z)\|$ for all $z \in Z$ and $k \in \{0, \cdots, n-1\}$;
(6) $z \in Z$, $k \in \{0, \cdots, n\}$ and $f_k(z) \neq 0$ imply $\sum_{j=0}^k \|d_j(z)\| = a$.

(4) For $T \in \mathscr{D}$ and $1 \leqq p \leqq \infty$ define $\|T\|_p = \sup\{\|T\tilde{f}\|_p \mid \tilde{f} \in S_E, \|\tilde{f}\|_p \leqq 1\}$.

Since $\mu(G_f(a))$ is finite and $f_0(z) = 0$ for $z \notin G_f(a)$, we deduce that $f_0 \in \mathscr{L}^1_E$; it follows that $d_1, \cdots, d_n, f_1, \cdots, f_n$ belong also to $\mathscr{L}^1_E$. Let us also remark that:

(7) If $C \in \mathscr{E}$ and $\sum_{j=0}^n \| d_j(z) \| = a$ for each $z \in C$ then

$$\int_C (a - \| d_0(z) \|) d\mu(z) \leqq \int_Z \| f_0(z) \| \, d\mu(z).$$

In fact, using (5) we deduce

$$\begin{aligned} &\int_C (a - \| d_0(z) \|) d\mu(z) \\ &\leqq \sum_{j=1}^n \int_Z \| d_j(z) \| \, d\mu(z) \\ &= \sum_{j=1}^n \left(\int_Z \| U_j f_{j-1}(z) \| d\mu(z) - \int_Z \| f_j(z) \| \, d\mu(z) \right) \\ &\leqq \sum_{j=1}^n \left(\int_Z \| f_{j-1}(z) \| \, d\mu(z) - \int_Z \| f_j(z) \| \, d\mu(z) \right) \leqq \int_Z \| f_0(z) \| \, d\mu(z). \end{aligned}$$

For every $j \in \{1, \cdots, n\}$, $i \in \{1, \cdots, n\}$ such that $j \geqq i$ we shall define (by induction) an operator $U_{(j,i)}$. We write $U_{(j,i)} = U_j$ if $j = i$. Suppose now that $j > i$ and that $U_{(j-1,i)}$ has been defined; we write then $U_{(j,i)} = U_{(j,j)} U_{(j-1,i)}$.

Using (3) and the equation $\tilde{f} = \tilde{d}_0 + \tilde{f}_0$ we obtain easily (by induction) for all $k \in \{1, \cdots, n\}$ the formula

$$U_{(k,1)} \tilde{f} = \sum_{j=1}^{k} U_{(k,j)} \tilde{d}_{j-1} + \tilde{d}_k + \tilde{f}_k. \tag{8}$$

Throughout this paper we consider only tribes ($=\sigma$-algebras) $\mathscr{F} \subset \mathscr{E}$ such that the restriction of μ to $\mathscr{F}$ is *totally σ-finite*. Let now $\mathscr{F} \subset \mathscr{E}$ be a tribe. It is easily seen that there is an operator $E_{\mathscr{F}} \in \mathscr{D}$ (define first $E_{\mathscr{F}} \tilde{f}$ for simple functions $f \in \mathscr{S}_E$) such that: (i) for each $f \in \mathscr{V}$ there exists a representative in the class $E_{\mathscr{F}} \tilde{f}$ which is measurable with respect to $\mathscr{F}$; (ii) for each $f \in L^1_E$ and $A \in \mathscr{F}$

$$\int_A E_{\mathscr{F}} f \, d\mu = \int_A f d\mu.$$

The operator $E_{\mathscr{F}}$ is called the *conditional expectation* with respect to $\mathscr{F}$.

Let now $(\mathscr{F}_n)_{n \in N}$ be a decreasing sequence of tribes contained in $\mathscr{E}$ and $(f_n)_{n \in N}$ a sequence of functions belonging to $\mathscr{V}$. We say that $(f_n)_{n \in N}$ is a *decreasing martingale* with respect to the sequence $(\mathscr{F}_n)_{n \in N}$ if for each $n \in N$, $E_{\mathscr{F}_{n+1}} \tilde{f}_n = \tilde{f}_{n+1}$. In a similar manner we define an *increasing martingale.*

PART I

1. A maximal ergodic theorem. Let $T_0, T_1, \cdots, T_r \in \mathscr{D}$ $(r \geqq 1)$ and consider the conditions:

(9) $T_0 = I$;

(10) $T_{j+1}T_j = T_{j+1}$ for $j \in \{0,\cdots,r-1\}$.

We define $T_j^0 = I$ for all $j \in \{0,\cdots,r\}$. We shall now state and prove our first maximal theorem:

THEOREM 1. *Let $T_0, T_1,\cdots,T_r \in \mathscr{D}$ be $r+1$ operators satisfying the conditions* (9) *and* (10). *For each $f \in \mathscr{V}$ and each $a > 0$ define*

$$G_f^*(a) = \{z \mid \sup_{j \in \{0,\dots,r\}, n \in N} \| (T_j^0 + T_j^1 + \cdots + T_j^n)f(z)/(n+1) \| > a\}.$$

Then for each set $F \in \mathscr{E}$ verifying (*except for sets of measure zero*) *the relations $G_f(a) \subset F \subset G_f^*(a)$ we have*

$$a\mu(F) \leqq \int_F \|f(z)\| \, d\mu(z) < \infty.$$

Let $f \in \mathscr{V}$; for each $a > 0$ and each $s \in N$ define

$$G_f^*(a,s) = \{z \mid \sup_{j \in \{1,\dots,r\}, n \in \{0,\dots,s\}} \|(T_j^0 + T_j^1 + \cdots + T_j^n)f(z)/(n+1)\| > a\}.$$

Remark that (except for sets of measure zero) $G_f(a) = G_f^*(a,0) \subset G_f^*(a,s)$ for each $s \in N$, the sequence $(G_f^*(a,s))_{s \in N}$ is increasing, and that $\bigcup_{s \in N} G_f^*(a,s) = G_f^*(a)$. For each $s \in N$ let $F_s = G_f^*(a,s) \cap F$. To prove the theorem, it is enough to show that (see the paragraph on notations and terminology)

$$\int_{F_-} (a - \| d_0(z) \|) d\mu(z) \leqq \int_Z \|f_0(z)\| \, d\mu(z) \tag{11}$$

for each $s \in N$. In fact, (11) implies

$$\int_F (a - \| d_0(z) \|) d\mu(z) \leqq \int_Z \|f_0(z)\| \, d\mu(z)$$

and the desired conclusion follows from Proposition 1 below.

Let $s \in N$, $s \geqq 1$ and define

$$\begin{aligned} U_0 &= T_0, \\ U_1 &= T_1, \cdots, U_s = T_1, \\ &\cdots\cdots\cdots\cdots \\ U_{(j-1)s+1} &= T_1, \cdots, U_{(j-1)s+s} = T_j, \\ &\cdots\cdots\cdots\cdots \\ U_{(r-1)s+1} &= T_r, \cdots, U_{(r-1)s+s} = T_r, \end{aligned}$$

and the corresponding functions $d_0,\cdots,d_{(r-1)s+s}$ and $f_0,\cdots,f_{(r-1)s+s}$ (see the paragraph on notations and terminology). Let $j \in \{1,\cdots,r\}$; we then have for $t = 0$

$$T_j^0 \tilde{f} = \tilde{f} = \tilde{d}_0 + \tilde{f}_0 \tag{12}$$

and for $1 \leqq t \leqq s$

$$T_j^t \tilde{f} = T_j^t\left(\sum_{q=0}^{(j-1)s} \tilde{d}_q \right) + \sum_{u=1}^{t} T_j^{t-u}(\tilde{d}_{(j-1)s+u}) + \tilde{f}_{(j-1)s+t}. \tag{13}$$

Formula (12) is obvious. Formula (13) follows easily from the definition of the operators $U_0, U_1, \cdots, U_{(r-1)s+s}$, from formula (8) and conditions (10). In fact (we write $A = \emptyset$ if $t = 1$ and $A = \{v \mid 1 < v \leqq t\}$ if $t > 1$):

$$\begin{aligned} T_j^t \tilde{f} &= \sum_{u=1}^{(j-1)s+t} U_{((j-1)s+t,u)} \tilde{d}_{u-1} + \tilde{d}_{(j-1)s+t} + \tilde{f}_{(j-1)s+t} \\ &= \sum_{u=1}^{(j-1)s+1} U_{((j-1)s+t,u)} \tilde{d}_{u-1} + \sum_{v \in A} U_{((j-1)s+t,(j-1)s+v)} \tilde{d}_{(j-1)s+v-1} \\ &\quad + \tilde{d}_{(j-1)s+t} + \tilde{f}_{(j-1)s+t}. \end{aligned}$$

The conditions (10) imply that $U_{((j-1)s+t,u)} = T_j^t$ for all u such that $1 \leqq u \leqq (j-1)s+1$; if $A \neq \emptyset$ and $v \in A$ then $U_{((j-1)s+t,(j-1)s+v)} = T_j^{t-(v-1)}$. We deduce then the formula (13).

Let us remark now that for $n = 0$

$$\sum_{t=0}^{n} T_j^t \tilde{f} = \tilde{d}_0 + \tilde{f}_0 \tag{14}$$

and for $1 \leqq n \leqq s$

$$\begin{aligned} \sum_{t=0}^{n} T_j^t \tilde{f} = \tilde{d}_0 &+ \tilde{f}_0 + \sum_{q=1}^{n} \tilde{a}_{(j-1)s+q} + \sum_{q=1}^{n} \tilde{f}_{(j-1)s+q} \\ &+ \sum_{t=1}^{n} T_j^t \left(\sum_{q=0}^{(j-1)s+n-t} \tilde{d}_q \right). \end{aligned} \tag{15}$$

Formula (14) is obvious. To prove (15) we shall reason by induction. For $n = 1$ we have from (12) and (13)

$$T_j^0 \tilde{f} + T_j^1 \tilde{f} = \tilde{d}_0 + \tilde{f}_0 + \tilde{d}_{(j-1)s+1} + \tilde{f}_{(j-1)s+1} + T_j \left(\sum_{q=0}^{(j-1)s} \tilde{d}_q \right)$$

and thus (15) is true for $n = 1$. Suppose now that (15) is valid for $1, 2, \cdots, n$ ($n < s$); we shall show that the formula holds also for $n + 1$. In fact, by (13)

$$\begin{aligned} T_j^{n+1} \tilde{f} = T_j^{n+1}(\tilde{d}_0 + \cdots + \tilde{d}_{(j-1)s}) &+ T_j^n \tilde{d}_{(j-1)s+1} + \cdots + T_j \tilde{a}_{(j-1)s+n} \\ &+ \tilde{d}_{(j-1)s+n+1} + \tilde{f}_{(j-1)s+n+1} \, . \end{aligned}$$

If we add $T_j^{n+1} \tilde{f}$ to (15) we deduce that formula (15) remains valid for $n + 1$. Hence (15) is true for all $n \in \{1, \cdots, s\}$.

Now for each $1 \leqq j \leqq r$ and each $0 \leqq n \leqq s$, let

$$A(j,n) = \{ z \mid \| (T_j^0 + \cdots + T_j^n) f(z)/(n+1) \| > a \} \cap F_s.$$

Then $F_s = \bigcup_{1 \leqq j \leqq r,\ 0 \leqq n \leqq s} A(j,n)$. If $n = 0$, we have on $A(j,0)$ (see (14))

$$a < \left\| \sum_{t=0}^{n} T_j^t f(z) \right\| = \| f(z) \| = \| d_0(z) \| + \| f_0(z) \| \leqq a + \| f_0(z) \|$$

almost everywhere; if $1 \leqq n \leqq s$, we have on $A(j,n)$ (see (15) and (2))

$$(n+1)a < \Big\| \sum_{t=0}^{n} T_j^t f(z) \Big\| \leqq (n+1)a + \|f_0(z)\| + \sum_{q=1}^{n} \|f_{(j-1)s+q}(z)\|$$

almost everywhere. Thus on F_s, $\sum_{q=0}^{(r-1)s+s} \|f_q(z)\| > 0$ almost everywhere; hence $\sum_{q=0}^{(r-1)s+s} \| d_q(z) \| = a$ (use (6)) and formula (11) follows then immediately from (7). The proof of the theorem will then be completed if we prove the following (see also [5]):

PROPOSITION 1. *Let g, h be two positive functions belonging to $\mathscr{L}_R^p$ for some $1 \leqq p < \infty$, a a strictly positive number, $A = \{z \mid g(z) + h(z) > a\}$ and $Y \in \mathscr{E}$ such that $A \subset Y$. Suppose that*: (1.1) $h(z) = 0$ *if* $z \notin A$; (1.2) $a - g$ *is integrable on Y and*

$$\int_Y (a - g(z))d\mu(z) \leqq \int_Z h(z)d\mu(z).$$

Then

$$a\mu(Y) \leqq \int_Y (g(z) + h(z))d\mu(z) < \infty. \tag{16}$$

From (1.2) and (1.1) we deduce (h is obviously integrable since A has finite measure)

$$\int_Y (a - g(z))d\mu(z) \leqq \int_Y h(z)d\mu(z).$$

Hence

$$\int_Y (g(z) + h(z) - a)d\mu(z) \geqq 0. \tag{17}$$

Now let $0 < b < a$ and $B = \{z \mid g(z) + h(z) > b\}$. Clearly $Y - B \subset Y - A$. For $z \in B$ we have $a - (g(z) + h(z)) \geqq a - b$ and for $z \notin A$ we have $a - (g(z) + h(z)) \geqq 0$. It follows that

$$(a-b)\mu(Y-B) \leqq \int_{Y-B} (a - (g(z)+h(z))d\mu(z) \leqq \int_{Y-A} (a-(g(z)+h(z))d\mu(z).$$

The last integral is finite since on $Y - A$, $a - (g + h) = a - g$ and since (by (1.2)) $a - g$ is integrable on $Y - A$. Thus $\mu(Y - B)$ is finite. Since $\mu(B)$ is obviously finite, we deduce that $\mu(Y)$ is finite, and (16) follows immediately from (17).

From Theorem 1 we deduce the following two consequences:

COROLLARY 1. *Let $T \in \mathscr{D}$. For each $f \in \mathscr{V}$ and each $a > 0$ define*

$$E_f^*(a) = \{z \mid \sup_{n \in N} \| (T^0 + T^1 + \cdots + T^n)f(z)/(n+1) \| > a\}.$$

Then

$$a\mu(E_f^*(a)) \leqq \int_{E_f^*(A)} \|f(z)\| d\mu(z) < \infty.$$

Corollary 1 follows from Theorem 1 if we take $r = 1$, $T_0 = I$, $T_1 = T$ and $F = E_f^*(a)$.

COROLLARY 2. *Let $(T_j)_{j \in \{1,\dots,r\}}$ be a decreasing sequence of projections belonging to $\mathscr{D}$. For each $f \in \mathscr{V}$ and each $a > 0$ define*

$$H_f^*(a) = \{z \mid \sup(\|f(z)\|, \|T_1 f(z)\|, \cdots, \|T_r f(z)\|) > a\}$$

Then

$$a\mu(H_f^*(a)) \leqq \int_{H_f^*(a)} \|f(z)\| \, d\mu(z) < \infty.$$

It is clear that $(T_j)_{j \in \{0,1,\dots,r\}}$ (where $T_0 = I$) satisfies the conditions (9) and (10) and that $G_f(a) \subset H_f^*(a)$. We now have $H_f^*(a) = G_f(a) \cup G_1 \cup \cdots \cup G_r$ where $G_j = \{z \mid \|T_j f(z)\| > a\}$ for all $j \in \{1, \cdots, r\}$. For each $j \in \{1, \cdots, r\}$ we have, almost everywhere on G_j,

$$\lim_{n \in N} \|(I + T_j^1 + \cdots + T_j^n) f(z)\| / (n+1) = \lim_{n \in N} \|f(z) + nT_j f(z)\| / (n+1) = \|T_j f(z)\| > a;$$

hence (except for a set of measure zero) $G_j \subset G_f^*(a)$. Thus Corollary 2 is an immediate consequence of Theorem 1.

Corollary 1 gives the classical maximal ergodic theorem (see [2; 39; 22; 20; 13; 14; 5; 6]). From Corollary 1 and the Theorem of the Appendix we deduce the corresponding pointwise ergodic theorem:

COROLLARY 3. *Let $T \in \mathscr{D}$ and suppose that the space E is reflexive. There is then a projection $T_\infty \in \mathscr{D}$ such that, for each $f \in \mathscr{V}$, the sequence*

$$((T^0 + T^1 + \cdots + T^n) f(z)/(n+1))_{n \in N}$$

converges almost everywhere to $T_\infty f(z)$.

Corollary 2 gives in particular a maximal theorem for martingales (see [30; 37; 10; 11; 31]). Using Corollary 2 and the Theorem of the Appendix we deduce the following:

COROLLARY 4. *Let $(T_n)_{n \in N}$ be a decreasing sequence of projections belonging to $\mathscr{D}$ and suppose that the space E is reflexive. There is then a projection $T_\infty \in \mathscr{D}$ such that, for each $f \in \mathscr{V}$, the sequence*

$$(T_n f(z))_{n \in N}$$

converges almost everywhere to $T_\infty f(z)$.

The Theorem of the Appendix can be used here, since $(T_n)_{n \in N}$ is obviously a system of almost invariant integrals (with respect to the semi-group spanned by I and by the family $(T_n)_{n \in N}$) in S_E and every space L_E^p, $1 \leqq p < \infty$ (see also [27]).

In particular Corollary 4 contains (when the space E is reflexive) the pointwise convergence theorem concerning the (abstract) decreasing martingale (see [11; 15; 31; 7; 36]). However, if for each $n \in N$ the operator T_n is the conditional expectation with respect to $\mathscr{F}_n$, where $(\mathscr{F}_n)_{n\in N}$ is a decreasing sequence of tribes contained in $\mathscr{E}$, then the conclusion of Corollary 4 remains valid without supposing E reflexive; in fact it is enough to remark that, by Corollary 4, $(T_nf(z))_{n\in N}$ converges almost everywhere for each simple function $f\in\mathscr{S}_E$, to apply Corollary 2, and (see the Appendix) to use the Banach theorem (see also [7; 36]).

REMARKS. (1) We use Corollary 2 to show that, for each $f\in\mathscr{V}$, $\sup_{n\in N}\|T_nf(z)\|$ is finite almost everywhere. This is obvious if $f\in\mathscr{L}^1_E$. If $f\in\mathscr{L}^p_E$, with $1<p<\infty$, we remark that (for each $r\in N^*$) we have

$$a\mu(H_f^*(a))\leqq\int_{H_f^*(a)}\|f(z)\|\,d\mu(z)\leqq\|f\|_p\mu(H_f^*(a))^{1/p'}$$

where $1/p+1/p'=1$. We deduce then that $a\mu(H_f^*(a))^{1/p}\leqq\|f\|_p$ and hence that $\mu(H_f^*(a))\leqq\|f\|_p^p/a^p$. (2) It is easy to see that Theorem 1 remains also valid for an "infinite sequence of operators."

COROLLARY 5. *Let $(T_n)_{n\in N}$ be an increasing sequence of projections belonging to $\mathscr{D}$ and suppose that the space E is reflexive. There is then a projection $T_\infty\in\mathscr{D}$ such that, for each $f\in\mathscr{V}$, the sequence*

$$(T_nf(z))_{n\in N}$$

converges almost everywhere to $T_\infty f(z)$.

Using Corollary 2 (see also the Remark (1) above) we deduce that for each $f\in\mathscr{V}$, $\sup_{n\in N}\|T_nf(z)\|$ is finite almost everywhere.

Since the space L^2_E is reflexive (see [34] or [24]) it follows (use [1] of Eberlein's ergodic theorem([5])) that the sequence $(T_n)_{n\in N}$ convegres strongly, in L^2_E, to a projection T_∞. Obviously $T_nT_\infty=T_\infty T_n=T_n$ for each $n\in N$. Let $X=\bigcup_{n\in N}T_n(L^2_E)$ and $Y=(I-T_\infty)(L^2_E)$; then $X+Y$ is dense in L^2_E.

Let now $\tilde f\in X+Y$; then $\tilde f=\tilde u+\tilde v$ with $\tilde u\in X$ and $\tilde v\in Y$ and hence, for each $n\in N$, $T_n\tilde f=T_n\tilde u+T_n\tilde v$. If $\tilde u$ belongs to $T_p(L^2_E)$ then $T_n\tilde u=T_p\tilde u$ for $n\geqq p$; on the other hand, for each $n\in N$, $T_n\tilde v=T_nT_\infty\tilde v=0$. Hence $(T_nf(z))_{n\in N}$ converges almost everywhere to a limit for each $\tilde f\in X+Y$. Using the Banach theorem (see the Appendix) we deduce that $(T_nf(z))_{n\in N}$ converges almost everywhere for each $\tilde f\in L^2_E$. Now S_E is contained and dense in every L^q_E, $1\leqq q<\infty$; using once more the Banach theorem we deduce that $(T_nf(z))_{n\in N}$ converges, almost everywhere, for every $f\in\mathscr{V}$.

By an argument used in the proof of the Theorem in the Appendix we see that (the restriction to S_E of) T_∞ belongs to $\mathscr{D}$ and that for each $f\in\mathscr{V}$, $T_\infty f$ is equal almost everywhere to the limit of $(T_nf(z))_{n\in N}$.

Corollary 5 contains (when the space E is reflexive) a pointwise convergence theorem concerning the (abstract) increasing martingale. However if for each $n\in N$ the operator T_n is the conditional expectation with respect to $\mathscr{E}_n$, where

([5]) We apply Eberlein's ergodic theorem to the decreasing sequence of projections $(I-T_n)_{n\in N}$.

$(\mathscr{E}_n)_{n\in N}$ is an increasing sequence of tribes contained in $\mathscr{E}$ then the conclusion of Corollary 5 remains valid without supposing E reflexive (see the discussion following Corollary 4 above).

Under the hypotheses of Corollary 5 for each $f\in\mathscr{L}_E^p$, $1<p<\infty$, the sequence $(T_nf)_{n\in N}$ converges to $T_\infty f$ in $\mathscr{L}_E^p$ (use the fact that L_E^p is reflexive and [1] or Eberlein's ergodic theorem); moreover if $\mu(Z)$ is finite, then for each $f\in\mathscr{L}_E^1$, the sequence $(T_nf)_{n\in N}$ converges to $T_\infty f$ in $\mathscr{L}_E^1$. These results remain valid without supposing E reflexive in the case when the operators $(T_n)_{n\in N}$ are the conditional expectations corresponding to an increasing sequence of tribes contained in $\mathscr{E}$.

2. **Another maximal theorem.** Theorem 2 below gives again as immediate consequences a maximal ergodic theorem and a maximal theorem for martingales. Although the proofs of Theorem 1 and Theorem 2 are based on the same idea (see also [5; 6]), the proof of Theorem 2 is perhaps shorter. We wish to remark however that Theorem 1 gives (at least in certain cases) sharper direct estimates.

Let $(T_n)_{n\in N}$ be a sequence of operators belonging to $\mathscr{D}$; we suppose below that $T_0=I$. Let now $(a_{nj})_{n\in N, j\in\{0,\cdots,n\}}$ be a matrix of numbers such that $|a_{nj}|\leqq B$ for all $n\in N$ and $j\in\{0,\cdots,n\}$.

For every $j\in N$, $i\in N$ such that $j\geqq i$ we shall define (see the paragraph on notations and terminology) an operator $T_{(j,i)}$. We write $T_{(j,i)}=T_j$ if $j=i$. Suppose now that $j>i$ and that $T_{(j-1,i)}$ has been defined; we write then $T_{(j,i)}=T_{(j,j)}T_{(j-1,i)}$.

Consider now the condition

$$\left\| \sum_{i=1}^{n}\sum_{j=i}^{n} a_{nj}T_{(j,i)}\tilde{g}_i\right\|_\infty \leqq C\left\|\sum_{i=1}^{n}\|g_i\|\right\|_\infty \tag{18}$$

for each $n\geqq 1$ and $g_1,\cdots,g_n\in\mathscr{S}_E$ (for each $i\in\{1,\cdots,n\}$, $\|g_i\|$ is the mapping $z\to\|g_i(z)\|$).

For each $n\in N$ define

$$T^{(n)}=\sum_{j=0}^{n}a_{nj}T_{(j,0)}.$$

We may now state and prove the:

THEOREM 2. *Let (T_j), (a_{nj}) and $(T^{(n)})$ be as above and suppose that condition* (18) *is satisfied. For each $f\in\mathscr{V}$ and each $a>0$ define*

$$K_f^*(a)=\left\{z\;\Big|\;\sup_{n\in N}\|T^{(n)}f(z)\|>(C+B)a\right\}.$$

Then for each set $F\in\mathscr{E}$ verifying (except for sets of measure zero) the relations $G_f(a)\subset F\subset G_f(a)\cup K_f^*(a)$ we have

$$a\mu(F)\leqq\int_F\|f(z)\|\,d\mu(z)<\infty.$$

Let $f \in \mathscr{V}$ and $a > 0$. For each $n \in N$ define

$$A(n) = \{z \mid \| T^{(n)} f(z) \| > (C+B)a\};$$

obviously $\bigcup_{n \in N} A(n) = K_f^*(a)$.

We have (see the paragraph on notations and terminology)

$$T^{(0)} \tilde{f} = a_{00} \tilde{d}_0 + a_{00} \tilde{f}_0$$

and hence

$$(C+B)a < Ba + |a_{00}| \, \|f_0(z)\|$$

almost everywhere on $A(0)$; hence $\|f_0(z)\| \neq 0$ almost everywhere on $A(0)$. For $n \in N$, $n \geqq 1$ we have (we write $T_{(j,i)} = 0$ if $i > j \geqq 1$)

$$\begin{aligned} T^{(n)} \tilde{f} &= \sum_{j=0}^{n} a_{nj} T_{(j,0)} \tilde{f} = a_{n0} \tilde{f} + \sum_{j=1}^{n} a_{nj} T_{(j,1)} \tilde{f} \\ &= \sum_{j=1}^{n} a_{nj} \sum_{i=1}^{j} T_{(j,i)} \tilde{d}_{i-1} + \sum_{j=0}^{n} a_{nj} \tilde{d}_j + \sum_{j=0}^{n} a_{nj} \tilde{f}_j \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} a_{nj} T_{(j,i)} \tilde{d}_{i-1} + \sum_{j=0}^{n} a_{nj} \tilde{d}_j + \sum_{j=0}^{n} a_{nj} \tilde{f}_j \\ &= \sum_{i=1}^{n} \sum_{j=i}^{n} a_{nj} T_{(j,i)} \tilde{d}_{i-1} + \sum_{j=0}^{n} a_{nj} \tilde{d}_j + \sum_{j=0}^{n} a_{nj} \tilde{f}_j \end{aligned}$$

and hence, from (18) and (2)

$$(C+B)a < Ca + Ba + \sum_{j=0}^{n} |a_{nj}| \, \|f_j(z)\|$$

almost everywhere on $A(n)$; hence $\sum_{j=0}^{n} \|f_j(z)\| \neq 0$ almost everywhere on $A(n)$. Since $n \in N$ was arbitrary we obtain $\sum_{j \in N} \|f_j(z)\| \neq 0$ almost everywhere on $K_f^*(a)$. From (2) and (4) wededuce that $\|f_0(z)\| \neq 0$ on $G_f(a)$; hence $\sum_{j \in N} \|f_j(z)\| \neq 0$ almost everywhere on the set F.

For each $n \in N$ let

$$F_n = \left\{ z \,\middle|\, \sum_{j=0}^{n} \|f_j(z)\| \neq 0 \right\} \cap F.$$

Remark that the sequence $(F_n)_{n \in N}$ is increasing and that (except for sets of measure zero) $F = \bigcup_{n \in N} F_n$. By (6) we have, for each $n \in N$, $\sum_{j=0}^{n} \| d_j(z) \| = a$ almost everywhere on F_n; by (7) we have

$$\int_{F_n} (a - \| d_0(z) \|) \, d\mu(z) \leqq \int_Z \|f_0(z)\| \, d\mu(z) \tag{19}$$

for each $n \in N$. But (19) implies

$$\int_F (a - \| d_0(z) \|) d\mu(z) \leqq \int_Z \| f_0(z) \| \, d\mu(z)$$

and the desired conclusion follows from Proposition 1 above.

COROLLARY 6. (6.1) *If* $(T_j)_{j\in N}(T_0 = I)$ *is a decreasing sequence of projections belonging to* $\mathscr{D}$, $a_{nj} = 0$ *if* $j \neq n$ *and* $a_{nn} = 1$, *then condition* (18) *is verified with* $C = 1$. (6.2) *If* $T_j = T \in \mathscr{D}$ *for all* $j \neq 0$ *and* $a_{nj} = 1/(n+1)$ *for all* (n,j), *then condition* (18) *is verified with* $C = 1$.

PART II

1. An almost everywhere convergence theorem for increasing sequences of projections. Throughout this section we shall suppose the Banach space E *reflexive*.

Let $H \subset L_E^1$; we shall say that the set H is *uniformly integrable* if:

(20) for every $\varepsilon > 0$ there is a set $Z(\varepsilon) \in \mathscr{E}$ of finite measure such that

$$\int_{CZ(\varepsilon)} \| f(z) \| \, d\mu(z) \leqq \varepsilon \qquad \text{for all } \tilde{f} \in H;$$

(21) for each $\varepsilon > 0$ there is $\eta_\varepsilon > 0$ such that $A \in \mathscr{E}$ and $\mu(A) \leqq \eta_\varepsilon$ imply

$$\int_A \| f(z) \| \, d\mu(z) \leqq \varepsilon \qquad \text{for all } \tilde{f} \in H.$$

With this definition we may state the following:

PROPOSITION 2. *If* $H \subset L_E^1$ *is a uniformly integrable bounded set then* H *is weakly relatively compact.*

We shall not prove here this result. It can be obtained using for instance, the method of proof of Theorem 1 of [18, pp. 400–403].

We shall now state and prove:

THEOREM 3. *Let* $(T_n)_{n \in N}$ *be an increasing sequence of projections belonging to* $\mathscr{D}$ *and* $(f_n)_{n\in N}$ *a sequence of functions belonging to* $\mathscr{V}$. *Suppose that* $T_n \tilde{f}_{n+1} = \tilde{f}_n$ *for all* $n \in N$. (3.1) *Assume that* $f_n \in \mathscr{L}_E^1$ *for all* $n \in N$, $\sup_{n\in N} \| f_n \|_1$ *is finite and the set of elements of the sequence* $(\tilde{f}_n)_{n\in N}$ *is uniformly integrable. Then the sequence* $(f_n(z))_{n\in N}$ *converges almost everywhere to a function* $f_\infty \in \mathscr{L}_E^1$ *and* $T_n \tilde{f}_\infty = \tilde{f}_n$ *for all* $n \in N$. (3.2) *Assume that, for some* $1 < p < \infty$, $f_n \in \mathscr{L}_E^p$ *for all* $n \in N$ *and* $\sup_{n\in N} \| f_n \|_p$ *is finite. Then the sequence* $(f_n(z))_{n\in N}$ *converges almost everywhere to a function* $f_\infty \in \mathscr{L}_E^p$ *and* $T_n \tilde{f}_\infty = \tilde{f}_n$ *for all* $n \in N$.

Remark (as it can be easily seen by induction) that for all $n \in N$ and $m > n$ we have

$$T_n \tilde{f}_m = \tilde{f}_n. \tag{22}$$

Assume now that $f_n \in \mathscr{L}_E^1$ for all $n \in N$, $\sup_{n\in N} \| f_n \|_1$ is finite and the set of elements of the sequence $(\tilde{f}_n)_{n\in N}$ is uniformly integrable. By Proposition 2, the sequence $(\tilde{f}_n)_{n \in N}$ has a weak cluster point (for the topology $\sigma(L_E^1, (L_E^1)'))$ $\tilde{f}_\infty \in L_E^1$. Since each T_n, $n \in N$, is also weakly continuous (see [4, p. 103]) we deduce from (22) $T_n \tilde{f}_\infty = \tilde{f}_n$ for all $n \in N$. It follows then from Corollary 5, that the sequence $(f_n(z))_{n \in N}$ converges almost everywhere to the function (we use here the notations of Corollary 5) $T_\infty f_\infty \in \mathscr{L}_E^1$. (Since the elements of the sequence $(\tilde{f}_n)_{n \in N}$ are uniformly integrable it follows that $(f_n)_{n \in N}$ converges also in $\mathscr{L}_E^1$ to $T_\infty f_\infty$; whence $T_\infty \tilde{f}_\infty = \tilde{f}_\infty$.)

Assume that, for some $1 < p < \infty$, $f_n \in \mathscr{L}_E^p$ for all $n \in N$ and $\sup_{n \in N} \|f_n\|_p$ is finite. Since the space L_E^p is reflexive (see [34] or [24]), the sequence $(\tilde{f}_n)_{n \in N}$ has a weak cluster point (for the topology $\sigma(L_E^p, (L_E^p)'))$ $\tilde{f}_\infty \in L_E^p$. As above we deduce $T_n \tilde{f}_\infty = \tilde{f}_n$ for all $n \in N$. It follows then from Corollary 5 that the sequence $(f_n(z))_{n \in N}$ converges almost everywhere to the function (we use here the notations of Corollary 5) $T_\infty f_\infty \in \mathscr{L}_E^p$. (Using [1] or Eberlein's ergodic theorem (see foot-(note 5) we deduce that $(f_n)_{n \in N}$ converges in $\mathscr{L}_E^p$ to $T_\infty f_\infty$; whence $T_\infty \tilde{f}_\infty = \tilde{f}_\infty$.)

REMARKS. (1) As it was remarked in the proof of Theorem 3, the sequence $(\tilde{f}_n)_{n \in N}$ converges to $\tilde{f}_\infty$ also in L_E^1 or L_E^p respectively. (2) In a particular form Theorem 3 is obtained in [7; 36].

2. **Abstract measures on product spaces.** Let X be a set $\mathscr{F}$ a tribe of parts of X and $X^\infty = \prod_{s \in N} X_s$, where $X_s = X$ for all $s \in N$. For each $n \in N$, denote by X^n the product set $\prod_{s \in \{0, \cdots, n\}} X_s$ and by $\mathscr{F}^n$ the tribe spanned by the set of all parts of X^n of the form $\prod_{s \in \{0, \cdots, n\}} A_s$ where $A_s \in \mathscr{F}$ for all $s \in \{0, \ldots, n\}$.

For each $n \in N$ denote by $\mathrm{pr}_{\{0, \cdots, n\}}$ the projection of X^∞ onto X^n. It is obvious that, for every $n \in N$,

$$\mathrm{pr}_{\{0, \cdots, n\}}^{-1} \quad (\mathscr{F}^n) = \{\mathrm{pr}_{\{0, \cdots, n\}}^{-1} \quad (B) \mid B \in \mathscr{F}^n\}$$

is a tribe (namely the tribe spanned by the parts of X^∞ of the form $\prod_{s \in N} A_s$, where $A_s \in \mathscr{F}$ if $s \in \{0, \cdots, n\}$ and $A_s = X$ if $s \notin \{0, \cdots, n\}$).

Let $\mathscr{C} = \bigcup_{n \in N} \mathrm{pr}_{\{0, \cdots, n\}}^{-1} (\mathscr{F}^n)$; then $\mathscr{C}$ is a clan (=boolean algebra). Denote by $\mathscr{T}$ the tribe spanned by $\mathscr{C}$.

Consider now a mapping ϕ of $\mathscr{C}$ into R having the following two properties (for each $n \in N$ we denote by ϕ_n the restriction of ϕ to the tribe $\mathrm{pr}_{\{0, \cdots, n\}}^{-1} (\mathscr{F}^n)$):

(23) $$0 \leqq \phi(A) \leqq \phi(X^\infty) < \infty \qquad \text{for all } A \in \mathscr{C};$$

(24) For each $n \in N$, ϕ_n is countably additive.

We shall introduce now the following definition:

(25) *We say that the object* $\{X, \mathscr{F}\}$ *has the property* (CAE) *if every mapping* ϕ *of* $\mathscr{C}$ *into* R *having the properties* (23) *and* (24) *admits a countably additive extension to* $\mathscr{T}$.

We shall use below the following:

PROPOSITION 3. *If X is a Polish space (=metrizable, complete for some* *distance compatible with its topology and of countable type) and* $\mathscr{F}$ *the tribe of Borel parts of* X *then* $\{X,\mathscr{F}\}$ *has the property* (CAE).

For a proof of this known result see for instance [23; 31] (see also [3; 28]).

Let now E be a *Banach space.* Let ψ be a mapping of $\mathscr{C}$ into E (as above we denote with ψ_n the restriction of ψ to the tribe $\overset{-1}{\mathrm{pr}}_{\{0,\dots n\}}(\mathscr{F}^n)$, for each $n \in N$). Consider now the following conditions concerning the mapping ψ:

(26) ψ_n is countably additive, for each $n \in N$;

(27) ψ_n has finite total variation $\mathfrak{v}_n$, for each $n \in N$ (see the Remark below);

(28) $\sup_{n \in N} \mathfrak{v}_n(A)$ is finite for each $A \in \mathscr{C}$ (here the supremum is taken over all $n \in N$ such that $A \in \overset{-1}{\mathrm{pr}}_{\{0,\dots n\}}(\mathscr{F}^n)$).

Remark. If $\mathscr{U}$ is a tribe and h a mapping of $\mathscr{U}$ into a Banach space E then the *total variation* $\mathfrak{v}$ of h is defined (on $\mathscr{U}$) as follows: for $A \in \mathscr{U}$ we write $\mathfrak{v}(A) = \sup \sum \| h(A_i) \|$, where the supremum is taken over all finite families (A_i) of disjoint parts of A belonging to $\mathscr{U}$. We say that h has *finite total variation* if $\mathfrak{v}(A)$ is finite for all $A \in \mathscr{U}$. If h is countably additive and has finite total variation then $\mathfrak{v}$ is also *countably additive.*

We shall prove now:

Proposition 4. *Suppose that* $\{X,\mathscr{F}\}$ *has the property* (CAE) *and let* ψ *be a mapping of* $\mathscr{C}$ *into a Banach space E having the properties* (26), (27) *and* (28). *Then* ψ *has a (unique) countably additive extension to* $\mathscr{T}$ *and this extension has finite total variation.*

Let $A \in \mathscr{C}$. There is then $p \in N$ such that $A \in \overset{-1}{\mathrm{pr}}_{\{0,\dots,p\}}(\mathscr{F}^p)$; hence $\mathfrak{v}_n(A)$ is defined for all $n \geqq p$. Remark that the sequence $(\mathfrak{v}_n(A))_{n \geqq p}$ is increasing and bounded and define

$$\mathfrak{v}^\infty(A) = \lim_{n \to \infty, n \geqq p} \mathfrak{v}_n(A). \tag{29}$$

It is obvious that $\mathfrak{v}^\infty(A)$ does not depend on p and that $\mathfrak{v}^\infty(A) \geqq \| \psi(A) \|$. Also $\mathfrak{v}_n^\infty$ (= the restriction of $\mathfrak{v}^\infty$ to the tribe $\overset{-1}{\mathrm{pr}}_{\{0,\dots,n\}}(\mathscr{F}^n)$) is a countably additive mapping [19, p. 170].

Since $\mathfrak{v}^\infty(X^\infty)$ is finite $\mathfrak{v}^\infty$ has, by (CAE), a (unique) countably additive extension to $\mathscr{T}$; we shall denote this extension by the same letter[6]. Define now a semi-distance d on $\mathscr{T}$ by the equations $d(A,B) = \mathfrak{v}^\infty((A - B) \cup (B - A))$ for all A, $B \in \mathscr{T}$; for the topology defined by this semi-distance $\mathscr{C}$ is dense in $\mathscr{T}$ [19, p. 56–57]. For $A \in \mathscr{C}$, $B \in \mathscr{C}$ we have $\| \psi(A) - \psi(B) \| \leqq \| \psi(A - A \cap B) \| + \| \psi(B - A \cap B) \| \leqq \mathfrak{v}^\infty(A - A \cap B) + \mathfrak{v}^\infty(B - A \cap B) = d(A,B)$. It follows that ψ can be extended by continuity to $\mathscr{T}$; denote its extension to $\mathscr{T}$ by the same letter. The extension ψ verifies again the relation

(6) Once $\mathfrak{v}^\infty$ is extended to $\mathscr{T}$, the final conclusion of the proposition can be easily obtained (use for instance [32] or [33]). However, for completeness we give a direct argument below.

$$\|\psi(A) - \psi(B)\| \leqq d(A,B) \tag{30}$$

for all $A \in \mathscr{T}$, $B \in \mathscr{T}$. In particular (let $B = \emptyset$ in formula (30)) $\|\psi(A)\| \leqq \mathfrak{v}^\infty(A)$ for all $A \in \mathscr{T}$. Thus to complete the proof of our proposition it is enough to show that ψ is additive on $\mathscr{T}$. Since the "intersection" and the "complement" are continuous functions [19, p. 168] and since $\mathscr{C}$ is dense in $\mathscr{T}$, this follows from the additivity of the restriction of ψ to $\mathscr{C}$. Hence the proposition is completely proved.

3. **The almost everywhere convergence theorem for abstract increasing martingales.** Let E be a Banach space of *countable type, dual of a Banach space.* We shall take below $X = E$ and $\mathscr{F}$ = the tribe of all Borel parts of E. For each $n \in N$, we shall denote by pr_n the projection of X^∞ onto $X_n = E$. With these notations we may state and prove the following:

PROPOSITION 5. *Let α be a positive measure on $\mathscr{T}$ of finite total mass. Suppose that:* (5.1) $\mathrm{pr}_n \in \mathscr{L}^1_E(X^\infty, \mathscr{T}, \alpha)$ *for each* $n \in N$; (5.2) $(\mathrm{pr}_n)_{n \in N}$ *is an increasing martingale with respect to the sequence of tribes* $(\overset{-1}{\mathrm{pr}}_{\{0,\cdots,n\}}(\mathscr{F}^n))_{n \in N}$; (5.3) $\sup_{n \in N} \|\mathrm{pr}_n\|_1$ *is finite. Then* $(\mathrm{pr}_n)_{n \in N}$ *converges almost everywhere.*

For each $n \in N$ and $A \in \overset{-1}{\mathrm{pr}}_{\{0,\cdots,n\}}(\mathscr{F}^n)$ define

$$\psi_n(A) = \int_A \mathrm{pr}_n(x)d\alpha(x);$$

then ψ_n is a countably additive mapping of $\overset{-1}{\mathrm{pr}}_{\{0,\cdots,n\}}(\mathscr{F}^n)$ into E. It is obvious that ψ_n has finite total variation $\mathfrak{v}_n$ and that $\mathfrak{v}_n(A) \leqq \sup_{m \in N}\|\mathrm{pr}_m\|_1$ for all $A \in \overset{-1}{\mathrm{pr}}_{\{0,\cdots,n\}}(\mathscr{F}^n)$.

For every $n \in N$ and $A \in \overset{-1}{\mathrm{pr}}_{\{0,\cdots,n\}}(\mathscr{F}^n)$ we have

$$\psi_{n+1}(A) = \int_A \mathrm{pr}_{n+1}(x)d\alpha(x) = \int_A \mathrm{pr}_n(x)d\alpha(x) = \psi_n(A)$$

(we use (5.2)); whence ψ_{n+1} is an extension of ψ_n to $\overset{-1}{\mathrm{pr}}_{\{0,\cdots,n+1\}}(\mathscr{F}^{n+1})$. We may now define a mapping ψ of $\mathscr{C}$ into E as follows: for $A \in \overset{-1}{\mathrm{pr}}_{\{0,\cdots,n\}}(\mathscr{F}^n)$ we write $\psi(A) = \psi_n(A)$. It is easy to verify that ψ_n is the restriction of ψ to $\overset{-1}{\mathrm{pr}}_{\{0\cdots n\}}(\mathscr{F}^n)$, for each $n \in N$, and that ψ has the properties (26), (27), (28). By Propositions 3 and 4 ψ has a (unique) countably additive extension to $\mathscr{T}$ (we shall denote this extension by the same letter); moreover this extension has finite total variation, $\mathfrak{v}^\infty$.

Define $\gamma = \alpha + \mathfrak{v}^\infty$; for each $n \in N$ denote by γ_n, α_n the restrictions of γ, α to $\overset{-1}{\mathrm{pr}}_{\{0,\cdots,n\}}(\mathscr{F}^n)$ respectively.

Denote by $\mathfrak{c}$ the density of ψ with respect to $\gamma : \psi = \mathfrak{c} \cdot \gamma$; for every $n \in N$ denote by $\mathfrak{c}_n$ the density of ψ_n with respect to $\gamma_n : \psi_n = \mathfrak{c}_n \cdot \gamma_n$ (see [24]). By Corollary 5 (see also the remarks following the proof of Corollary 5) $(\mathfrak{c}_n)_{n \in N}$ converges almost everywhere (with respect to γ) as well as in $\mathscr{L}^1_E(X^\infty, \mathscr{T}, \gamma)$ to a limit $\mathfrak{c}_\infty$; whence we may suppose that $\mathfrak{c}(x) = \mathfrak{c}_\infty(x)$ for each $x \in X^\infty$.

Denote by $\mathfrak{a}$ the density of α with respect to $\gamma : \alpha = \mathfrak{a} \cdot \gamma$; for every $n \in N$ denote by $\mathfrak{a}_n$ the density of α_n with respect to $\gamma_n : \alpha_n = \mathfrak{a}_n \cdot \gamma_n$. As above we see that $(\mathfrak{a}_n(x))_{n \in N}$ converges almost everywhere (with respect to γ) to $\mathfrak{a}(x)$. Remark here that if $B = \{x \mid \mathfrak{a}(x) = 0\}$ then $\alpha(B) = \int_B \mathfrak{a}(x) d\gamma(x) = 0$; whence $\mathfrak{a}(x) > 0$ almost everywhere with respect to α.

For every $n \in N$ and $A \in \mathrm{pr}^{-1}_{\{0,\cdots,n\}}(\mathscr{F}^n)$ we have

$$\psi_n(A) = \int_A \mathrm{pr}_n(x) d\alpha_n(x) = \int_A \mathrm{pr}_n(x)\mathfrak{a}_n(x) d\gamma_n(x);$$

we deduce $\mathrm{pr}_n(x)\mathfrak{a}_n(x) = \mathfrak{c}_n(x)$ almost everywhere with respect to γ_n and hence almost everywhere with respect to γ; since $\gamma \geqq \alpha$ we deduce that $\mathrm{pr}_n(x)\mathfrak{a}_n(x) = \mathfrak{c}_n(x)$ almost everywhere with respect to α. Since $(\mathfrak{c}_n(x))_{n \in N}$ converges almost everywhere, with respect to α, and $(\mathfrak{a}_n(x))_{n \in N}$ converges almost everywhere, with respect to α, to a strictly positive function $\mathfrak{a}(x)$, it follows that $(\mathrm{pr}_n(x))_{n \in N}$ converges almost everywhere, with respect to α. Hence the proposition is completely proved.

REMARK. The assumption that the Banach space E is of countable type, dual of a Banach space was used in fact in the proof of Proposition 5 only to enable us to apply Proposition 4 and the following result: Every countably additive mapping of $\mathscr{T}$ into E with finite total variation has a Bochner measurable density with respect to its total variation.

We return now to our measure space $(Z, \mathscr{E}, \mu)$ introduced in the section on notations and terminology. We denote by E a Banach space which is either *reflexive* or of *countable type* and *dual of a Banach space*. Let $(\mathscr{E}_n)_{n \in N}$ be an increasing sequence of tribes contained in $\mathscr{E}$ and let $\mathscr{E}_\infty$ be the tribe spanned by their union.

THEOREM 4. *Let $(f_n)_{n \in N}$ be a sequence of functions belonging to $\mathscr{L}^1_E$. Suppose that*: (4.1) *$(f_n)_{n \in N}$ is a martingale with respect to $(\mathscr{E}_n)_{n \in N}$*; (4.2) *$\sup_{n \in N} \|f_n\|_1$ is finite. Then there is a function $f_\infty \in \mathscr{L}^1_E$ (measurable with respect to $\mathscr{E}_\infty$) such that $(f_n(z))_{n \in N}$ converges almost everywhere to $f_\infty(z)$.*

Let us remark that the measurability of f_∞ with respect to $\mathscr{E}_\infty$ is a consequence of the almost everywhere convergence of the sequence $(f_n(z))_{n \in N}$. The relation $f_\infty \in \mathscr{L}^1_E$ follows from (4.2) and Fatou's lemma. Also it is obviously enough to prove the theorem when E is of countable type and dual of a Banach space. Finally we may suppose that $\mu(Z)$ is finite.

Consider the mapping $g : z \to (f_n(z))_{n \in N}$ of Z into X^∞ (we take $X = E$ and $\mathscr{F}$ = the tribe of all Borel parts of E). Since f_n is, for each $n \in N$, Bochner measurable it follows that $g^{-1}(A) \in \mathscr{E}$ for all $A \in \mathscr{T}$. For every $A \in \mathscr{T}$ write $\alpha(A) = \mu(g^{-1}(A))$; then α is a well defined positive measure on $\mathscr{T}$ of finite tota mass.

Let now $n \in N$. For each $x' \in E'$ ($=$ the dual of E) the function $x' \circ \mathrm{pr}_n$ is obviously measurable with respect to $\mathrm{pr}^{-1}_{\{0,\dots,n\}}(\mathscr{F}^n)$; since E is of countable type we deduce that pr_n is Bochner measurable with respect to $\mathrm{pr}^{-1}_{\{0,\dots,n\}}(\mathscr{F}^n)$. On the other hand $\|\mathrm{pr}_n\| \circ g(z) = \|\mathrm{pr}_n \circ g(z)\| = \|f_n(z)\|$ for every $z \in Z$; whence $\mathrm{pr}_n \in \mathscr{L}^1_E(X^\infty, \mathscr{T}, \alpha)$ and

$$\int_X \|\mathrm{pr}_n(x)\| \, d\alpha(x) = \int_Z \|f_n(z)\| \, d\mu(z).$$

Since $n \in N$ was arbitrary it follows that the sequence $(\mathrm{pr}_n)_{n \in N}$ verifies the conditions (5.1) and (5.3) of Proposition 5.

We shall show now that (5.2) is also verified. Let $n \in N$ and $A \in \mathrm{pr}^{-1}_{\{0,\dots,n\}}(\mathscr{F}^n)$. Then(7) $g^{-1}(A) \in \mathscr{E}_n$ and

$$\int_A \mathrm{pr}_{n+1}(x) d\alpha(x) = \int_{g^{-1}(A)} \mathrm{pr}_{n+1}(g(z)) d\mu(z) = \int_{g^{-1}(A)} f_{n+1}(z) d\mu(z)$$
$$= \int_{g^{-1}(A)} f_n(z) d\mu(z) = \int_{g^{-1}(A)} \mathrm{pr}_n(g(z)) d\mu(z) = \int_A \mathrm{pr}_n(x) d\alpha(x);$$

whence $(\mathrm{pr}_n)_{n \in N}$ is a martingale with respect to the sequence $(\mathrm{pr}^{-1}_{\{0,\dots,n\}}(\mathscr{F}^n))_{n \in N}$ of tribes.

By Proposition 5 there is a set $\mathscr{N} \in \mathscr{T}$ such that $\alpha(\mathscr{N}) = 0$ and $(\mathrm{pr}_n(x))_{n \in N}$ converges for $x \notin \mathscr{N}$. It follows that $\mu(g^{-1}(\mathscr{N})) = 0$ and that $(f_n(z))_{n \in N}$ converges for each $z \notin g^{-1}(\mathscr{N})$. Hence the theorem is completely proved.

Remarks. (1) In [7; 36] it is proved, under the hypothesis that E is reflexive, that the sequence $(f_n(z))_{n \in N}$ of Theorem 4 converges weakly almost everywhere; in [12] it is proved that the sequence $(f_n(z))_{n \in N}$ converges weakly almost everywhere if, for almost every $z \in Z$, the set $\{f_n(z) \mid n \in N\}$ is weakly relatively compact. The proof of Theorem 4 is suggested by an argument in [26, p. 72–73]. (2) Theorem 4 together with the counter-example constructed in [7] show that the Banach space $L^1_R([0,1], \mathscr{L}, \mu)$ (here $\mathscr{L}$ is the tribe of Lebesgue measurable parts of $[0,1]$ and μ is the Lebesgue measure on $\mathscr{L}$) is not isomorphic, as a Banach space, to a dual of a Banach space (see [17; 9]). (3) Let us also remark that Theorem 4 holds whenever the Banach space E is such that Proposition 5 is true.

Appendix

Let $\mathscr{H} \subset \mathscr{D}$ be a set of operators and let $\mathscr{H}^*$ be the smallest semi-group (for multiplication) containing $\mathscr{H}$ and I. Let A be a *countable* directed set and $(T_\alpha)_{\alpha \in A}$ a family of operators belonging to $\mathscr{D}$. We have then the following theorem, analogous for instance to a result in [38]. For completeness, we sketch here a proof based on the results in [16].

Theorem. *Suppose that*: (1) *E is reflexive*; (2) *$(T_\alpha)_{\alpha \in A}$ is a system of almost invariant integrals for $\mathscr{H}^*$ in the normed space S_E as well as in some space*

(7) We may suppose that, for each $n \in N, f_n$ is measurable with respect to $\mathscr{E}_n \subset \mathscr{E}$.

L_E^p with $1 < p < \infty$; (3) $\sup_{\alpha \in A} \| T_\alpha f(z) \|$ is finite almost everywhere, for each $f \in \mathscr{V}$. There exists then a projection $T_\infty \in \mathscr{D}$ such that, for each $f \in \mathscr{V}$, the family $(T_\alpha f(z))_{\alpha \in A}$ converges almost everywhere to $T_\infty f(z)$.

Denote by X the closed vector space of all $\tilde{f} \in L_E^p$ such that $S\tilde{f} = \tilde{f}$ for all $S \in \mathscr{H}^*$ and by Y the closed vector space spanned by $\bigcup_{S \in \mathscr{H}^*}(I - S)(L_E^p)$. Since L_E^p is reflexive (see [34] or [24]) it follows that L_E^p is the (topological) direct sum of X and Y (see [8]). Now let Y_0 be the vector space spanned by $\bigcup_{S \in \mathscr{H}^*} (I - S)(S_E)$; obviously $Z = X + Y_0$ is dense in L_E^p.

Let $\tilde{f} \in Z$; then $\tilde{f} = \tilde{u} + \tilde{v}$ with $\tilde{u} \in X$ and $\tilde{v} \in Y_0$. Then for each $\alpha \in A$, $T_\alpha \tilde{f} = T_\alpha \tilde{u} + T_\alpha \tilde{v} = \tilde{u} + T_\alpha \tilde{v}$. Now $\tilde{v} = \sum_{i=1}^n c_i(I - S_i)\tilde{w}_i$ for some constants $c_1, \cdots, c_n$, $S_1, \cdots, S_n \in \mathscr{H}^*$ and $\tilde{w}_1, \cdots, \tilde{w}_n \in S_E$. Since $(T_\alpha)_{\alpha \in A}$ is a system of almost invariant integrals for $\mathscr{H}^*$, in the normed space S_E, it follows that $(T_\alpha f(z))_{\alpha \in A}$ converges almost everywhere. Using the Banach theorem (see the remark below) we deduce that $(T_\alpha f(z))_{\alpha \in A}$ converges almost everywhere for each $\tilde{f} \in L_E^p$. Now S_E is contained and dense in every L_E^q, $1 \leqq q < \infty$; using once more the Banach theorem we deduce that $(T_\alpha f(z))_{\alpha \in A}$ converges, almost everywhere, for every $f \in \mathscr{V}$.

For each $\tilde{f} \in S_E$ denote by $T_\infty f$ the (almost everywhere) limit of $(T_\alpha f(z))_{\alpha \in A}$. It is easy to see that $T_\infty \in \mathscr{D}$ (use Fatou's lemma) and that for each $f \in \mathscr{V}$, $T_\infty f$ is equal (almost everywhere) to the limit of $(T_\alpha f(z))_{\alpha \in A}$. By direct computation, or using [16], we see that T_∞ is a projection.

REMARKS. (1) It is obvious that if $f \in \mathscr{L}_E^p$ the family $(T_\alpha f)_{\alpha \in A}$ converges to $T_\infty f$ in $\mathscr{L}_E^p$. If $\mu(Z)$ is finite, we deduce that for each $f \in \mathscr{L}_E^q$, $1 \leqq q < \infty$, $(T_\alpha f)_{\alpha \in A}$ converges to $T_\infty f$ in $\mathscr{L}_E^q$. (2) The Banach theorem, in the form needed here, may be proved exactly as in [14, pp. 332–334].

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