

AN ISOPERIMETRIC INEQUALITY FOR MOMENTS OF INERTIA OF PLANE CONVEX SETS

BY
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1. **Introduction.** Let \mathbf{K} be the class of all closed plane convex sets K with given area v and C_K be the centroid of K . If the line $l(P, \theta)$ lies in the plane of K through the point P and with direction θ , $0 \leq \theta \leq 2\pi$, we denote the moment of inertia of K about the line $l(P, \theta)$ by $I(K, P, \theta)$. The maximum moment of inertia $I_M(K, P)$ and the minimum moment of inertia $I_m(K, P)$ about the point P are defined by:

$$I_M(K, P) = \max_{\theta} I(K, P, \theta) = I(K, P, \theta_M),$$

$$I_m(K, P) = \min_{\theta} I(K, P, \theta) = I(K, P, \theta_m),$$

where θ_M and θ_m are called the principal directions of the moments of inertia of K with respect to the point P . It is known [1] that $I_M(K, P)$ and $I_m(K, P)$ exist for a given K and a fixed point P , and $|\theta_M - \theta_m| = \pi/2$.

The isoperimetric inequality to be established in this paper arises from the problem proposed by Truesdell and formulated by Keller, namely: find the shape of the strongest column with convex normal sections and with a given volume and height. To solve this extremal problem it is necessary to solve the following isoperimetric problem first: determine the shape of the convex set $K_{\Delta} \in \mathbf{K}$ such that

$$I_m(K_{\Delta}, C_{K_{\Delta}}) = \max_{\mathbf{K}} I_m(K, C_K) = \max_{\mathbf{K}} \min_{\theta} I(K, C_K, \theta).$$

This problem was investigated previously by Keller and others [2]. It was conjectured by Keller and Ungar that the solution figure is an equilateral triangle $K_{\Delta} \in \mathbf{K}$. Keller also pointed out that "there is little doubt" about the correctness of their conjecture and he used it to solve Truesdell's problem [2]. It is the purpose of this paper to give a proof of the conjecture. We now state it as

THEOREM. *If $K \in \mathbf{K}$ then*

$$I_m(K, C_K) = \min_{\theta} I(K, C_K, \theta) \leq \frac{v^2}{6\sqrt{3}}.$$

The equality sign holds, if, and only if, K is an equilateral triangle $K_{\Delta} \in \mathbf{K}$.

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As $I_m(K, C_K)$ is small if the diameter of K is too large, we may consider only the sets of \mathbf{K} with diameter less than a fixed number. Hence, the existence of solution follows from the Blaschke's compactness theorem [3]. The difficulty in proving our theorem is that convexity is a necessary condition for the existence of solution, but it is difficult to give it an analytic characterization so that Euler's variational technique would become applicable. Accordingly, the present proof is carried out by direct comparison of $I_m(K_\Delta, C_{K_\Delta})$ with $I_m(K, C_K)$ for $K \in \mathbf{K}$. Therefore, this proof does not depend upon the existence proof of solution. It should be mentioned again that without loss of generality we shall restrict \mathbf{K} to be the class of closed bounded plane convex sets with given area v .

2. Moment of inertia and area-preserving affine transformation. Since an affine transformation is linear, it preserves convexity. As we shall see, the area-preserving affine transformation will play an important role in this proof. First, we prove the following lemma which is simple and essential to further developments.

LEMMA 1. *If A is nonsingular affine transformation, then $A(C_K) = C_{A(K)}$. That is, the centroid is affinely invariant.*

Proof. Let the matrix of A relative to a fixed rectangular Cartesian coordinate system be

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}.$$

If (x_{C_K}, y_{C_K}) are the coordinates of C_K , then the coordinates of $A(C_K)$ are given by:

$$(1) \quad \hat{x}_{C_K} = a_1 x_{C_K} + b_1 y_{C_K}, \quad \hat{y}_{C_K} = a_2 x_{C_K} + b_2 y_{C_K}.$$

On the other hand, the coordinates of $C_{A(K)}$ are defined by:

$$(2) \quad \begin{aligned} \hat{x}_{C_{A(K)}} &= \frac{1}{V(A(K))} \iint_{A(K)} \hat{x} d\hat{x} d\hat{y} = \frac{1}{V(K)} \iint_K (a_1 x + b_1 y) dx dy \\ &= a_1 x_{C_K} + b_1 y_{C_K}, \end{aligned}$$

$$\hat{y}_{C_{A(K)}} = \frac{1}{V(A(K))} \iint_{A(K)} \hat{y} d\hat{x} d\hat{y} = a_2 x_{C_K} + b_2 y_{C_K},$$

where $V(A(K))$ and $V(K)$ denote the area of $A(K)$ and K respectively. Now (1) and (2) indicate that $A(C_K)$ and $C_{A(K)}$ coincide and the lemma is proved.

To see the effect of an area-preserving affine transformation A upon the moments of inertia $I(K, C_K, \theta)$ of a given set $K \in \mathbf{K}$, we prove the following two lemmas:

LEMMA 2. *If $K \in \mathbf{K}$ and $I_M(K, C_K) > I_m(K, C_K)$, then there exists a set $A(K) \in \mathbf{K}$ such that*

$$I_m(A(K), C_{A(K)}) > I_m(K, C_K).$$

Proof. Choose C_K as the origin of a rectangular Cartesian coordinate system and choose the x - and y -axes coincided with the principal directions of $I(K, C_K, \theta)$ so that

$$I_m(K, C_K) = \int_K \int x^2 dx dy, \quad I_M(K, C_K) = \int_K \int y^2 dx dy, \quad \int_K \int xy dx dy = 0.$$

Let

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

be the matrix of an area-preserving affine transformation A relative to the fixed coordinate system. It follows from Lemma 1 that

$$\begin{aligned} (3) \quad I(A(K), C_{A(K)}, 0) &= \int_{A(K)} \int \hat{y}^2 d\hat{x} d\hat{y} = \int_K \int (a_2 x + b_2 y)^2 dx dy \\ &= a_2^2 I_m(K, C_K) + b_2^2 I_M(K, C_K), \end{aligned}$$

$$\begin{aligned} (4) \quad I\left((AK), C_{A(K)}, \frac{\pi}{2}\right) &= \int_{A(K)} \int \hat{x}^2 d\hat{x} d\hat{y} = \int_K \int (a_1 x + b_1 y)^2 dx dy \\ &= a_1^2 I_m(K, C_K) + b_1^2 I_M(K, C_K), \end{aligned}$$

and

$$(5) \quad \int_{A(K)} \int \hat{x} \hat{y} d\hat{x} d\hat{y} = a_1 a_2 I_m(K, C_K) + b_1 b_2 I_M(K, C_K).$$

So far A is only restricted to be area-preserving, i.e.,

$$(6) \quad \det(A) = -b_1 a_2 + a_1 b_2 = 1.$$

We may, therefore, require that

$$(7) \quad a_1 a_2 I_m(K, C_K) + b_1 b_2 I_M(K, C_K) = 0.$$

That is, the x - and y -axes of the same coordinate system are also the principal directions of $I(A(K), C_{A(K)}, \theta)$. From (6) and (7), we have

$$(8) \quad a_2 = \frac{-b_1 \beta}{a_1^2 + b_1^2 \beta}, \quad b_2 = \frac{a_1}{a_1^2 + b_1^2 \beta},$$

where

$$(9) \quad \beta \equiv I_M(K, C_K)/I_m(K, C_K) \equiv 1 + \varepsilon, \quad \varepsilon > 0.$$

It may be noted that $\beta > 1$ and the denominator in (8) does not vanish, if

$$a_1^2 + b_1^2 \neq 0.$$

From (3), (4) and (8), we see that the affine image $A(K)$ of K will be the set $A(K)$ to be exhibited if, and only if, the following two inequalities are satisfied:

$$(10) \quad \begin{aligned} a_1^2 + b_1^2\beta &> 1, \\ b_1^2\beta^2 + a_1^2\beta &> (a_1^2 + b_1^2\beta)^2, \quad \text{or } \beta > a_1^2 + b_1^2\beta. \end{aligned}$$

That is, if

$$(11) \quad 1 + \varepsilon > a_1^2 + b_1^2 + b_1^2\varepsilon > 1,$$

where we have made use of (9) and the fact $a_1^2 + b_1^2\beta > 0$. Now if we put

$$a_1^2 + b_1^2 = 1, \quad 0 < b_1^2 < 1,$$

then the inequalities in (11) are automatically satisfied and hence our lemma is proved.

From now on we denote by $J(K, P)$ the polar moment of inertia of K about the line through the point P and perpendicular to the plane of K . Also we note that

$$(12) \quad J(K, P) = I(K, P, \theta) + I\left(K, P, \theta + \frac{\pi}{2}\right), \quad 0 \leq \theta \leq 2\pi.$$

LEMMA 3. *If $K \in \mathbf{K}$, $I_m(K, C_K) = I_M(K, C_K) = I$, then for every area-preserving affine transformation A with C_K as fixed point,*

$$J(A(K), C_{A(K)}) \geq J(K, C_K).$$

Proof. Let

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

be the matrix of A relative to a fixed coordinate system with C_K as origin. Proceeding precisely in the same way as what we have done in the previous proof, we obtain

$$(13) \quad \begin{aligned} -b_1a_2 + a_1b_2 &= 1, \\ I(A(K), C_{A(K)}, 0) &= (a_2^2 + b_2^2)I, \\ I\left(A(K), C_{A(K)}, \frac{\pi}{2}\right) &= (a_1^2 + b_1^2)I. \end{aligned}$$

It is geometrically clear that $J(K, P)$ is invariant under any rigid rotation about the line through the point P and perpendicular to the plane of K . Furthermore

$$J(K, P) = I_m(K, P) + I_M(K, P).$$

Thus to evaluate $J(A(K), C_{A(K)})$, we may assume that the x - and y -axes are also the principal axes of $I(A(K), C_{A(K)}, \theta)$. Hence, by (5)

(14) $a_1 a_2 + b_1 b_2 = 0,$

$$I_m(A(K), C_{A(K)}) = (a_2^2 + b_2^2)I,$$

(15) $I_M(A(K), C_{A(K)}) = (a_1^2 + b_1^2)I.$

Solving the first equation in (13) and the equation (14) for a_2 and b_2 , inserting the resulting expressions for a_2 and b_2 into the first equation in (15) and then adding the corresponding sides of the equations in (15), we obtain

$$J(A(K), C_{A(K)}) = \left(\frac{1}{a_1^2 + b_1^2} + a_1^2 + b_1^2 \right) J(K, C_K)/2.$$

This shows that

$$J(A(K), C_{A(K)}) \geq J(K, C_K),$$

and the lemma is proved. It may be noted that the equality sign holds, if, and only if, A is a rigid rotation about C_K . We shall also use this fact in proving the theorem.

3. **An extremal problem in polar moments of inertia.** In proving the theorem we shall use Lemma 6 heavily. However, to prove Lemma 6 the following two lemmas are going to be used.

LEMMA 4. Let $(-s, 0), (b, h), (c, h), (s, 0)$ with $s > 0, c > b, h > 0$ be respectively the coordinates of the vertices A, B, C, D of a quadrilateral. If the point E with coordinates (ξ, h) and the point \hat{P}_s with coordinates (α, β) are given and if $c > \xi > b, \beta \leq 0$, then either

(16) $J(\triangle ABD, \hat{P}_s) > J(\triangle AED, \hat{P}_s),$ or
 $J(\triangle ACD, \hat{P}_s) > J(\triangle AED, \hat{P}_s).$

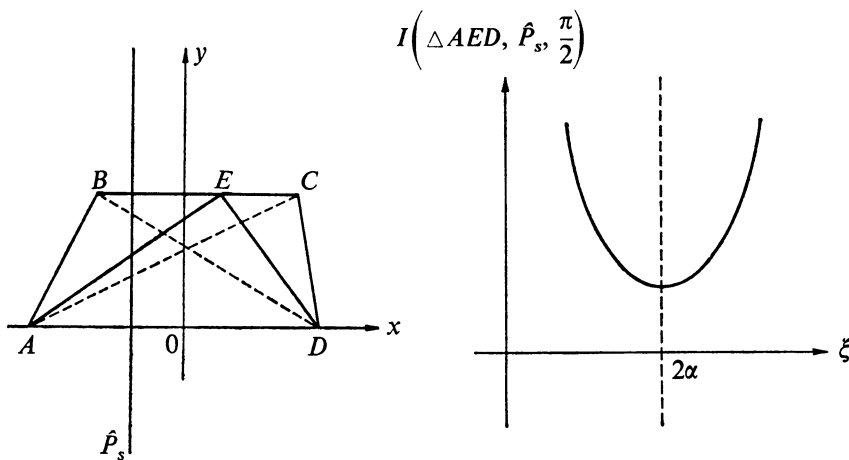


FIGURE 1

Proof. First, we compare the moments of inertia of the three triangles about the line through the point \hat{P}_s , and perpendicular to AD . To this end we consider how $I(\triangle AED, \hat{P}_s, \pi/2)$ varies with the abscissa ξ of the point E when E moves along the segment BC . To simplify computations we are going to use the following known results [1]: If a particle whose mass is one-third that of the triangle be placed at the middle of each side, the moment of inertia of the triangle about any line is the same as that of the system of the three particles. Clearly, the areas of the three triangles, $\triangle AED$, $\triangle ABD$, and $\triangle ACD$, are all equal. Without loss of generality, we may assume the mass of each triangle to be 3. Thus

$$I\left(\triangle AED, \hat{P}_s, \frac{\pi}{2}\right) = \alpha^2 + \left(\alpha - \frac{\xi - s}{2}\right)^2 + \left(\alpha - \frac{\xi + s}{2}\right)^2.$$

That is, $I(\triangle AED, \hat{P}_s, \pi/2)$ as a function of ξ for a fixed value of α is a concave-upward parabola. It achieves minimum at $\xi = 2\alpha$. Consequently, on any closed interval of ξ its maximum is at the end point of the interval, so that we always have either

$$(17) \quad \begin{aligned} I\left(\triangle ABD, \hat{P}_s, \frac{\pi}{2}\right) &> I\left(\triangle AED, \hat{P}_s, \frac{\pi}{2}\right), \text{ or} \\ I\left(\triangle ACD, \hat{P}_s, \frac{\pi}{2}\right) &> I\left(\triangle AED, \hat{P}_s, \frac{\pi}{2}\right). \end{aligned}$$

Since the segments AD and BC are parallel,

$$I(\triangle AED, A, 0) = I(\triangle ABD, A, 0) = I(\triangle ACD, A, 0).$$

Moreover, the centroids of $\triangle AED$, $\triangle ABD$, $\triangle ACD$ are collinear and they are on the line parallel to the segment AD . It follows from the ‘‘parallel-axis’’ theorem [1] that

$$(18) \quad I(\triangle ABD, \hat{P}_s, 0) = I(\triangle ACD, \hat{P}_s, 0) = I(\triangle AED, \hat{P}_s, 0).$$

Now (16) is a direct consequence of (17), (18) and (12). The lemma is now proved.

Lemma 5 is an approximation theorem of a special convex set by convex polygons, but the approximation polygons have to satisfy certain constraints. Since a theorem precisely of this form is not available, we state it as a lemma and omit the proof which is similar to those as given in [3]. In what follows, we use ∂K to denote the frontier of K and $V(K)$ the area of K . If A, B are two points or one of them is a point, we use $\rho(A, B)$ to denote the usual distance between A and B . If K is a set, we use $U(K, \delta)$ to denote the set:

$$U(K, \delta) = \{X | \rho(X, K) < \delta\}.$$

Let K_1, K_2 be two convex sets. Let δ_1 be the lower bound of the positive numbers δ such that $U(K_1, \delta) \supset K_2$ and δ_2 be the lower bound of the positive numbers δ

such that $U(K_2, \delta) \supset K_1$. The distance ρ^* between the two convex sets K_1, K_2 is defined by the formula:

$$\rho^*(K_1, K_2) = \delta_1 + \delta_2.$$

LEMMA 5. If $\hat{K} \in \mathbf{K}$, $\partial \hat{K}$ contains the segments $\hat{P}_s P_1$ and $\hat{P}_s P_2$ and no larger segments $\hat{P}_s P_1^*$, $\hat{P}_s P_2^*$ and if S_1, S_2 are support lines of \hat{K} at P_1, P_2 respectively, then for every $\varepsilon > 0$, there is a polygon H with the properties: (i) $\rho^*(\hat{K}, H) < \varepsilon$, (ii) ∂H contains $\hat{P}_s P_1, \hat{P}_s P_2$, (iii) S_1, S_2 are lines of support of H at P_1, P_2 respectively.

LEMMA 6. Let $\triangle \hat{P}_s A_1 A_2$ be a triangle with $\rho(\hat{P}_s, A_1) > \rho(\hat{P}_s, A_2)$ and let A_1^* be the point on $\hat{P}_s A_1$ and $\rho(\hat{P}_s, A_1^*) = \rho(\hat{P}_s, A_2)$. If \hat{K} is a convex set with area equal to that of $\triangle \hat{P}_s A_1 A_2$ and $\partial \hat{K} \supset \hat{P}_s A_2, \partial \hat{K} \supset \hat{P}_s A_1^*$, then

$$J(\hat{K}, \hat{P}_s) < J(\triangle \hat{P}_s A_1 A_2, \hat{P}_s),$$

unless \hat{K} is the triangle $\hat{P}_s A_1 A_2$ or its mirror image.

Proof. First we show that the triangle $\hat{P}_s A_1 A_2$ has greater polar moment of inertia than that of any other triangles which have the same area and which contain $\hat{P}_s A_2$ and $\hat{P}_s A_1^*$ in their sides. Consider any such triangle $\hat{P}_s A_1'' A_2''$. Put

$$\rho(\hat{P}_s, A_2) = b, \quad \rho(\hat{P}_s, A_1) = c, \quad \rho(\hat{P}_s, A_2'') = b'', \quad \rho(\hat{P}_s, A_1'') = c'',$$

and $\angle A_1 \hat{P}_s A_2 = \theta$. By hypothesis

$$(22) \quad b'' \geq b, \quad c'' \geq b,$$

and

$$(23) \quad bc \sin \theta = b''c'' \sin \theta.$$

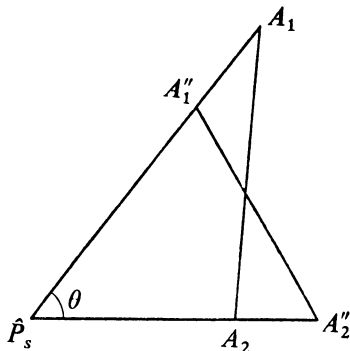


FIGURE 2

Straight-forward computations analogous to what has been done in Lemma 4 give

$$\begin{aligned}
 J(\Delta \hat{P}_s A_1 A_2, \hat{P}_s) &= \frac{1}{2}(b^2 + c^2 - 2bc \cos \theta), \\
 (24) \quad J(\Delta \hat{P}_s A_1'' A_2'', \hat{P}_s) &= \frac{1}{2}(b''^2 + c''^2 - 2b''c'' \cos \theta).
 \end{aligned}$$

It follows from (23) and (24) that

$$J(\Delta \hat{P}_s A_1 A_2, \hat{P}_s) \geq J(\Delta \hat{P}_s A_1'' A_2'', \hat{P}_s),$$

if, and only if,

$$(25) \quad b^2 + c^2 \geq b''^2 + c''^2.$$

In view of (23), the inequality in (25) holds, if and only if,

$$b + c \geq b'' + c''.$$

That is,

$$b'' \leq c.$$

But this inequality is a direct consequence of (23) and the second inequality in (22). Thus our lemma is proved for the case of triangles.

If \hat{K} is not a triangle, let $\hat{P}_s A_1', \hat{P}_s A_2'$ be the maximum linear segments of $\partial \hat{K}$ through the point \hat{P}_s . By hypothesis, we have

$$\rho(\hat{P}_s, A_1') \geq \rho(\hat{P}_s, A_2^*), \quad \rho(\hat{P}_s, A_2') \geq \rho(\hat{P}_s, A_2).$$

Let the point $A \in \partial \hat{K}$ be such that a line of support of \hat{K} at A does not meet $\hat{P}_s A_1'$ and $\hat{P}_s A_2'$. If both the arcs AA_1', AA_2' of $\partial \hat{K}$ are linear segments, then the lemma follows by a single application of Lemma 4. For if we move the point A along the line parallel to the segment $A_1' A_2'$ until it just meets the extension of $\hat{P}_s A_1'$ or $\hat{P}_s A_2'$, then \hat{K} is deformed into a triangle with the same area and the polar moment of inertia of this triangle about \hat{P}_s is greater than $J(\hat{K}, \hat{P}_s)$ but less than $J(\Delta \hat{P}_s A_1 A_2, \hat{P}_s)$. Otherwise, we assume first \hat{K} to be a polygon. Now $\hat{P}_s A$ separates \hat{K} into two convex polygons \hat{K}_1, \hat{K}_2 which contain the points A_1', A_2' respectively. If \hat{K}_1 has more than 4 vertices, we repeatedly apply Lemma 4 to reduce \hat{K}_1 to new polygons in such a way that (i) each of the new convex polygons with \hat{P}_s, A_1', A as vertices and with area equal to $V(\hat{K}_1)$, (ii) each of the new polygons has greater polar moment of inertia about \hat{P}_s and has one vertex less than the previous one. Similarly, if \hat{K}_2 has more than 4 vertices, we apply Lemma 4 to make the same reduction. In this way, we shall arrive at the single convex polygon $\hat{K}', \hat{P}_s A_1' A_1'' A A_2'' A_2' \hat{P}_s$, such that

$$V(\Delta A_1' A_1'' A) = V(\hat{K}_1 - \Delta \hat{P}_s A_1' A), \quad V(\Delta A_2' A_2'' A) = V(\hat{K}_2 - \Delta \hat{P}_s A_2' A),$$

and the points A_1'', A_2'' lie somewhere on the segments $B_1' B_1'' // A_1' A, B_2' B_2'' // A_2' A$ respectively. From the very method of reduction, we have

$$J(\hat{K}, \hat{P}_s) < J(\hat{K}', \hat{P}_s).$$

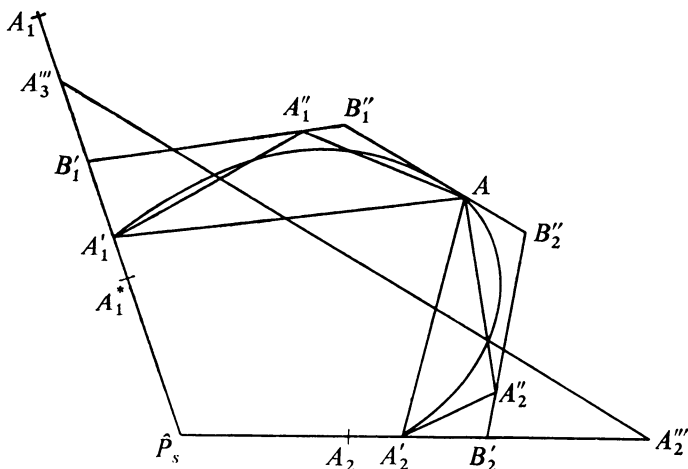


FIGURE 3

In the general case, we apply Lemma 5 to approximate \hat{K}_1, \hat{K}_2 by convex polygons with sufficient accuracy and then repeatedly apply Lemma 4 to these polygons in the way as just described. This will lead to a similar polygon \hat{K}' . Furthermore, a limiting process based on Lemma 5 ensures that

$$J(\hat{K}, \hat{P}_s) \leq J(\hat{K}', \hat{P}_s),$$

provided the approximations of \hat{K}_1, \hat{K}_2 by polygons are sufficiently close. Since \hat{K}' has at least 4 vertices and at most 6 vertices, we apply now Lemma 4 to \hat{K}' once, twice or three times and we come to a triangle $\hat{P}_s A_1''' A_2'''$ such that

$$\rho(\hat{P}_s, A_1''') \geq \rho(\hat{P}_s, A_1'), \quad \rho(\hat{P}_s, A_2''') \geq \rho(\hat{P}_s, A_2'), \quad V(\Delta \hat{P}_s A_1''' A_2''') = V(\hat{K}),$$

and

$$J(\hat{K}, \hat{P}_s) < J(\Delta \hat{P}_s A_1''' A_2''', \hat{P}_s).$$

The lemma now follows from the proof for the case of triangles.

4. Proof of theorem. By establishing the previous lemmas we are now ready to prove the theorem. By Lemma 2 the solution figure to the isoperimetric problem must have the property that its moment of inertia about all lines in its plane and passing through its centroid are all equal. Accordingly, we may restrict \mathbf{K} to be the class of convex sets with this additional property.

Let $K \in \mathbf{K}$ with $I_m(K, C_K) = I_M(K, C_K)$. The convexity of K implies the existence of a six-partite point [3; 4; 5]. We denote this point by P_s . If C_K is a six-partite point, we always choose C_K as the point P_s under consideration. Also we choose C_K as the fixed point of an affine transformation. Under an area-preserving affine transformation A , the image $\hat{P}_s = A(P_s)$ will be the six-partite point of $\hat{K} = A(K)$. Furthermore, we adjust the transformation A so that the three division lines, which pass through \hat{P}_s and divide the area of \hat{K} into six equal parts, will make an angle $\pi/3$ with one another.

We are going to compare $J(\hat{K}, \hat{P}_s)$ with $J(K_\Delta, C_{K_\Delta})$ where K_Δ is an equilateral triangle in \mathbf{K} . To this end we denote in the clock-wise order the vertices of K_Δ by A_1, A_3 and A_5 , the intersection points of $\partial\hat{K}$ and its division lines by P_1, P_2, P_3, P_4, P_5 and P_6 , and the three medians of K_Δ by A_1A_4, A_2A_5, A_3A_6 .

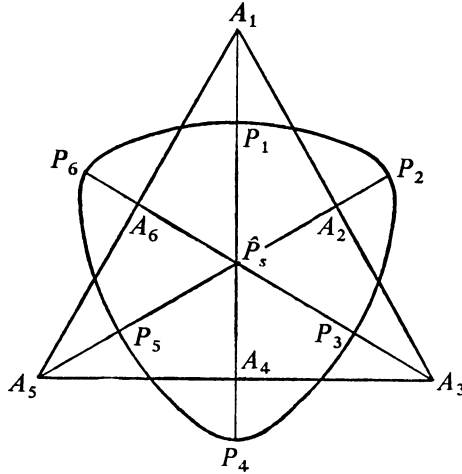


FIGURE 4

It may be noted that C_{K_Δ} is a six-partite point of K_Δ and the three medians divide the area of K_Δ into six equal parts. Place K_Δ upon \hat{K} so that C_{K_Δ} coincides with \hat{P}_s and the three medians A_1A_4, A_2A_5, A_3A_6 coincides with the division lines P_1P_4, P_2P_5, P_3P_6 respectively.

Now we assert that

$$(26) \quad \rho(\hat{P}_s, P_i) \geq h/3, \quad i = 1, 2, \dots, 6,$$

where h denotes the height of K_Δ . To prove this assertion we denote by $\hat{K}_{i,i+1}$ the sector of \hat{K} bounded by segments $\hat{P}_sP_i, \hat{P}_sP_{i+1}$ and $\partial\hat{K}$ for $i = 1, 2, \dots, 5$, and denote by $\hat{K}_{1,6}$ the sector of \hat{K} bounded by segments \hat{P}_sP_1 and \hat{P}_sP_6 and $\partial\hat{K}$. Suppose now $\rho(\hat{P}_s, P_2) < h/3$. Since $V(\hat{K}_{1,2}) = V(\Delta \hat{P}_s A_1 A_2)$, $V(\hat{K}_{2,3}) = V(\Delta \hat{P}_s A_2 A_3)$, there are a point $P_{1,2} \in \hat{K}_{1,2}$ but not in $\Delta \hat{P}_s A_1 A_2$ and a point $P_{2,3} \in \hat{K}_{2,3}$ but not in $\Delta \hat{P}_s A_2 A_3$. Now the convexity of \hat{K} implies that the triangle $\hat{P}_s P_{1,2} P_{2,3}$ is contained in \hat{K} . But $\Delta \hat{P}_s P_{1,2} P_{2,3}$ contains P_2 in its interior. This contradicts that $P_2 \in \partial\hat{K}$. Hence $\rho(\hat{P}_s, P_2) \geq h/3$. Similar arguments prove that (26) holds for $i = 4, 6$. To prove (26) for $i = 1, 3, 5$, we rotate K_Δ about the point \hat{P}_s through an angle $\pi/3$ and then apply the same arguments.

Consider now the triangle $\hat{P}_s A_1 A_2$ and the convex set $\hat{K}_{1,2}$. The inequalities in (26) and the relation $V(\Delta \hat{P}_s A_1 A_2) = V(\hat{K}_{1,2})$ imply that $\hat{K}_{1,2}$ satisfies all the conditions in Lemma 6. Hence

$$J(\hat{K}_{1,2}, \hat{P}_s) \leq J(\Delta \hat{P}_s A_1 A_2, \hat{P}_s),$$

where the equality sign can hold only when $\hat{K}_{1,2}$ is $\Delta \hat{P}_s A_1 A_2$ or its mirror image.

Since Lemma 6 can be applied to any one of the six equal areas of \hat{K} , it follows that

$$J(\hat{K}, \hat{P}_s) \leq J(\triangle A_1 A_2 A_3, \hat{P}_s) = J(K_\Delta, C_{K_\Delta}),$$

where the equality sign can hold only when \hat{K} is K_Δ . If \hat{K} is K_Δ , then $K = A^{-1}(\hat{K})$ is a triangle, C_K, P_s, \hat{P}_s coincide by our choice of the transformation A . Hence, by Lemma 3,

$$J(K, C_K) \leq J(\hat{K}, \hat{P}_s) = J(K_\Delta, C_{K_\Delta}),$$

where the equality sign can hold only when K is K_Δ . We need to consider the only case:

$$J(\hat{K}, \hat{P}_s) < J(K_\Delta, C_{K_\Delta}).$$

By "Parallel-axis" theorem for polar moment of inertia,

$$J(\hat{K}, C_{\hat{K}}) \leq J(\hat{K}, \hat{P}_s),$$

and the equality sign holds only if \hat{P}_s is $C_{\hat{K}}$. Hence

$$J(\hat{K}, C_{\hat{K}}) < J(K_\Delta, C_{K_\Delta}).$$

Since \hat{K} is such that $I_m(K, C_K) = I_m(K, C_K)$, we have, by Lemma 3,

$$J(K, C_K) \leq J(\hat{K}, C_{\hat{K}}) < J(K_\Delta, C_{K_\Delta}).$$

Thus,

$$I_m(K, C_K) = \frac{1}{2} J(K, C_K) < \frac{1}{2} J(K_\Delta, C_{K_\Delta}) = I_m(K_\Delta, C_{K_\Delta}).$$

Clearly, K is not K_Δ in this case. For if K were K_Δ , then the transformation A becomes a rigid rotation and we must have $J(\hat{K}, \hat{P}_s) = J(K_\Delta, C_{K_\Delta})$. The proof to our theorem is now completed.

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