ALMOST LOCALLY POLYHEDRAL CURVES IN EUCLIDEAN n-SPACE

BY

J. C. CANTRELL AND C. H. EDWARDS, JR.(1)

Fox and Artin [2] have given examples of wild arcs and curves in E^3 which fail to be locally polyhedral at only one or two points. It is shown in this paper that no such "simple" examples of wild curves are to be expected in dimensions higher than three. In particular, it is proved that a wild simple closed curve in Euclidean n-space E^n, $n > 3$, must fail to be locally polyhedral at each point of a Cantor set. Examples of such wild curves in E^n have been given by Blankinship [1].

A set K in E^n is called *tame* if there is a homeomorphism h of E^n onto itself such that $h(K)$ is polyhedral (relative to the standard triangulation of E^n). Otherwise K is wild. K is said to be *locally polyhedral* at the point $p \in K$ if there exists a neighborhood N of p such that $\mathrm{Cl}(N \cap K)$ is a polyhedron. The map $h: E^n \to E^n$ is said to be *locally semilinear* at x if there is a neighborhood N of x such that $h \mid N$ is semilinear.

In this paper, $S(p, \varepsilon)$ denotes the set of points $x \in E^n$ whose distance $\rho(x, p)$ from p is less than ε.

The local connectivity of an arc gives the following lemma.

LEMMA 1. *Suppose that A is an arc in E^n with p an interior point of A. Given $\varepsilon > 0$, there exists $\delta > 0$ such that, if L is any subarc of A whose endpoints lie in $S(p, \delta)$, then $L \subset S(p, \varepsilon)$.*

LEMMA 2(2). *Suppose that C is a simple closed curve in E^n, $n > 3$, and that B is the set of points at which C fails to be locally polyhedral. If p is an isolated point of B, then, given $\varepsilon > 0$, there exists a homeomorphism h of E^n onto E^n such that*

(a) *h is the identity on $E^n - S(p, \varepsilon)$,*

(b) *h is locally semilinear except at p,*

(c) *$h(C)$ is locally polyhedral at $h(p)$.*

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Proof. Let q_1 and q_2 be two points of $C - S(p,\varepsilon)$ and let L_1, L_2, L_3 be the three subarcs of C with endpoints q_1 and p, q_2 and p, q_1 and q_2, respectively, such that $C = L_1 \cup L_2 \cup L_3$ and each subarc meets either of the other subarcs in a single endpoint.

Let $\varepsilon_1 = \varepsilon$ and let N_1 be the closed cubical neighborhood of p of diameter ε_1. Let δ_1 be given by Lemma 1 for $\varepsilon_1/3$, p, and $A = L_1 \cup L_2$. Let $\varepsilon_2 = \min[\delta_1, \rho(p,L_3)]$ and let N_2 be the closed cubical neighborhood of p of diameter ε_2. For $i > 2$ let δ_{i-1} be given by Lemma 1 for $\varepsilon_{i-1}/3$, p, and $A = L_1 \cup L_2$. Let $\varepsilon_i = \delta_{i-1}$ and let N_i be the closed cubical neighborhood of p of diameter ε_i.

By making use of a semilinear deformation in a small neighborhood of Bd N_{2i}, if necessary, we may assume that $C \cap \text{Bd } N_{2i}$ is a finite set of points, and that no pair of components of $C - N_{2i}$ share a common endpoint. For each positive integer i, let $u_{i1}, \cdots, u_{ik(i)}$ be the closures of those components of $C - N_{2i}$ which have both endpoints on Bd N_{2i}. Observe that each of these sets is contained in the half open annulus Int N_{2i-1} − Int N_{2i}. Let w_{i1} be a polyhedral arc in Bd N_{2i} which connects the endpoints of u_{i1} and, except for those two points, is disjoint from C. The resulting simple closed curve $d_{i1} = u_{i1} \cup w_{i1}$ bounds a polyhedral 2-cell D_{i1} in $\text{Int}\, N_{2i-1}$, since $n > 3$ [3, p. 32]. Hence we may assume that D_{i1} intersects C only in w_{i1}. We may further assume that D_{i1} and Bd N_{2i} are in relative general position, so that each component of Int $D_{i1} \cap \text{Bd } N_{2i}$ is either a simple closed curve or an open arc (whose closure may be either an arc intersecting w_{i1} in its endpoints or a simple closed curve intersecting w_{i1} in a single point).

Let e be *a* simple closed curve component of Int $D_{i1} \cap \text{Bd } N_{2i}$ which contains no other such component in its interior (relative to D_{i1}). Let Y be the subdisk of D_{i1} bounded by e. Let E and F be the components of $E^n - \text{Bd } N_{2i}$ and labeled so that $Y \subset \text{Cl } E$.

Let r be a point in F, sufficiently close to Bd N_{2i} for X (the join of r and e) to meet $D_{i1} \cup C$ only in e. Define $D'_{i1} = (D_{i1} - Y) \cup X$ and deform D'_{i1} semilinearly away from Bd N_{2i} in a sufficiently small neighborhood of e so that no new intersections are introduced. The disk D''_{i1} thus obtained is bounded by d_{i1} and intersects Bd N_{2i} in exactly those components, other than e, in which D_{i1} intersected Bd N_{2i}. Since the open arc components of Int $D_{i1} \cap$Bd N_{2i} can be eliminated by a similar process of exchanging disks, after a finite number of steps we obtain a disk D^*_{i1} which, except for w_{i1}, is contained in the open annulus $\text{Int}(N_{2i-1} - N_{2i})$ and intersects C only in u_{i1}.

Let $\eta > 0$ be such that the η-neighborhood S_{i1} of D^*_{i1} is contained in $\text{Int}(N_{2i-1} - N_{2i+1})$ and intersects $C - N_{2i}$ only in u_{i1}. By a sequence of simplicial moves across the 2-simplexes of D^*_{i1} the arc u_{i1} may be moved onto the arc w_{i1}. By making use of a corresponding space homeomorphism [5, Lemma 3], we may deform u_{i1} onto w_{i1} and then into $\text{Int}\, N_{2i}$ by a semilinear space homeomorphism which is the identity outside S_{i1}.

The components $u_{i2}, \cdots, u_{ik(i)}$ are successively moved into $\text{Int}(N_{2i} - N_{2i+1})$ by a technique similar to that used on u_{i1}. We are careful in each move to leave the remaining components fixed. This is to keep from introducing new intersections with Bd N_{2i}. We denote the composition of these moves by f_i and observe that f_i is a semilinear space homeomorphism which is fixed outside $\mathrm{Int}(N_{2i-1} - N_{2i+1})$. Also, if a_i is the first point of $L_1 \cap \mathrm{Bd}\, N_{2i}$ relative to the order of L_1 from q_1 to p and if b_i is the first point of $L_2 \cap \mathrm{Bd}\, N_{2i}$ relative to the order of L_2 from q_2 to p, then $f_i(C) \cap \mathrm{Bd}\, N_{2i} = a_i \cup b_i$.

We define a mapping f of E^n onto E^n by the equations

$$f(x) = x \quad \text{if } x \in E^n - N_1,$$

$$f(x) = f_i(x) \quad \text{if } x \in N_{2i-1} - N_{2i+1}, \quad i = 1, 2, \cdots,$$

$$f(p) = p.$$

It is clear since, for each i, f_i is fixed on $\mathrm{Bd}\, N_{2i-1} \cup \mathrm{Bd}\, N_{2i+1}$ and f_i eliminates all but two points of intersection of C and $\mathrm{Bd}\, N_{2i}$, that f is a homeomorphism, semilinear except at p, and that $f(C) \cap \mathrm{Bd}\, N_{2i} = a_i \cup b_i$, $i = 1, 2, \cdots$.

We now consider the curve $f(C)$. Let $L_{1i} = f(L_1) \cap \mathrm{Cl}(N_{2i} - N_{2i+2})$ and $L_{2i} = f(L_2) \cap \mathrm{Cl}(N_{2i} - N_{2i+2})$. Let x_i be the point of intersection of the linear segment $\overline{a_1 p}$ with $\mathrm{Bd}\, N_{2i}$ and let y_i be the point of intersection of $\overline{b_1 p}$ with $\mathrm{Bd}\, N_{2i}$. Let ϕ_i be a semilinear space homeomorphism which is fixed outside $N_{2i-1} - N_{2i+1}$ and which carries $\mathrm{Bd}\, N_{2i}$ onto $\mathrm{Bd}\, N_{2i}$, with $\phi_i(a_i) = x_i$ and $\phi_i(b_i) = y_i$. Since $a_1 = x_1$ and $b_1 = y_1$, we will assume that ϕ_1 is the identity homeomorphism. The simple closed curve $\overline{x_i x_{i+1}} \cup \phi_{i+1}\phi_i(L_{1i})$ bounds a polyhedral disk D_i, which may be taken to be disjoint from $\overline{y_i y_{i+1}} \cup \phi_{i+1}\phi_i(L_{2i})$. Furthermore, in the light of the elimination o compo¤ent scheme u:ed above, D_i may be selected so that $D_i \cap (\mathrm{Bd}\, N_{2i} \cup \mathrm{Bd}\, N_{2i+2}) = x_i \cup x_{i+1}$. The arc $\phi_{i+1}\phi_i(L_{1i})$ is then moved across the disk D_i onto the arc $\overline{x_i x_{i+1}}$ by a semilinear space homeomorphism ψ_{i1}, which is the identity outside $N_{2i} - N_{2i+2}$ and on $\phi_{i+1}\phi_i(L_{2i})$. Similarly $\phi_{i+1}\phi_i(L_{2i})$ is moved onto $\overline{y_i y_{i+1}}$ by a space homeomorphism ψ_{i2}, which is fixed outside $N_{2i} - N_{2i+2}$ and on $\overline{x_i x_{i+1}}$. The composition $\psi_{i2}\psi_{i1}$ is denoted by ψ_i.

A mapping g of E^n onto E^n is defined by the equations

$$g(x) = x \quad \text{if } x \in E^n - N_2,$$

$$g(x) = \psi_i \phi_{i+1} \phi_i(x) \quad \text{if } x \in N_{2i} - N_{2i+2} \quad i = 1, 2, \cdots,$$

$$g(p) = p.$$

The fact that $\psi_i \phi_{i+1} \phi_i$ and $\psi_{i+1} \phi_{i+2} \phi_{i+1}$ agree on $\mathrm{Bd}\, N_{2i+2}$ (each reduces to ϕ_{i+1}) insures that g is a homeomorphism. It is clear that g is semilinear except at p and that the subarc of $f(L_1)$ from a_1 to p is carried onto the segment $\overline{a_1 p}$ and that the subarc of $f(L_2)$ from b_1 to p is carried onto the segment $\overline{b_1 p}$.

Finally the desired homeomorphism h is taken to be the composition $h = gf$, so that the proof of Lemma 2 is complete.

Notice that the essential point upon which the proof of Lemma 2 depends is the fact that polyhedral simple closed curves cannot knot or link in E^n if $n > 3$, whereas the construction of the typical example of a wild curve in E^3, locally polyhedral except at a single point, involves knotting or linking in a neighborhood of the exceptional point.

The following lemma will serve as the first step in an inductive proof of the principal theorem.

LEMMA 3. *If C is a simple closed curve in E^n, $n > 3$, denote by B the set of points at which C fails to be locally polyhedral, and by B' the derived set of B. Then, given $\varepsilon > 0$ and a compact set F not meeting $E = B - B'$, there is is a homeomorphism h of E^n onto itself such that*

(a) $h(x) = x$ *if* $x \in B' \cup [E^n - S(B,\varepsilon)]$,

(b) h *is locally semilinear on* $E^n - B$,

(c) $h(C)$ *is locally polyhedral at each point of* $h(C) - B'$,

(d) $\rho(x, h(x)) \leqq \min\{\varepsilon, \varepsilon \cdot \rho(x,F)\}$ *for each* $x \in E^n$.

Proof. Since the discrete set E is at most countable [4, p. 62] enumerate its points in a sequence $\{a_i\}_1^\infty$ (assume that E is infinite since otherwise Lemma 3 follows immediately from Lemma 2). Choose a sequence $\{U_i\}_1^\infty$ of mutually disjoint open sets with $a_i \in U_i$, $U_i \cap B' = \square$, and diam $U_i < \{\varepsilon \cdot \rho(U_i, F)\}$ for each i, and a monotone decreasing sequence $\{\varepsilon_i\}_1^\infty$ of positive numbers with each $\varepsilon_i < \varepsilon/2$ and $\lim_{i\to\infty} \varepsilon_i = 0$, such that $\mathrm{Cl}\, S(a_i, \varepsilon_i) \subset U_i$ for each i.

Using Lemma 2, choose for each $i = 1, 2, \cdots$ a homeomorphism h_i of E^n onto itself such that $h_i(x) = x$ if $x \in E^n - S(a_i, \varepsilon_i)$, h_i is locally semilinear except at a_i, and $h_i(C)$ is locally polyhedral at $h_i(a_i)$.

Then define

$$h(x) = \begin{cases} x & \text{if } x \in E^n - \bigcup_{i=1}^{\infty} S(a_i, \varepsilon_i), \\ h_i(x) & \text{if } x \in S(a_i, \varepsilon_i). \end{cases}$$

Obviously $h(x) = x$ if $x \in B' \cup [E^n - S(B,\varepsilon)]$ and the fact that diam $U_i < \varepsilon \cdot \rho(U_i, F)$ for each i implies that $\rho(x, h(x)) \leqq \min\{\varepsilon, \varepsilon \cdot \rho(x,F)\}$ for every $x \in E^n$. Since $h_i | S(a_i, \varepsilon_i)$ is 1-1 onto, it is clear that h is a 1-1 map of E^n onto itself. Since h is the identity outside some n-cell, to show that h is a homeomorphism it suffices to show that it is continuous.

If $x \in \mathrm{Cl}\, S(a_i, \varepsilon_i)$ for some i, then there is a neighborhood U of x such that $U \cap S(a_j, \varepsilon_j) = \square$ if $i \neq j$. Therefore $h = h_i$ on U so that h is continuous at x and is also locally semilinear at x, unless x is a_i.

If $x \in E^n - \mathrm{Cl}\, E - \bigcup_{i=1}^\infty \mathrm{Cl}\, S(a_i, \varepsilon_i)$, then there is a neighborhood V of x such that $V \cap \bigcup_{i=1}^\infty S(a_i, \varepsilon_i) = \square$. To the contrary, suppose that for each positive integer k there is an integer i_k such that $S(x, 1/k) \cap S(a_{i_k}, \varepsilon_{i_k})$ contains a point x_k.

Since x is not in any $\operatorname{Cl} S(a_i, \varepsilon_i)$, it follows easily that $\lim_{k\to\infty} i_k = \infty$, so that $\lim_{k\to\infty} \varepsilon_{i_k} = 0$.

Because $\rho(x_k, a_{i_k}) < \varepsilon_{i_k}$, while $\lim_{k\to\infty} x_k = x$, it therefore follows that $\lim_{k\to\infty} a_{i_k} = x$, contradicting the fact that $x \notin \operatorname{Cl} E$. Consequently h agrees with the identity in some neighborhood of x, so that h is both continuous and locally semilinear at x. It is now clear that h is locally semilinear on $E^n - B$.

It remains to consider the case of a point x in $\operatorname{Cl} E - \bigcup_{i=1}^{\infty} \operatorname{Cl} S(a_i, \varepsilon_i)$. Let $\{x_k\}_1^{\infty}$ be an arbitrary sequence of points converging to x. Since $h(x) = x$ and $h(x_k) = x_k$ if $x_k \notin \bigcup_{i=1}^{\infty} S(a_i, \varepsilon_i)$, suppose that for each k there is an i_k such that $x_k \in S(a_{i_k}, \varepsilon_{i_k})$. Since x is not in any $\operatorname{Cl} S(a_i, \varepsilon_i)$, it is clear that $\lim_{k\to\infty} i_\cdot = \infty$ so that $\lim_{k\to\infty} \varepsilon_{i_k} = 0$. Now $h(x_k) \in S(a_{i_k}, \varepsilon_{i_k})$, since h carries $S(a_{i_k}, \varepsilon_{i_k})$ onto itself, so that $\rho(x_k, h(x_k)) < 2\varepsilon_{i_k}$. But $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} \varepsilon_{i_k} = 0$. Hence $\lim_{k\to\infty} h(x_k) = x = h(x)$ so that h is continuous at x.

Consequently h is a homeomorphism of E^n onto itself. Since it is clear from the construction of h that $h(C)$ is locally polyhedral at each point of $h(C) - B'$, this completes the proof of Lemma 3.

THEOREM. *Let C be a simple closed curve in E^n, $n > 3$, and denote by B_0 the set of points at which C is not locally polyhedral. If B_0 is countable then C is tame.*

Proof. Obviously B_0 is a compact subset of C. Denote by B_1 the derived set B_0' of B_0. Supposing that B_α has been defined for every ordinal number α preceding the ordinal number β, B_β is defined as follows: If β is an ordinal of the first kind, i.e., $\beta = \alpha + 1$ for some α, define $B_\beta = B_\alpha'$; while if β is an ordinal of the second kind, define $B_\beta = \bigcap_{\alpha<\beta} B_\alpha$. Since $\{B_\alpha\}$ is a decreasing transfinite sequence of compact subsets of C, there exists a first ordinal number γ preceding the first uncountable ordinal Ω, such that $B_\gamma = B_\delta$ for every $\delta > \gamma$ [4, p. 67]. If the compact perfect set B_γ were nonempty, then it would be uncountable [4, p. 92], thereby contradicting the hypothesis that B_0 is countable. Consequently $B_\gamma = \square$. It therefore follows that, in order to prove the Theorem, it suffices to show that the following lemma holds for every countable ordinal α.

LEMMA A_α. *Let C and B_0 be as in the Theorem. Given a compact set F not intersecting $B_0 - B_\alpha$ and a positive number ε, there is a homeomorphism h of E^n onto itself such that*

(a) $h(x) = x$ *if* $x \in B_\alpha \cup [E^n - S(B_0, \varepsilon)]$,

(b) *h is locally semilinear on* $E^n - B_0$,

(c) *$h(C)$ is locally polyhedral at each point of* $h(C) - B_\alpha$,

(d) $\rho(x, h(x)) \leq \min\{\varepsilon, \varepsilon \cdot \rho(x, F)\}$ *for each* $x \in E^n$.

To prove Lemma A_α for every ordinal number α of the first and second classes, it is sufficient by the transfinite induction principle to show that

(1) Lemma A_1 is true.

(2) The truth of Lemma A_α implies the truth of Lemma $A_{\alpha+1}$ for every $\alpha < \Omega$.

(3) If $\{\alpha_i\}_{i=1}^\infty$ is an increasing sequence of countable ordinals, with $\lim_{i\to\infty}\alpha_i = \alpha_0$, such that Lemma A_{α_i} is true for each $i = 1,2,\cdots$, then it follows that Lemma A_{α_0} is true.

Lemma A_1 has already been proved as Lemma 3. Now suppose that Lemma A_α is true and let $\varepsilon > 0$ and a compact set F not meeting $B_0 - B_{\alpha+1}$ be given. By Lemma A_α there is a homeomorphism h_1 of E^n onto itself such that (a) $h_1(x) = x$ if $x \in B_\alpha \cup [E^n - S(B_0,\varepsilon)]$, (b) h_1 is locally semilinear on $E^n - B_0$, (c) $h_1(C)$ is locally polyhedral at each point of $h_1(C) - B_\alpha$, and (d) $\rho(x,h_1(x)) \leqq \min\{\varepsilon/2, \varepsilon\cdot\rho(x,F)/3\}$. Then since the simple closed curve $h_1(C)$ is locally polyhedral except at the points of B_α, by Lemma A_1 there is a homeomorphism h_2 of E^n onto itself such that (a) $h_2(x) = x$ if $x \in B_{\alpha+1} \cup [E^n - S(B_\alpha,\varepsilon)]$, (b) h_2 is locally semilinear on $E^n - B_\alpha$, (c) $h_2h_1(C)$ is locally polyhedral at each point of $h_2h_1(C) - B_{\alpha+1}$, and (d) $\rho(x,h_2(x)) \leqq \min\{\varepsilon/2, \varepsilon\cdot\rho(x,F)/3\}$. It follows that $h = h_2h_1$ satisfies the requirements of Lemma $A_{\alpha+1}$, so that the truth of Lemma A_α implies the truth of Lemma $A_{\alpha+1}$.

Now suppose that $\{\alpha_i\}_1^\infty$ is an increasing sequence of countable ordinals, with $\lim_{i\to\infty}\alpha_i = \alpha_0$, such that Lemma A_{α_i} is true for each $i = 1,2,3,\cdots$. Let $\varepsilon > 0$ and a compact set F not meeting $B_0 - B_{\alpha_0}$ be given. Using Lemma A_{α_1}, let h_1 be a homeomorphism of E^n onto itself such that (a) $h_1(x) = x$ if $x \in B_{\alpha_1} \cup [E^n - S(B_0,\varepsilon/2)]$, (b) h_1 is locally semilinear on $E^n - B_0$, (c) $h_1(C)$ is locally polyhedral at each point of $h_1(C) - B_{\alpha_1}$, and (d) $\rho(x,h_1(x)) \leqq \min\{\varepsilon/2, \varepsilon\cdot\rho(x,F)/3, \rho(x,B_{\alpha_0})/4\}$. In general, having defined $h_1, h_2, \cdots, h_{j-1}$ such that $h_{i-1}\cdots h_2h_1(C)$ is locally polyhedral except possibly at the points of $B_{\alpha_{i-1}}$, use Lemma A_{α_1} to define a homeomorphism h_i of E^n onto itself such that (a) $h_i(x) = x$ if $x \in B_{\alpha_1} \cup [E^n - S(B_{\alpha_{i-1}},\varepsilon/2^i)]$, (b) h_i is locally semilinear on $E^n - B_{\alpha_{i-1}}$, (c) $h_ih_{i-1}\cdots h_1(C)$ is locally polyhedral except at the points of the α_ith derived set of $B_{\alpha_{i-1}}$, and (d) $\rho(x,h_i(x)) \leqq \min\{\varepsilon/2^i, \varepsilon\cdot\rho(x,F)/3^i, \rho(x,B_{\alpha_0})/4^i\}$. Notice that $(B_{\alpha_{i-1}})_{\alpha_i} = B_{\alpha_{i-1}+\alpha_i} \subset B_{\alpha_i}$ since $\alpha_{i-1} + \alpha_i \geqq \alpha_i$, so that $h_i\cdots h_1(C)$ is locally polyhedral except possibly at the points of B_{α_i}. Finally define $h(x) = \lim_{i\to\infty} h_i\cdots h_1(x)$ for each $x \in E^n$. The map h is well-defined because h_i moves no point more than $\varepsilon/2^i$ and $\sum_{i=1}^\infty \varepsilon/2^i = \varepsilon < \infty$. Routine series calculations show that $\rho(x,h(x)) \leqq \min\{\varepsilon, \varepsilon\cdot\rho(x,F), \frac{1}{2}\rho(x,B_{\alpha_0})\}$ so that condition (d) of Lemma A_{α_0} is satisfied. It is obvious from the construction that h satisfies condition (a) of Lemma A_{α_0}.

To show that h is continuous at each point of $E^n - B_{\alpha_0}$, consider an arbitrary point $x \in E^n - B_{\alpha_0}$. Since $\rho(x,h(x)) \leqq \frac{1}{2}\rho(x,B_{\alpha_0})$, and since $\{\mathrm{Cl}\, S(B_{\alpha_{i-1}}, \varepsilon/2^i\}_1^\infty$ is a decreasing sequence of compact sets intersecting in B_{α_0}, there exists an integer $j(x)$ such that $h(x) \in E^n - \mathrm{Cl}\, S(B_{\alpha_{j(x)-1}}, 1/2^{j(x)})$. Therefore there is a neighborhood U of x such that $h = h_{j(x)}\cdots h_2h_1$ on U, so that h is continuous at x, and is also locally semilinear at x if $x \notin B_0$. It follows also, if $x \in C$, that $h(C)$ is locally polyhedral at $h(x)$, since $x \notin B_{j(x)-1}$, so that conditions (b) and (c) of Lemma A_{α_0} are satisfied.

To see that h is 1-1 on E^n, consider any two distinct points x and y of E^n. If x, $y \in B_{\alpha_0}$, then $h(x) = x \neq y = h(y)$ since h is the identity on B_{α_0}. If $x \in B_{\alpha_0}$ and $y \in E^n - B_{\alpha_0}$, then $h(x) = x$ while $h(y) \in E^n - B_{\alpha_0}$ as above, so that $h(x) \neq h(y)$. Finally suppose that $x, y \in E^n - B_{\alpha_0}$. Then denote by k the maximum of $j(x)$ and $j(y)$. It follows that $h(x) = h_k \cdots h_1(x)$ and $h(y) = h_k \cdots h_1(y)$, so that $h(x) \neq h(y)$ because the h_i are homeomorphisms.

Now suppose that $z \in E^n - B_{\alpha_0}$ and choose an integer k such that $z \in E^n - S(B_{\alpha_k}, \varepsilon/2^{k+1})$, so that $h_i(z) = z$ if $i > k$, and let $y = h_1^{-1} \cdots h_k^{-1}(z)$. Then $h(y) = z$ so that h maps $E^n - B_{\alpha_0}$ onto itself. Since h is the identity on B_{α_0}, it follows that h maps E^n *onto* E^n.

In order to prove that h is a homeomorphism of E^n onto itself, it remains to prove that h is continuous at each point of B_{α_0} (h being the identity except on a compact set). Since h is the identity on B_{α_0}, it suffices to show that, given a point $x \in B_{\alpha_0}$ and an arbitrary neighborhood U of x, there is a neighborhood V of x such that $h(V) \subset U$. First choose $\gamma > 0$ such that $S(x, \gamma) \subset U$, and let $V = S(x, 2\gamma/3)$. Then, given $y \in V$, $\rho(y, B_{\alpha_0}) < 2\gamma/3$ and $\rho(y, h(y)) \leqq \frac{1}{2}\rho(y, B_{\alpha_0}) < \gamma/3$ so that $\rho(x, h(y)) \leqq \rho(x, y) + \rho(y, h(y)) < 2\gamma/3 + \gamma/3 = \gamma$, and hence $h(y) \in S(x, \gamma) \subset U$. Thus $h(V) \subset U$. Therefore h is continuous at x.

Consequently h is a homeomorphism of E^n onto itself satisfying the conditions of Lemma A_{α_0}. It now follows by transfinite induction that Lemma A_α holds for each countable ordinal number α. This completes the proof of the Theorem.

COROLLARY. *If the simple closed curve C in E^n, $n > 3$, is wild, then C fails to be locally polyhedral at each point of some Cantor set.*

Proof. If B denotes the set of points at which C is not locally polyhedral then, by the Theorem, B is uncountable. As in the proof of the Theorem, choose a countable ordinal number α such that $B_\beta = B_\alpha$ if $\beta > \alpha$. Then B_α is an uncountable compact perfect set, at each point of which C fails to be locally polyhedral. If B_α is totally disconnected, then it is a Cantor set. Otherwise B_α contains an arc, which in turn contains a Cantor set.

REFERENCES

1. W. A. Blankinship, *Generalization of a construction of Antoine*, Ann. of Math. (2) **53** (1951), 276–297.

2. R. Fox and E. Artin, *Some wild cells and spheres in 3-dimensional space*, Ann. of Math. (2) **49** (1948), 979–990.

3. V. K. A. M. Gugenheim, *Piecewise linear isotopy and embedding of elements and spheres* (I), Proc. London Math. Soc. **3** (1953), 29–53.

4. W. Sierpinski, *General topology*, Univ. of Toronto Press, Toronto, 1952.

5. E. C. Zeeman, *Unknotting spheres*, Ann. of Math. (2) **72** (1960), 350–361.

UNIVERSITY OF TENNESSEE,
KNOXVILLE, TENNESSEE
UNIVERSITY OF WISCONSIN,
MADISON, WISCONSIN