THE CRITICAL POINTS OF A LINEAR COMBINATION OF GREEN'S FUNCTIONS(1)

BY
MORRIS MARDEN

1. **Introduction.** During the past three decades, Professor J. L. Walsh of Harvard University has proved a number of interesting theorems about the location of the critical points of the Green's function, with pole at infinity, for an infinite two-dimensional region with a finite boundary(2). Among these is the following:

THEOREM 1.1 (WALSH(3)). *Let R be an infinite region whose boundary B is a finite Jordan configuration and let $G(x, y)$ be the Green's function for R with pole at infinity. Then the critical points of $G(x, y)$ in R lie in the convex hull $H(B)$ of B, with none on the boundary of $H(B)$ unless the points of B are collinear.*

This result is analogous to the well-known theorem on polynomials:

THEOREM 1.2 (LUCAS(4)). *The critical points of a polynomial $f(z)$ lie in the convex hull H of the zeros of $f(z)$ with none on the boundary of H unless it is a multiple zero of $f(z)$ or unless the zeros of $f(z)$ are collinear.*

One of the generalizations of Lucas' theorem is the following:

THEOREM 1.3 (MARDEN(5)). *If an nth degree polynomial $f(z)$ has p zeros $(2 \leqq p \leqq n)$ in a circle C of radius a, it has at least $p - 1$ critical points in the concentric circle C' of radius $a \csc[\pi/2(n - p + 1)]$.*

This raises the question as to the possibility of an analogous generalization of Theorem 1.3. In the ensuing sections we shall investigate this question and deve-

Received by the editors May 10, 1962 and, in revised form, June 8, 1962.

(1) The research for this paper was done with the support of the National Science Foundation under contract number NSF-G16315. The results were presented at the International Congress of Mathematicians, Stockholm, August 15–22, 1962.

(2) A detailed exposition of this subject may be found on pages 241–268 of J. L. Walsh's, *The location of the critical points of analytic and harmonic functions*, Amer. Math. Soc. Colloq. Publ. Vol. 34, Amer. Math. Soc., Providence, R. I., 1950.

(3) J. L. Walsh, ibid., p. 249.

(4) M. Marden, *The geometry of the zeros of a polynomial in a complex variable*, Mathematical Surveys, No. 3, Amer. Math. Soc., Providence, R. I., 1949, pp. 15–17.

(5) M. Marden, ibid., pp. 89–90.

lop several such analogies. They will involve a linear combination of $G(x, y)$ with the Green's functions for infinite lemniscatic regions whose poles lie in R. The theorems will be of the following type.

THEOREM 1.4. *Let R be an infinite region bounded by a finite Jordan configuration B. Let $p_k(z)$ be a polynomial of degree n_k with all its zeros in R and let L denote the intersection of the m lemniscatic regions*

$$L_k: |p_k(z)| > \rho_k, \qquad \rho_k > 0, \qquad k = 1, 2, \cdots, m \tag{1.1}$$

where the ρ_k are chosen so that $L \cap R \neq 0$. Let the Green's function with pole at infinity be denoted by $G(x, y)$ for R and by $g_k(x, y)$ for L_k. Then the linear combination

$$\Phi(x, y) = G(x, y) + \sum_{k=1}^{m} \lambda_k g_k(x, y), \qquad \text{all } \lambda_k > 0, \tag{1.2}$$

has at most $N = n_1 + n_2 + \cdots + n_m$ critical points (counted with their multiplicities) outside a certain star-shaped region S containing $H(B)$, dependent upon N, but not upon the location of the zeros of the $p_k(z)$.

In §§3, 4, 5, we shall prove this theorem for various cases and further specify the region S. The proof in each case is made possible by the establishment of an identity which relates $G(x, y)$ to any $N + 1$ critical points of $\Phi(x, y)$.

2. **Function $\Psi(z)$.** For the development of theorems like 1.4 we need the following[6].

THEOREM 2.1. *In Theorem 1.4, let*

$$p_k(z) = (z - \zeta_{1k})(z - \zeta_{2k}) \cdots (z - \zeta_{n_k k}).$$

Then the critical points of $\Phi(x, y)$ in $R \cap L$ are the zeros of the function

$$\Psi(z) = \int_B \frac{d\mu}{z - t} + \sum_{k=1}^{m} \frac{\lambda_k}{n_k} \sum_{j=1}^{n_k} \frac{1}{z - \zeta_{jk}}, \tag{2.1}$$

where

$$d\mu = (1/2\pi)(\partial G/\partial \nu) ds > 0,$$

and where ν is the interior normal to B.

As its form suggests, the conjugate imaginary of $\Psi(z)$ may be interpreted as the force due to a distribution $d\mu$ on B plus the forces due to N discrete particles of

(6) Generalization of Theorem 1, J. L. Walsh, ibid., p. 247.

mass (λ_k/n_k) at the points ζ_{jk}, the force attracting according to the inverse distance law.

To establish the lemma, we use the representations(7)

$$G(x, y) = \int_B \log r d\mu + \gamma, \qquad \gamma = \text{const.}, \tag{2.2}$$

$$g_k(x, y) = (1/n_k) \log [|p_k(z)|/\rho_k] \tag{2.3}$$

where r is the distance from point (x, y) to a variable point on B. Thus

$$\Phi(x, y) = \gamma + \int_B \log r d\mu + \sum_{k=1}^{m} (\lambda_k/n_k) \log [|p_k(z)|/\rho_k].$$

On adding to $\Phi(x, y)$, $\sqrt{-1}$ times its harmonic conjugate and taking the derivative, we obtain the function $\Psi(z)$.

3. **Case $m = 1, L_1$: circular region.** We begin with the computationally simplest case in which we shall obtain the following result.

THEOREM 3.1. *Let R be an infinite region bounded by a finite Jordan configuration B and containing the point ζ. Let the Green's function with pole at infinity be denoted by $G(x, y)$ for R and by $g(x, y)$ for the circular region $C:|z - \zeta| > \rho$, where the radius ρ is taken so that $R \cap C \neq 0$. Then at most a simple critical point of the function*

$$\Phi(x, y) = G(x, y) + \lambda g(x, y), \qquad \lambda > 0,$$

lies outside the star-shaped region S comprised of all points from which $H(B)$ subtends an angle of at least $\pi/2$.

In this case $g(x, y) = \log (|z - \zeta|/\rho)$ so that (2.1) reduces to

$$\Psi(z) = \int_B \frac{d\mu}{z - t} + \frac{\lambda}{z - \zeta}. \tag{3.1}$$

Let us assume, contrary to Theorem 3.1, that $\Phi(x,y)$ has two distinct critical points z_0 and z_1 outside S. Then

$$I_j + \frac{\lambda}{z_j - \zeta} = 0, \qquad j = 0, 1, \tag{3.2}$$

where

$$I_j = \int_B \frac{d\mu}{z_j - t}. \tag{3.3}$$

(7) J. L. Walsh, ibid., p. 246 and p. 248.

On defining

$$I_{ij} = \int_B \frac{d\mu(t)}{(z_i - t)(z_j - t)}, \tag{3.4}$$

we find

$$I_i - I_j = (z_j - z_i)I_{ij}. \tag{3.5}$$

By eliminating ζ from equations (3.2), we obtain the equation

$$I_0 I_1 + \lambda I_{01} = 0 \tag{3.6}$$

which is a relation between $G(x, y)$ and any two critical points z_0 and z_1 of $\Phi(x, y)$.

If $\Phi(x, y)$ has a multiple critical point at z_0, the corresponding relation

$$I_0^2 + \lambda I_{00} = 0 \tag{3.6$'$}$$

may be obtained from (3.6) by allowing $z_1 \to z_0$, or equivalently by eliminating ζ from the equations $\Psi(z_0) = 0$ and $\Psi'(z_0) = 0$.

Let T_j denote a point such that

$$\arg(T_j - z_j) = \sup_{t \in B} \arg(t - z_j), \qquad \text{mod } 2\pi. \tag{3.7}$$

Since z_0 and z_1 lie outside S, B subtends an angle less then $\pi/2$ at z_0 and z_1 and thus

$$0 \leqq \arg \frac{T_j - z_j}{t - z_j} < \frac{\pi}{2}, \qquad \text{for all } t \in B.$$

Hence, the vectors

$$v_1 = [(T_0 - z_0)I_0][(T_1 - z_1)I_1],$$

$$v_2 = (T_0 - z_0)(T_1 - z_1)I_{01}$$

have the property

$$0 \leqq \arg v_j < \pi, \qquad j = 1, 2$$

and hence $v_1 + v_2 \neq 0$. That is, the assumption that two critical points z_0 and z_1 lie outside S implies that (3.6) or (3.6)$'$ is not satisfied and thus leads to a contradiction. Therefore, at most, a simple critical point may lie outside S.

Immediate consequences of Theorem 3.1 are:

COROLLARY 3.1. *If B lies inside a circle of radius a, then the $\Phi(x, y)$ of Theorem* 3.1 *has at most a simple critical point outside the concentric circle of radius $\sqrt{2}a$.*

COROLLARY 3.2. *If B lies on a line segment A, the function $\Phi(x, y)$ of Theorem* 3.1 *has at most a simple critical point exterior to the circle C having A as diameter.*

4. **Case** $\lambda_k = \lambda n_k$, **all** k. We shall next prove the following generalization of Theorem 3.1.

THEOREM 4.1. *Let R be an infinite region bounded by a finite Jordan configuration B and let $p_k(z)$ be polynomials of degree n_k with all their zeros in R. Let L be the intersection of the m lemniscatic regions*

$$L_k: |p_k(z)| > \rho_k, \qquad \rho_k > 0, \qquad k = 1, 2, \cdots, m \tag{4.1}$$

where ρ_k are chosen so that $R \cap L \neq 0$. Let the Green's function with pole at infinity be denoted by $G(x, y)$ for R and by $g_k(x, y)$ for L_k. Then the linear combination

$$\Phi(x, y) = G(x, y) + \sum_{k=1}^{m} \lambda n_k g_k(x, y), \qquad \lambda > 0 \tag{4.2}$$

has at most N critical points (counted with their multiplicities) outside the star-shaped region S comprised of all points from which $H(B)$ subtends an angle of at least $\pi/(N+1)$.

Proof. Let us assume on the contrary that $\Phi(x, y)$ has $N+1$ distinct critical points $z_0, z_1, \cdots, z_N$ outside S. These critical points satisfy the equations obtained from (2.1):

$$I_p + \lambda \sum_{j=1}^{N} \frac{1}{z_p - \zeta_j} = 0, \qquad p = 0, 1, \cdots, N \tag{4.3}$$

where the ζ_j ($j = 1, 2, \cdots, N$) are the ζ_{jk} relabelled with a single subscript and I_p is defined by (3.3).

We shall now establish three lemmas involving the integrals

$$I_{i_1 i_2 \ldots i_k} = \int_B \frac{d\mu(t)}{(z_{i_1} - t)(z_{i_2} - t) \cdots (z_{i_k} - t)}. \tag{4.4}$$

LEMMA 4.1. *Under the hypotheses of Theorem 4.1, the distinct critical points $z_0, z_1, \cdots, z_N$ of $\Phi(x, y)$ satisfy the equation*

$$\sum_{k=0}^{N} P_k \lambda^k = 0 \tag{4.5}$$

where

$$\begin{aligned} P_k &= \sum I_{i_k} I_{i_{k+1}} \cdots I_{i_N} D_{i_0} D_{i_1} \cdots D_{i_{k-1}} \mathscr{V}_N, \\ D_j &= (\partial/\partial z_j), \end{aligned} \tag{4.6}$$

$$(4.7)\qquad \mathscr{V}_N = V_{01\ldots N} = \begin{vmatrix} z_0^N & z_0^{N-1} & \cdots & 1 \\ z_1^N & z_1^{N-1} & \cdots & 1 \\ \cdot & \cdot & \cdots & \cdot \\ z_N^N & z_N^{N-1} & \cdots & 1 \end{vmatrix},$$

and where in (4.6) *the sum is taken with the set* $(i_0, i_1, \cdots, i_N)$ *running through all possible permutations of the set* $(0, 1, \cdots, N)$.

The proof of Lemma 4.1 involves eliminating $\zeta_1, \zeta_2, \cdots, \zeta_N$ from equations (4.3). To facilitate this step, we introduce the function

$$\phi(z) = (z - \zeta_1)(z - \zeta_2)\cdots(z - \zeta_N)$$
$$= \sigma_0 z^N - \sigma_1 z^{N-1} + \cdots + (-1)^N \sigma_N, \qquad \sigma_0 = 1,$$

so that after multiplication by $\phi(z_j)$ equations (4.3) become

$$\lambda\phi'(z_j) + I_j\phi(z_j) = 0, \qquad j = 0, 1, \cdots, N.$$

Thus, with $\lambda_i = \lambda$ for $i \neq N$ and $\lambda_N = 0$,

$$(4.8)\qquad \sum_{i=0}^{N} (-1)^i [I_j z_j + \lambda_i(N - i)] z_j^{N-i-1} \sigma_i = 0, \qquad j = 0, 1, \cdots, N.$$

It is sufficient now to eliminate the σ_j, requiring that

$$(4.9)\qquad \Delta = \det \| [I_j z_j + \lambda_i(N - i)] z_j^{N-i-1} \| = 0; \qquad i, j = 0, 1, \cdots, N.$$

But Δ may be written in terms of the derivative operators D_j and the Vandermonde determinant $\mathscr{V}_N$ as

$$(4.10)\qquad \Delta = (I_0 + \lambda D_0)(I_1 + \lambda D_1)\cdots(I_N + \lambda D_N)\mathscr{V}_N.$$

As (4.10) is the same as (4.5) and (4.6), Lemma 4.1 has been established.

LEMMA 4.2. *In the notation of Lemma* 4.1, *let*

$$(4.11)\qquad E_{i_1 i_2 \ldots i_k} = \frac{D_{i_1} D_{i_2} \cdots D_{i_k} \mathscr{V}_N}{\mathscr{V}_N},$$

$$0 \leqq k \leqq N, \qquad 0 \leqq i_1 < i_2 < \cdots < i_k \leqq N.$$

Then

$$(4.12)\qquad E_{i_1 i_2 \ldots i_k} = \sum \frac{k_1!(k_2 - k_1)!\cdots(k_p - k_{p-1})!}{\prod_{r=1}^{k_1}(z_{i_r} - z_{j_1}) \prod_{r=k_1+1}^{k_2}(z_{i_r} - z_{j_2}) \cdots \prod_{r=k_{n-1}+1}^{k_n}(z_{i_r} - z_{j})},$$

where the sum is taken for all k_r *and* j_r *such that*

$$0 = k_0 < k_1 < k_2 < \cdots < k_p = k;$$

and such that $j_1, j_2, \cdots, j_p$ *run independently through all the values* $0, 1, 2, \cdots, N$ *with the exception of* $i_1, i_2, \cdots, i_k$.

In the proof of Lemma 4.2, we may use the symmetry of $E_{i_1 i_2 \ldots i_k}$ in the subscripts. It is sufficient therefore to compute $E_{012\ldots k}$. After cancellation of the factor $V_{k+1,k+2,\ldots,N}$ from the numerator and denominator, equation (4.11) reduces in this case to

$$E_{012\ldots k} = \frac{D_0 D_1 \cdots D_k [P(z_0)P(z_1) \cdots P(z_k)\mathscr{V}_k]}{P(z_0)P(z_1) \cdots P(z_k)\mathscr{V}_k},$$

where

$$P(z) = (z - z_{k+1})(z - z_{k+2}) \cdots (z - z_N).$$

Equation (4.12) now follows from equations (5.2) and (3) of a previous paper(8) in which the Lagrange Interpolation Formula was extended to functions of several variables.

LEMMA 4.3. *Under the hypotheses of Theorem* 4.1 *and in the notation of Lemma* 4.2 *the distinct critical points* $z_0, z_1, \cdots, z_N$ *of* $\Phi(x, y)$ *satisfy the equation*

$$\sum_{k=0}^{N} \lambda^k \sum \prod_{j=0}^{p-1} (k_{j+1} - k_j)!\, I_{i_{k_j+j} i_{k_j+j+1} \cdots i_{k_{j+1}+j}} \prod_{u=k+p}^{N} I_{i_u} = 0. \tag{4.13}$$

This is an identity connecting $G(x, y)$ with any $N+1$ distinct critical points of $\Phi(x, y)$.

To prove this lemma, we may substitute from (4.12) into (4.6) and then both interchange the summation order and renumber the indices so as to obtain:

$$P_k = \sum k_1!(k_2 - k_1)! \cdots (k - k_{p-1})!\, I_{i_{k+p}} I_{i_{k+p+2}} \cdots I_{i_N} F_0 F_1 \cdots F_{p-1} \tag{4.14}$$

where for $j = 0, 1, \cdots, p-1$,

$$F_j = \sum_{\nu=k_j+j+1}^{k_{j+1}+j+1} \frac{I_{i_\nu}}{\phi_j'(z_{i_\nu})} \tag{4.15}$$

and

$$\phi_j(z) = -\prod_{\nu=k_j+j+1}^{k_{j+1}+j+1} (z_{i_\nu} - z).$$

(8) M. Marden, *Kakeya's problem on the zeros of the derivative of a polynomial*, Trans. Amer. Math. Soc. **40** (1939), 355–368.

On the other hand, the function

$$I(z) = \int_B \frac{d\mu}{z - t}$$

has, due to (4.4), the successive divided differences:

$$\begin{aligned}
[z_1 z_2] &= \frac{I_2 - I_1}{z_2 - z_1} = -I_{12}, \\
[z_1 z_2 z_3] &= \frac{(-I_{13}) - (-I_{12})}{z_3 - z_2} = I_{123}, \\
\cdots & \qquad \cdots \\
[z_1 z_2 \cdots z_k] &= (-1)^{k-1} I_{123\ldots k}.
\end{aligned}$$

By a well-known formula for divided differences,

$$\begin{aligned}
[z_1 z_2 \cdots z_k] = & \frac{I_1}{(z_1 - z_2)(z_1 - z_3)\cdots(z_1 - z_k)} \\
& + \frac{I_2}{(z_2 - z_1)(z_2 - z_4)\cdots(z_2 - z_k)} \\
& + \cdots + \frac{I_k}{(z_k - z_1)(z_k - z_2)\cdots(z_k - z_{k-1})}.
\end{aligned}$$

Hence, from (4.15) follows

$$F_j = I_{i_{k_j+j+1} i_{k_j+j+2} \ldots i_{k_{j+1}+j+1}}$$

and from (4.5) and (4.14) follows (4.13).

Proof of Theorem 4.1. Let us suppose contrary to Theorem 4.1 that $\Phi(x, y)$ has $N+1$ distinct critical points $z_0, z_1, \cdots, z_N$ outside S. Then at each z_j $(j = 0,1,\cdots,N)$, B subtends an angle less than $\pi/(N+1)$. This means that, if points T_j be introduced as in (3.7),

$$0 \leqq \arg \frac{T_j - z_j}{t - z_j} < \frac{\pi}{N+1}$$

for $j = 0, 1, \cdots, N$ and for all $t \in B$. Thus

$$0 \leqq \arg \frac{(z_{i_1} - T_{i_1})(z_{i_2} - T_{i_2})\cdots(z_{i_k} - T_{i_k})}{(z_{i_1} - t)(z_{i_2} - t)\cdots(z_{i_k} - t)} < \frac{\pi k}{N+1} < \pi.$$

Hence,

$$0 \leqq \arg\left[(z_{i_1} - T_{i_1})(z_{i_2} - T_{i_2})\cdots(z_{i_k} - T_{i_k}) I_{i_1 i_2 \ldots i_k}\right] < \pi k/(N+1).$$

Multiplied by the factor $(z_0 - T_0)(z_1 - T_1)\cdots(z_N - T_N)$, each term in (4.13) therefore represents a vector which, if drawn from the origin, would lie in the sector

$$0 \leqq \theta < \pi$$

and the same would hold for their sum P_k. Each term in the sum on the left side of (4.13) would consequently also be represented by a vector in this sector, so that equation (4.13) would not be satisfied. In view of this contradiction, the assumption that $\Phi(x, y)$ has at least $N + 1$ critical points outside S has been shown to be incorrect and that therefore at most N critical points may lie outside S as was to be proved.

If $\Phi(x, y)$ has multiple critical points, an identity connecting $G(x, y)$ with critical points of $\Phi(x, y)$ of total multiplicity $N+1$ may be derived from (4.13) by allowing the appropriate z_j to coalesce. This identity has the same form as (4.13), with the subscripts not necessarily distinct, and hence the proof of Theorem 4.1 remains valid when the critical points are not necessarily simple.

When N is specialized to be zero, the region S reduces to $H(B)$, the convex hull of B. Thus Theorem 4.1 is a generalization of Theorem 1.1.

5. **Arbitrary** λ_k. In the general case, we may try to proceed as above. On assuming that $\Phi(x, y)$ has $N + 1$ critical points $z_0, z_1, \cdots, z_N$ outside S and by substituting the z_k into $\Psi(z)$ of formula (3.1) we may obtain a system of $N + 1$ equations in the N unknowns, the ζ_{jk}. Our next step would be to eliminate the ζ_{jk} from these equations, thereby to get a direct relation between the configuration B and the critical points z_k. From this relation we should expect to determine S.

However, the elimination of the ζ_{jk} is in practice quite involved. We therefore limit our discussion of arbitrary λ_k to the case $m = 2$, $n_1 = n_2 = 1$. For this case we shall establish

THEOREM 5.1. *Let R be an infinite region bounded by a finite Jordan configuration B and containing the two distinct points ζ_1 and ζ_2. Denote by $G(x, y)$ the Green's function with pole at infinity for R and by $g_k(x, y)$ that for the circular region*

$$C_k\colon |z - \zeta_k| > \rho_k, \qquad \rho_k > 0, \qquad k = 1, 2,$$

where ρ_k is chosen so that $R \cap C_1 \cap C_2 \neq 0$. Then the function

$$\Phi(x, y) = G(x, y) + \lambda_1 g_1(x, y) + \lambda_2 g_2(x, y), \qquad \lambda_1 > 0, \quad \lambda_2 > 0, \tag{5.1}$$

has at most two critical points (counted with their multiplicities) outside of the star-shaped region S comprised of all points from which $H(B)$ subtends an angle of at least $\pi/6$.

As a first step in proving Theorem 5.1, we shall establish

LEMMA 5.1. *Any three distinct critical points* z_0, z_1, z_2 *of* $\Phi(x, y)$, *defined by* (5.1), *satisfy the equation*

$$
\begin{aligned}
(5.2)\qquad & I_0^2 I_1^2 I_2^2 + \lambda_1\lambda_2(I_0^2 I_{12}^2 + I_1^2 I_{20}^2 + I_2^2 I_{01}^2) + (\lambda_1 + \lambda_2)^2 \lambda_1 \lambda_2 I_{012}^2 \\
& + (\lambda_1 + \lambda_2) I_0 I_1 I_2 (I_0 I_{12} + I_1 I_{20} + I_2 I_{01}) + 4\lambda_1\lambda_2 I_0 I_1 I_2 I_{012} \\
& + (\lambda_1^2 + \lambda_2^2)(I_0 I_1 I_{02} I_{12} + I_1 I_2 I_{01} I_{02} + I_0 I_2 I_{10} I_{12}) \\
& + 2(\lambda_1 + \lambda_2)\lambda_1\lambda_2(I_0 I_{12} + I_1 I_{02} + I_2 I_{01}) I_{012} \\
& + (\lambda_2 - \lambda_1)^2(\lambda_1 + \lambda_2) I_{01} I_{12} I_{20} = 0.
\end{aligned}
$$

We note that equation (5.2) is an identity connecting $G(x, y)$ with any three distinct critical points of $\Phi(x, y)$.

If $\lambda_1 = \lambda_2 = \lambda$, equation (5.2) reduces to the equation

$$\{I_0 I_1 I_2 + \lambda(I_0 I_{12} + I_2 I_{01} + I_1 I_{20}) + 2\lambda^2 I_{012}\}^2 = 0$$

which is essentially the special case $N = 2$ of equations (4.5) and (4.13).

To derive equation (5.2), we begin with the equations

$$(5.3)\qquad I_k + \frac{\lambda_1}{z_j - \zeta_1} + \frac{\lambda_2}{z_j - \zeta_2} = 0, \qquad j = 0, 1, 2,$$

satisfied by the critical points z_0, z_1, z_2 and eliminate ζ_1. This leads to the quadratic equations

$$(5.4)\quad \left(\frac{\lambda_2}{z_i - \zeta_2} + I_i\right)\left(\frac{\lambda_2}{z_j - \zeta_2} + I_j\right) + \lambda_1\left[\frac{\lambda_2}{(z_i - \zeta_2)(z_j - \zeta_2)} + I_{ij}\right] = 0$$

where $i \neq j$, $i, j = 0, 1, 2$. If now we eliminate ζ_2 from a pair of equations (5.4), we obtain (5.2).

To prove Theorem 5.1, we suppose $\Phi(x, y)$ to have three distinct critical points z_0, z_1, z_2 outside S and choose the points T_j as in (3.7) so that

$$0 \leqq \arg \frac{T_j - z_j}{t - z_j} < \frac{\pi}{6}, \qquad j = 0, 1, 2.$$

If every term in (5.2) is multiplied by

$$(T_0 - z_0)^2 (T_1 - z_1)^2 (T_2 - z_2)^2,$$

each resulting product will have an argument in the range $0 \leqq \theta < \pi$ and hence the sum cannot vanish. Thus, at most two critical points of $\Phi(x, y)$ may lie outside S.

Relations corresponding to (5.2) may be derived for any critical points of $\Phi(x, y)$ having a total multiplicity of three if the appropriate z_j are allowed to coalesce. The new relations are however of the same form as (5.2) and hence the proof of Theorem 5.1 remains valid for multiple critical points.

It is to be noted that for the region S, comprised of all points from which $H(B)$ subtends an angle of at least ϕ, the angle ϕ is $\pi/6$ in Theorem 5.1 but is $\pi/3$ in the case $N = 2$ of Theorem 4.1. The region S of Theorem 5.1 therefore contains the region S of case $N = 2$, Theorem 4.1. This is to be expected since the λ_1 and λ_2 are arbitrary for Theorem 5.1, whereas $\lambda_1 = \lambda_2$ for Theorem 4.1.

6. **Extensions.** We may extend Theorems 3.1, 4.1, and 5.1 to certain infinite regions R with a finite boundary B, not necessarily a Jordan configuration, in the same manner as Walsh has extended Theorem 1.1.

If R possesses a Green's function with pole at infinity, we may approximate to B by the level curve

$$B_\mu: \qquad G(x, y) = \mu, \qquad \mu > 0,$$

and apply the above theorems to locate the critical points of the function

$$\Phi_\mu(x, y) = [G(x, y) - \mu] + \sum_{k=1}^{m} \lambda_k g_k(x, y).$$

If S_μ denotes the corresponding region S, we find that

$$H(B_\mu) \to H(B), \quad S_\mu \to S \text{ as } \mu \to 0.$$

If R does not possess a Green's function with pole at infinity and if its generalized Green's function with pole at infinity is not the infinite constant, we may approximate to R by regions R' contained in R, possessing Green's functions and monotonically approaching R.

Further extensions are possible to Green's function with a finite pole $\alpha \in R$, by inverting in a circle about α. For example, Theorem 4.1 may be transformed into the following.

THEOREM 6.1. *Let R be a region bounded by a finite Jordan configuration and containing the distinct points α and ζ_{jk} where*

$$j = 1, 2, \cdots, n_k; \; k = 1, 2, \cdots, m; \; N = n_1 + n_2 + \cdots + n_m.$$

Let M be the intersection of the m regions

$$M_k: |p_k(z)| > \rho_k |z - \alpha|^{n_k}, \qquad \rho_k > 0,$$

where $p_k(z) = (z - \zeta_{1k})(z - \zeta_{2k}) \cdots (z - \zeta_{n_k k})$, $k = 1, 2, \cdots, m$ and the ρ_k are chosen so that $R \cap M \neq 0$. Furthermore, let the positive constant ω and the point β be so chosen that the circular region

$$K: \qquad |z - \beta| \leqq \omega |z - \alpha|$$

contains point α but not configuration B. Then at most N critical points (counted with their multiplicities) of the function

$$\Phi(x, y) = G(x, y) + \sum_{k=1}^{m} \lambda n_k g_k(x, y), \qquad \lambda > 0,$$

lie in $M \cap R$ outside the circular region

$$K': \qquad |z - \beta| \leqq \omega |z - \alpha| \csc [\pi/2(N + 1)].$$

Proof. Let us make the transformation

$$w = (z - \beta)/(z - \alpha).$$

The region R is mapped into an infinite region, the regions M_k are mapped into the regions L_k and the regions K and K' are mapped into disks with the origin as common center and with radii ω and $\omega \csc [\pi/2(N + 1)]$.

The Green's functions with pole at α are transformed into those with pole at infinity. We may now apply Theorem 4.1 and then apply the inverse transformation to complete the proof of Theorem 6.1.

7. **Interpretation.** In order to obtain a clearer view of the analogy between Theorems 1.3 and 1.4, we shall now show that $\Phi(x, y)$ is essentially the Green's function, with pole at infinity, for a certain region R' containing $(R \cap L)$.

Let us suppose that R is the lemniscatic region $L_0: |p_0(z)| > \rho_0$ where ρ_0 is a positive constant and $p_0(z)$ is a polynomial of degree n_0. (If R is not a lemniscatic region, we may approximate to it by means of a lemniscatic region.) In this case

$$G(x, y) \equiv g_0(x, y) = (1/n_0) \log (|p_0(z)|/\rho_0).$$

After choosing the lemniscatic regions $L_k: |p_k(z)| > \rho_k$, $k = 1, 2, \cdots, m$, as in Theorem 1.4, we form the function

$$\Omega(x, y) = \Lambda^{-1}\Phi(x, y) = \Lambda^{-1} \sum_{k=0}^{m} \lambda_k g_k(x, y)$$

where $\lambda_0 = 1$, $\lambda_k > 0$ all k, and $\Lambda = \sum_{k=0}^{m} \lambda_k$. The function $\Omega(x, y)$ is harmonic at the finite points in $\bigcap_{k=0}^{m} L_k$ and behaves for large $|z|$ like $\log |z|$. Thus $\Omega(x, y)$ is the Green's function, with pole at infinity, for the infinite region R' bounded by the Jordan curves $\Omega(x, y) = 0$.

To study the relation of the regions R' with R and L, we choose any point (x_1, y_1) on the boundary of any region L_k but not simultaneously on the boundaries of all L_k. Since $\Omega(x_1, y_1) > 0$, such a point (x_1, y_1) lies in R'. Let us denote by Γ_k that subset of curves $\Omega(x, y) = 0$ which lies exterior to region L_k. If we set

$$M_{jk} = \max g_j(x, y) \text{ for all } (x, y) \notin L_k,$$

$$\rho'_k = \rho_k \exp \left[-n_k \lambda_k^{-1} \sum_{j=0,\, j \neq k}^{m} \lambda_j M_{jk} \right],$$

we may say further that Γ_k lies interior to the lemniscatic region

$$L'_k: |p_k(z)| > \rho'_k.$$

Thus $R \cap L$ is contained in R'.

In particular Γ_0 lies exterior to L_0 and so lies in $H(B)$, where B is the boundary of $R \equiv L_0$ and therefore lies also in the starshaped region S of Theorem 1.4.

Thus, by Theorem 1.4, the location of all but at most N critical points of the Green's function $\Omega(x, y)$ of region R' is determined relative to just the part Γ_0 of the boundary of R'. This result is analogous to Theorem 1.3 whereby the location of all but at most $n - p$ critical points of an nth degree polynomial of $f(z)$ is determined relative to just p of the n zeros of $f(z)$.

UNIVERSITY OF WISCONSIN,
MILWAUKEE, WISCONSIN