BILATERAL BIRTH AND DEATH PROCESSES(1)

BY
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1. **Introduction and summary.** A bilateral birth and death process is a continuous parameter Markov process with path functions $X(t)$ taking on integer values and with stationary transition probabilities

$$P_{ij}(t) = \Pr\{X(t+s) = j \mid X(s) = i\}$$

satisfying the order relations

$$\begin{aligned} P_{i,i+1}(t) &= \lambda_i t + o(t), \\ P_{i,i}(t) &= 1 - (\lambda_i + \mu_i)t + o(t), \\ P_{i,i-1}(t) &= \mu_i t + o(t), \end{aligned}$$

as $t \to 0$. The λ_i, μ_i are positive constants which determine the rate of transit on from state i to states $i+1$, $i-1$.

The Markov property and the given order relations lead to the backward differential equation

$$P'(t) = AP(t), \qquad t \geqq 0, \tag{1.1}$$

where $P(t) = (P_{ij}(t))$, $i,j = 0, \pm 1, \pm 2, \cdots$, and $A = (a_{ij})$ with $a_{i,i+1} = \lambda_i$, $a_{ii} = -(\lambda_i + \mu_i)$, $a_{i\,i-1} = \mu_i$, and $a_{ij} = 0$ if $|i-j| > 1$. Under additional regularity conditions, the forward equation

$$P'(t) = P(t)A, \qquad t \geqq 0, \tag{1.2}$$

can also be derived. The initial condition is

$$P(0) = I, \tag{1.3}$$

where I is the identity matrix. In order to be a transition probability matrix, $P(t)$ should satisfy

$$P(t) \geqq 0, \qquad P(t)e \leqq e, \tag{1.4}$$

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where e is a vector with all components equal to one. The inequality in the second member of (1.4) allows for the possibility that $X(t) = \pm\infty$ with positive probability. The Markov property also leads immediately to the Chapman-Kolmogorov equation or semi-group property

$$P(s+t) = P(s)P(t). \tag{1.5}$$

Associated with the matrix A are several systems of orthogonal polynomials. The purpose of this paper is to obtain an explicit expression for the Laplace transform of every solution of (1.1)–(1.5) in terms of these polynomials and some associated limit functions. The particularly simple form of the polynomials makes it possible to consider the asymptotic behavior of the Laplace transform near the origin and in this way to find limit theorems for the underlying process. Some of these limit theorems will be considered in a future paper.

The important properties of the associated systems of orthogonal polynomials and their limit functions are given in §2. These are used extensively in the remainder of the paper. In §3 the Laplace transform of the minimal solution of (1.1)–(1.5) is obtained. Necessary and sufficient conditions are given for this to be the unique solution. The Laplace transforms of all solutions are found in §4 for the cases where nonuniqueness exists.

The Laplace transform of the general solution of the analogues of (1.1)–(1.5) for the unilateral birth and death process was derived by Feller in [3]. Karlin and McGregor found expressions for Laplace transforms in terms of orthogonal polynomials for these unilateral processes in [5]. Feller has also treated bilateral pure birth processes in [2]. Some probabilistic results which exhibit the usefulness of the representations in terms of the polynomials are given by Karlin and McGregor in [6] and [7].

The methods used in the general construction are similar to those employed by Feller for the unilateral process in [3]. It would also have been possible to obtain these results from Feller's work on general Kolmogorov differential equations in [2], but the degree of difficulty would be about the same and under these circumstances it was considered more desirable to keep the paper as self-contained as possible.

2. **The related systems of orthogonal polynomials.** The purpose of §2 is to collect some results on the associated systems of orthogonal polynomials and their limit functions. Most of these results are well known and, in fact, some of them are summarized by Karlin and McGregor in [5]. Others are essentially contained in some of Feller's proofs in [3]. However, many of the statements do not seem to be readily available for reference so they are included here for completeness.

The fundamental systems of orthogonal polynomials $\{Q_n^\alpha(s)\}$ are defined by

$$sQ_n^\alpha(s) = \mu_n Q_{n-1}^\alpha(s) - (\lambda_n + \mu_n)Q_n^\alpha(s) + \lambda_n Q_{n+1}^\alpha(s),$$

for all n, $\alpha = 0, 1$, or more compactly by

$$sQ^\alpha(s) = AQ^\alpha(s), \qquad \alpha = 0, 1,$$

and the normalizing conditions

$$Q^\alpha_\beta(s) \equiv \delta_{\alpha\beta}, \qquad \alpha, \beta = 0, 1.$$

If $Q_0(s)$ and $Q_1(s)$ are specified, that completely determines a solution of $sQ(s) = AQ(s)$, so the general solution of this equation is

$$Q_n(s) = Q_0(s)Q^0_n(s) + Q_1(s)Q^1_n(s). \tag{2.0}$$

In much of the following, the polynomials will be considered as functions of n with s fixed and the dependence on s will be suppressed when it is convenient to do so. The domain of the polynomials that will be of interest is the positive reals, and in the results stated below it is to be assumed throughout that s, u and v are positive.

A sequence of positive constants π_n is defined by

$$\pi_0 = 1, \qquad \pi_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}, \quad \pi_{-n} = \frac{\mu_0\mu_{-1}\cdots\mu_{-n+1}}{\lambda_{-1}\lambda_{-2}\cdots\lambda_{-n}}, \qquad n > 0.$$

This sequence is defined in such a way that $\lambda_n\pi_n = \mu_{n+1}\pi_{n+1}$ for all n. Another important class of polynomials is defined by

$$H^\alpha_{n+1}(s) = \lambda_n\pi_n[Q^\alpha_{n+1}(s) - Q^\alpha_n(s)], \qquad \alpha = 0, 1. \tag{2.1}$$

The recurrence relation may now be written

$$sQ^\alpha_n\pi_n = \lambda_n\pi_n(Q^\alpha_{n+1} - Q^\alpha_n) - \lambda_{n-1}\pi_{n-1}(Q^\alpha_n - Q^\alpha_{n-1}) = H^\alpha_{n+1} - H^\alpha_n,$$

and, consequently

$$s\sum_{k=m}^{n} Q^\alpha_k\pi_k = \lambda_n\pi_n(Q^\alpha_{n+1} - Q^\alpha_n) - \lambda_{m-1}\pi_{m-1}(Q^\alpha_m - Q^\alpha_{m-1}) = H^\alpha_{n+1} - H^\alpha_m. \tag{2.2}$$

In particular,

$$s\sum_{k=1}^{n} Q^1_k\pi_k = \lambda_n\pi_n(Q^1_{n+1} - Q^1_n) - \lambda_0, \tag{2.3}$$

and it follows by induction that

$$1 = Q^1_1 < Q^1_2 < \cdots < Q^1_n < \cdots.$$

A similar method shows that

$$0 = Q^1_0 > Q^1_{-1} > \cdots > Q^1_{-n} > \cdots.$$

Therefore Q^1_n is an increasing sequence so that $H^1_n > 0$ for all n. Furthermore (2.2) and th positivity of Q^1_n for positive n imply that H^1_n is an increasing sequence for $n \geqq 1$. The corresponding results for the other systems are: Q_n^0 is a decreasing sequence, $H_n^0 < 0$, and $-H_n^0$ is a decreasing sequence for $n \leqq 1$.

It is clear from the monotone nature of the $\{Q_n^\alpha\}$ and $\{H_n^\alpha\}$ sequences that they will either converge or tend to infinity. By considering the sequences as polynomials, it is possible to apply the following result of Stieltjes in order to obtain a criterion for when the convergence takes place. A proof is given in [5, pp. 504–505].

The following statements are equivalent:

(1) As $n \to \infty$, $Q_n^\alpha(s)$ converges for every complex s, uniformly in every circle $|s| \leqq R$;

(2) $Q_n^\alpha(s)$ is bounded as $n \to \infty$ for at least one $s > 0$;

(3) The series

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n \pi_n} \sum_{i=1}^{n} \pi_i \tag{2.4}$$

converges.

For the $\{H_n^\alpha(s)\}$ systems the series (2.4) should be replaced by

$$\sum_{n=1}^{\infty} \pi_n \sum_{i=1}^{n-1} \frac{1}{\lambda_i \pi_i}, \tag{2.5}$$

the result being otherwise unchanged. For the $\{Q_n^\alpha(s)\}$ systems again, but for $n \to -\infty$, the criterion for convergence is the convergence of the series

$$\sum_{n=-\infty}^{-1} \frac{1}{\lambda_n \pi_n} \sum_{i=n+1}^{-1} \pi_i, \tag{2.6}$$

and for the $\{H_n^\alpha(s)\}$ systems as $n \to -\infty$, the series is

$$\sum_{n=-\infty}^{-1} \pi_n \sum_{i=n}^{-1} \frac{1}{\lambda_i \pi_i}. \tag{2.7}$$

When the limit functions exist they will be denoted by $Q_\infty^\alpha(s)$, $H_\infty^\alpha(s)$, $Q_{-\infty}^\alpha(s)$, and $H_{-\infty}^\alpha(s)$. Because $Q_n^1 \to \infty$ when it does not converge, the expression $\lim_{n\to\infty} 1/Q_n^1$ will always be written $1/Q_\infty^1$ with the obvious interpretation that this is to be zero when Q_n^1 diverges. Analogous remarks apply to the other systems and to $n \to -\infty$.

The convergence of the various sequences of polynomials will play an important role in the subsequent development. In order to be able to describe easily the various possibilities that arise, the boundary terminology of Feller [1; 2; 3] will be employed. The boundary at infinity will be called:

regular	if (2.4) and (2.5) converge,
exit	if (2.4) converges and (2.5) diverges,
entrance	if (2.4) diverges and (2.5) converges,
natural	if (2.4) and (2.5) diverge.

The corresponding definitions apply to the boundary at minus infinity with the series (2.4), (2.5) replaced by (2.6), (2.7), respectively.

If Q_n is defined by (2.0), then it follows from (2.2) that

$$s \sum_{k=m}^{n} Q_k \pi_k = H_{n+1} - H_m, \tag{2.8}$$

where $H_n = Q_0 H_n^0 + Q_1 H_n^1$. A computation shows that

$$\begin{aligned}(v-u)\sum_{k=m}^{n} Q_k^\alpha(u) Q_k^\beta(v)\pi_k = H_{n+1}^\beta(v)Q_n^\alpha(u) - H_{n+1}^\alpha(u)Q_n^\beta(v) \\ - H_m^\beta(v)Q_{m-1}^\alpha(u) + H_m^\alpha(u)Q_{m-1}^\beta(v)\end{aligned}$$

for α, $\beta = 0, 1$. Therefore, if

$$f_n^\alpha(s) = \sum_{\gamma=0}^{1} f_\gamma^\alpha(s) Q_n^\gamma(s), \quad g_n^\alpha(s) = \sum_{\gamma=0}^{1} f_\gamma^\alpha(s) H_n^\gamma(s), \qquad \alpha = 1, 2,$$

then

$$\begin{aligned}(v-u)\sum_{k=m}^{n} f_k^1(u) f_k^2(v)\pi_k &= g_{n+1}^2(v) f_n^1(u) - g_{n+1}(u) f_n^2(v) \\ &\quad - g_m^2(v) f_{m-1}^1(u) + g_m^1(u) f_{m-1}^2(v).\end{aligned} \tag{2.9}$$

Some further identities that are now readily obtained are:

$$\lambda_n \pi_n (Q_{n+1}^1 Q_n^0 - Q_{n+1}^0 Q_n^1) = H_{n+1}^1 Q_n^0 - H_{n+1}^0 Q_n^1 \equiv \lambda_0, \tag{2.10}$$

$$H_{n+1}^0 H_n^1 - H_{n+1}^1 H_n^0 = s\pi_n \lambda_0, \tag{2.11}$$

$$\lambda_0 \sum_{k=1}^{n} \frac{1}{\lambda_k \pi_k Q_k^1 Q_{k+1}^1} = -\frac{Q_{n+1}^0}{Q_{n+1}^1}, \quad \lambda_0 \sum_{k=m}^{-1} \frac{1}{\lambda_k \pi_k Q_k^0 Q_{k+1}^0} = -\frac{Q_m^1}{Q_m^0}, \tag{2.12}$$

$$\lambda_0 s \sum_{k=1}^{n} \frac{\pi_k}{H_k^1 H_{k+1}^1} = 1 + \frac{H_{n+1}^0}{H_{n+1}^1}, \quad \lambda_0 s \sum_{k=m}^{0} \frac{\pi_k}{H_k^0 H_{k+1}^0} = 1 + \frac{H_m^1}{H_m^0}. \tag{2.13}$$

The series in (2.12) converge, for consider

$$\begin{aligned}\lambda_0 \sum_{k=1}^{n} \frac{1}{\lambda_k \pi_k Q_k^1 Q_{k+1}^1} &= \sum_{k=1}^{n} \frac{H_1^1}{H_{k+1}^1}\left\{\frac{1}{Q_k^1} - \frac{1}{Q_{k+1}^1}\right\} < \sum_{k=1}^{n}\left\{\frac{1}{Q_k^1} - \frac{1}{Q_{k+1}^1}\right\} \\ &= 1 - \frac{1}{Q_{n+1}^1} < 1.\end{aligned}$$

A similar argument applies to the other series. Therefore, for $s > 0$, let

$$U_1(s) = \lambda_0 \sum_{k=1}^{\infty} \frac{1}{\lambda_k \pi_k Q_k^1(s) Q_{k+1}^1(s)} = -\lim_{n\to\infty} \frac{Q_n^0(s)}{Q_n^1(s)},$$

$$U_0(s) = \lambda_0 \sum_{k=-\infty}^{-1} \frac{1}{\lambda_k \pi_k Q_k^0(s) Q_{k+1}^0(s)} = -\lim_{n\to-\infty} \frac{Q_n^1(s)}{Q_n^0(s)}.$$

It is important to observe here that no assumption has been made concerning the limiting behavior of the individual $\{Q_n^\alpha\}$ sequences so that the functions U_0, U_1 always exist. A careful examination of the convergence argument gives the upper estimates

$$U_1 < 1 - \frac{1}{Q_\infty^1}, \qquad U_0 < 1 - \frac{1}{Q_{-\infty}^0}. \tag{2.14}$$

In particular, this implies the positivity of the function $U(s) = 1 - U_0(s)U_1(s)$.

The convergence of the series in (2.13) now follows, for, by (2.10),

$$1 - \lambda_0 s \sum_{k=1}^{n} \frac{\pi_k}{H_k^1 H_{k+1}^1} = -\frac{H_{n+1}^0}{H_{n+1}^1} = \frac{\lambda_0}{H_{n+1}^1 Q_n^1} - \frac{Q_n^0}{Q_n^1}. \tag{2.15}$$

The right hand side is known to have a limit as $n \to \infty$, and therefore

$$1 - \lambda_0 s \sum_{k=1}^{\infty} \frac{\pi_k}{H_k^1 H_{k+1}^1} = -\lim_{n\to\infty} \frac{H_n^0}{H_n^1} = \frac{\lambda_0}{H_\infty^1 Q_\infty^1} + U_1. \tag{2.16}$$

Similarly

$$1 - \lambda_0 s \sum_{k=-\infty}^{0} \frac{\pi_k}{H_k^0 H_{k+1}^0} = -\lim_{n\to-\infty} \frac{H_n^1}{H_n^0} = -\frac{\lambda_0}{H_{-\infty}^0 Q_{-\infty}^0} + U_0. \tag{2.17}$$

The following lemma will be useful throughout the paper in the proof of certain positivity results.

LEMMA 2.1. *Let $f_n = f_0 Q_n^0 + f_1 Q_n^1$. Then $f_n \geqq 0$ for all n if and only if*

$$f_1 - U_1 f_0 \geqq 0 \text{ and } f_0 - U_0 f_1 \geqq 0. \tag{2.18}$$

These conditions are equivalent to $f_\infty \geqq 0$ and $f_{-\infty} \geqq 0$, respectively, when these functions exist.

Proof. Suppose $f_n \geqq 0$ for all n. *Then, for $n \geqq 1$,*

$$f_1 + f_0 \frac{Q_n^0}{Q_n^1} = \frac{f_n}{Q_n^1} \geqq 0,$$

and the first inequality in (2.18) results from letting $n \to \infty$. The other inequality follows from the non-negativity of f_n/Q_n^0 for negative n. Now let (2.18) hold. Then $f_1 \geqq U_1 f_0 \geqq U_1 U_0 f_1$ or $f_1 U \geqq 0$. Therefore $f_1 \geqq 0$ since $U > 0$, and then $f_0 \geqq U_0 f_1 \geqq 0$. For $n \geqq 2, f_n = Q_n^1(f_1 + f_0 Q_n^0/Q_n^1)$, which shows that f_n is non-negative since Q_n^0/Q_n^1 decreases monotonically to $-U_1$. The proof for negative n is similar.

The functions

$$F_n^1(s) = U_1(s)Q_n^1(s) + Q_n^0(s), \qquad F^0(s) = Q_n^1(s) + U_0(s)Q_n^0(s)$$

are seen to be non-negative by an application of Lemma 2.1. Furthermore, if

$$G_n^1(s) = U_1(s)H_n^1(s) + H_n^0(s), \qquad G_n^0(s) = H_n(s) + U_0(s)H_n^0(s),$$

then by (2.8) and (2.9),

$$s \sum_{k=m}^{n} F_k^\alpha \pi_k = G_{n+1}^\alpha - G_m^\alpha, \tag{2.19}$$

$$\begin{aligned}(v-u)\sum_{k=m}^{n} F_k^\alpha(u)F_k^\beta(v)\pi_k &= G_{n+1}^\beta(v)F_n^\alpha(u) - G_{n+1}^\alpha(u)F_n^\beta(v)\\ &\quad - G_m^\beta(v)F_{m-1}^\alpha(u) + G_m^\alpha(u)F_{m-1}^\beta(v),\end{aligned} \tag{2.20}$$

for $\alpha, \beta = 0,1$. It follows from (2.10) that

$$G_{n+1}^1 F_n^0 - G_{n+1}^0 F_n^1 = -\lambda_0 U. \tag{2.21}$$

Some convergence properties of the $\{F_n^\alpha\}$, $\{G_n^\alpha\}$ will be the subject of the next two lemmas.

LEMMA 2.2.

$$\lim_{n\to\infty} F_n^1 = \begin{cases} 0 & \textit{if (2.4) converges,}\\ \lambda_0/H_\infty^1 & \textit{if (2.4) diverges.}\end{cases}$$

$$\lim_{n\to-\infty} F_n^0 = \begin{cases} 0 & \textit{if (2.6) converges,}\\ -\lambda_0/H_{-\infty}^0 & \textit{if (2.6) diverges.}\end{cases}$$

$$\lim_{n\to\infty} G_n^1 = -\lambda_0/Q_\infty^1; \qquad \lim_{n\to-\infty} G_n^0 = \lambda_0/Q_{-\infty}^0.$$

Proof. The proof will be given for F_n^1 and G_n^1; the other two results may be obtained in a similar manner. For $n \geqq 1$, by (2.12),

$$\begin{aligned}F_n^1 &= Q_n^1\left(U_1 + \frac{Q_n^0}{Q_n^1}\right) = \lambda_0 Q_n^1 \sum_{k=n}^{\infty} \frac{1}{H_{k+1}^1}\left\{\frac{1}{Q_k^1} - \frac{1}{Q_{k+1}^1}\right\}\\ &\leqq \lambda_0 \frac{Q_n^1}{H_{n+1}^1}\sum_{k=n}^{\infty}\left\{\frac{1}{Q_k^1} - \frac{1}{Q_{k+1}^1}\right\} = \frac{\lambda_0}{H_{n+1}^1} - \frac{\lambda_0 Q_n^1}{H_{n+1}^1 Q_\infty^1},\end{aligned} \tag{2.22}$$

and by (2.15), (2.16),

$$\begin{aligned}F_n^1 &= Q_n^1\left(-\lambda_0 s \sum_{k=n+1}^{\infty} \frac{\pi_k}{H_k^1 H_{k+1}^1} - \frac{\lambda_0}{H_\infty^1 Q_\infty^1} + \frac{\lambda_0}{Q_n^1 H_{n+1}^1}\right)\\ &\geqq -\lambda_0 s \sum_{k=n+1}^{\infty} \frac{\pi_k Q_k^1}{s Q_k^1 \pi_k}\left\{\frac{1}{H_k^1} - \frac{1}{H_{k+1}^1}\right\} - \frac{\lambda_0 Q_n^1}{H_\infty^1 Q_\infty^1} + \frac{\lambda_0}{H_{n+1}^1}\\ &= \frac{\lambda_0}{H_\infty^1} - \frac{\lambda_0 Q_n^1}{H_\infty^1 Q_\infty^1}.\end{aligned}$$

Now, when $n \to \infty$, the upper and lower estimates both converge to the stated limit for F_n^1. It follows from (2.10) that for $n \geqq 1$,

$$G_{n+1}^1 = \frac{H_{n+1}^1}{Q_n^1} F_n^1 - \frac{\lambda_0}{Q_n^1} \geqq - \frac{\lambda_0}{Q_n^1}. \tag{2.23}$$

Using the upper bound for F_n^1 obtained in (2.22),

$$G_{n+1}^1 \leqq \frac{H_{n+1}^1}{Q_n^1}\left(\frac{\lambda_0}{H_{n+1}^1} - \frac{\lambda_0 Q_n^1}{H_{n+1}^1 Q_\infty^1}\right) - \frac{\lambda_0}{Q_n^1} = - \frac{\lambda_0}{Q_\infty^1},$$

and therefore G_n^1 converges to $-\lambda_0/Q_\infty^1$.

An important consequence of this lemma is that $F_n^1(u)\,G_n^1(v) \to 0$ as $n \to \infty$, and that $F_n^0(u)\,G_n^0(v) \to 0$ as $n \to -\infty$. The other limits of the $\{F_n^\alpha\}$ sequences will exist only in certain cases. If Q_∞^1 exists, then $\lim_{n\to\infty} F_n^0 = Q_\infty^1 U$, and if $Q_{-\infty}^0$ exists, then $\lim_{n\to-\infty} F_n^1 = Q_{-\infty}^0 U$.

LEMMA 2.3. *If ∞ is an entrance boundary, then $\lim_{n\to\infty} G_{n+1}^1(v)\,Q_n^1(u) = 0$; if ∞ is an exit boundary, then $\lim_{n\to\infty} H_{n+1}^1(v)\,F_n^1(u) = 0$. The corresponding limit relations are true at $-\infty$, the correct superscripts now being zeroes.*

Proof. Suppose infinity is an entrance boundary. Then (2.23) implies $G_{n+1}^1 Q_n^1 = H_{n+1}^1 F_n^1 - \lambda_0$ which tends to zero by Lemma 2.2. It remains to prove that $Q_n^1(u)/Q_n^1(v)$ is bounded. On the other hand, if infinity is an exit boundary, it follows from (2.22) that

$$0 \leqq H_{n+1}^1 F_n^1 \leqq \lambda_0 - \frac{\lambda_0 Q_n^1}{Q_\infty^1}.$$

Since the right hand side tends to zero, in this case it remains to prove that $H_n^1(v)/H_n^1(u)$ is bounded. This boundedness will be proved for the $\{H_n^1\}$ system only, the result for the $\{Q_n^1\}$ being somewhat easier since $Q_n^1(0)$ may be used as a normalization. Since

$$\frac{H_n^1(v)}{H_n^1(u)} = \exp\left[\int_u^v \frac{H_n^{1'}(s)}{H_n^1(s)}\, ds\right]$$

it will suffice to obtain bounds on $H_n^{1'}(s)/H_n^1(s)$ which are uniform on a fixed (u,v) interval. Differentiation of (2.3) yields

$$\frac{H_n^{1'}(s)}{H_n^1(s)} = \frac{\sum_{k=1}^{n-1} Q_k^1(s)\pi_k + s\sum_{k=1}^{n-1} Q_k^{1'}(s)\pi_k}{s\sum_{k=1}^{n-1} Q_k^1(s)\pi_k + \lambda_0},$$

and it is now apparent that it will be sufficient to prove that $0 \leqq Q_k^{1'}(s)/Q_k^1(s) \leqq M$.

Since $\{Q_n^1(s)\}$ is an orthogonal system on $-\infty < s \leqq 0$ [5, p. 494], $Q_n^1(s)$ has $n-1$ distinct negative roots $\xi_1, \xi_2, \cdots, \xi_{n-1}$. Then

$$Q_n^1(s) = Q_n^1(0) \prod_{i=1}^{n-1} \left(1 - \frac{s}{\xi_i}\right)$$

so that

$$0 \leqq \frac{Q^{1\prime}(s)}{Q_n^1(s)} = \sum_{i=1}^{n-1} \frac{1}{s - \xi_i} \leqq -\sum_{i=1}^{n-1} \frac{1}{\xi_i} = \frac{Q^{1\prime}(0)}{Q_n^1(0)}.$$

Summing the differentiated version of (2.3) leads to

$$\frac{Q^{1\prime}(0)}{Q_n^1(0)} = \frac{\sum_{k=1}^{n-1} \frac{1}{\lambda_k \pi_k} \sum_{i=1}^{k} Q_i^1(0)\pi_i}{Q_n^1(0)} \leqq \sum_{k=1}^{\infty} \frac{1}{\lambda_k \pi_k} \sum_{i=1}^{k} \pi_i,$$

which completes the proof.

When the various limits involved exist, the following functions will be quite useful:

$$A_1(s) = Q_\infty^1(s)Q_{-\infty}^0(s) - Q_\infty^0(s)Q_{-\infty}^1(s), \qquad A_2(s) = H_{-\infty}^1(s)Q_\infty^0(s) - Q_\infty^1(s)H_{-\infty}^0(s),$$

$$A_3(s) = H_\infty^1(s)Q_{-\infty}^0(s) - Q_{-\infty}^1(s)H_\infty^0(s), \qquad A_4(s) = H_{-\infty}^1(s)H_\infty^0(s) - H_\infty^1(s)H_{-\infty}^0(s).$$

The final two lemmas of this section are concerned with the positivity and relative growth of these functions.

LEMMA 2.4. *For all $s > 0$, $A_1(s) > 0$, $A_2(s) > \lambda_0$, $A_3(s) > \lambda_0$, and $A_4(s) > 0$.*

Proof. The first result follows trivially upon writing $A_1 = Q_\infty^1 Q_{-\infty}^0 U$. The others make use of (2.16), (2.17) and the bounds in (2.14). For example,

$$A_2 = -Q_\infty^1 H_{-\infty}^0\left(1 - \frac{H_{-\infty}^1}{H_{-\infty}^0}\frac{Q_\infty^0}{Q_\infty^1}\right) = -Q_\infty^1 H_{-\infty}^0\left[1 - \left(U_0 - \frac{\lambda_0}{H_{-\infty}^0 Q_{-\infty}^0}\right)U_1\right].$$

The decreasing nature of the $\{-H_n^0\}$ sequence implies $\lambda_0 = -H_1^0 < -H_{-\infty}^0$ so that $U_0 - \lambda_0/H_{-\infty}^0 Q_{-\infty}^0 < U_0 + 1/Q_{-\infty}^0 < 1$. Therefore

$$A_2 > -Q_\infty^1 H_{-\infty}^0(1 - U_1) > -H_{-\infty}^0 > \lambda_0.$$

LEMMA 2.5. *Whenever the indicated functions exist*, $\lim_{s\to\infty} 1/A_1(s) = 0$, $\lim_{s\to\infty} A_1(s)/A_i(s) = 0$ *and* $\lim_{s\to\infty} A_i(s)/A_4(s) = 0$ *for* $i = 2, 3$.

Proof. Writing the sum for $U_1(s)$ as in the convergence argument, it may be seen that $U_1(s) < \lambda_0/H_2^1(s)$ so that $U_1(s) = o(1)$ as $s \to \infty$. Since $U_0(s) = o(1)$ as $s \to \infty$ by a similar argument, it becomes apparent upon writing the $A_i(s)$ as in the proof of the last lemma that

$$A_1(s) = Q^1_\infty(s)Q^0_{-\infty}(s)(1 + o(1)), \qquad A_2(s) = -\,Q^1_\infty(s)H^0_{-\infty}(s)(1 + o(1)),$$

$$A_3(s) = H^1_\infty(s)Q^0_{-\infty}(s)(1 + o(1)), \qquad A_4(s) = -\,H^1_\infty(s)H^0_{-\infty}(s)(1 + o(1)).$$

To complete the proof, it will suffice to show that $\lim_{s\to\infty} Q^0_{-\infty}(s)/H^0_{-\infty}(s) = 0$ and $\lim_{s\to\infty} Q^1_\infty(s)/H^1_\infty(s) = 0$.

Dividing (2.1) by $\lambda_n\pi_n$ and summing,

$$\frac{Q^1_\infty(s)}{H^1_\infty(s)} = \sum_{k=0}^{\infty} \frac{1}{\lambda_k\pi_k}\,\frac{H^1_{k+1}(s)}{H^1_\infty(s)} \leqq \sum_{k=0}^{\infty} \frac{1}{\lambda_k\pi_k}\,\frac{H^1_{k+1}(s)}{H^1_{k+2}(s)}.$$

Now the limit of the last series as $s \to \infty$ is zero by dominated convergence, the convergence of the series $\sum 1/\lambda_n\pi_n$ being a consequence of (2.4). The proof that the other limit obtains is similar.

3. **The minimal solution.** Let $P(t)$ be a solution of (1.1)–(1.5), and

$$R(s) = \int_0^\infty e^{-st}P(t)dt, \quad s > 0. \tag{3.1}$$

THEOREM 3.1. *The transformation* (3.1) *establishes a one-to-one correspondence between the set of all matrices* $P(t)$ *satisfying* (1.1)–(1.5) *and the set of all matrices* $R(s)$ *satisfying*

$$-I + sR(s) = AR(s) = R(s)A, \tag{3.2}$$

$$R(s) \geqq 0, \quad sR(s)e \leqq e, \tag{3.3}$$

$$(v - u)R(u)R(v) = R(u) - R(v), \tag{3.4}$$

where I *is the identity matrix and* e *is a vector with all components equal to one.*

This is a standard theorem and the proof will be omitted. A proof for the case of unilateral birth and death processes is given in [3].

The sequence $\{\pi_i\}$ was defined in such a way that if π denotes the diagonal matrix with diagonal elements $\pi_{ii} = \pi_i$, then the matrix πA is symmetric. The second equation in (3.2) may be written $-I + sR = R\pi^{-1}\pi A$, or $-I + sR^T = \pi A\pi^{-1}R^T$, or finally $-\pi^{-1} + s\pi^{-1}R^T = A\pi^{-1}R^T$. Thus $R\pi^{-1}$ and its adjoint satisfy the same equation. This suggests the introduction of a symmetric Green's function as a solution of (3.2) for $R\pi^{-1}$ and leads to

THEOREM 3.2. *The general solution of* (3.2) *is given by*

$$R_{ij} = \begin{cases} \pi_j \left\{ \dfrac{1}{\lambda_0} Q^0_i Q^1_j + \displaystyle\sum_{\alpha,\beta=0}^{1} \dfrac{R_{\alpha\beta}}{\pi_\beta} Q^\alpha_i Q^\beta_j \right\}, & j \leqq i \\[2ex] \pi_j \left\{ \dfrac{1}{\lambda_0} Q^1_i Q^0_j + \displaystyle\sum_{\alpha,\beta=0}^{1} \dfrac{R_{\alpha\beta}}{\pi_\beta} Q^\alpha_i Q^\beta_j \right\}, & j \geqq i. \end{cases}$$

Proof. The final four terms in the above expression give the general solution to the homogeneous version of (3.2). For, in order to satisfy this version, both the rows and columns of $R\pi^{-1}$ must satisfy the equation $sQ = AQ$. To see that the first term is a particular solution of (3.2), it will suffice to check the first equation since the two are equivalent for self-adjoint versions of $R\pi^{-1}$. The first equation is obviously satisfied so long as $i \neq j$. In this case, letting R^0 denote the first term,

$$-1 + sR_{ii}^0 = -1 + \frac{\pi_i}{\lambda_0} Q_i^0 \left\{ \mu_i Q_{i-1}^1 - (\lambda_i + \mu_i) Q_i^1 + \lambda_i Q_{i+1}^1 \right\}$$

$$= -1 + (AR^0)_{ii} + \frac{\lambda_i \pi_i}{\lambda_0} \left\{ Q_{i+1}^1 Q_i^0 - Q_{i+1}^0 Q_i^1 \right\} = (AR^0)_{ii},$$

where the last equality is a consequence of (2.10).

An examination of the behavior of the above solutions when i is large will show that there is only one which satisfies (3.3) in the event that the $\{Q_n^\alpha\}$ systems diverge at both plus and minus infinity. To see this, write

$$R_{i1} = Q_i^1 \left\{ R_{11} + \left(\frac{1}{\mu_1} + R_{01} \right) \frac{Q_i^0}{Q_i^1} \right\}$$

for $i \geqq 1$. Now if Q_i^1 diverges, the expression in braces must converge to zero so that $R_{11} = U_1(R_{01} + 1/\mu_1)$. Considering also R_{i0} and letting $i \to -\infty$ as well, four equations are obtained which determine $R_{\alpha\beta}$, $\alpha, \beta = 0, 1$, as

$$R_{11} = \frac{U_1}{\mu_1 U}; \ R_{01} = \frac{U_1 U_0}{\mu_1 U}; \ R_{10} = \frac{U_1 U_0}{\lambda_0 U}; \ R_{00} = \frac{U_0}{\lambda_0 U}. \tag{3.5}$$

Substitution of these values into the general solution of (3.2) given by Theorem 3.2 leads to a distinguished solution which will now be studied more carefully.

THEOREM 3.3. *The matrix* $R(s)$ *with*

$$R_{ij}(s) = \begin{cases} \dfrac{\pi_j}{\lambda_0 U(s)} F_i^1(s) F_j^0(s), & j \leqq i, \\[2ex] \dfrac{\pi_j}{\lambda_0 U(s)} F_i^0(s) F_j^1(s), & j \geqq i, \end{cases}$$

is a solution of (3.2)–(3.4).

Proof. It has already been shown that $R(s)$ satisfies (3.2), and clearly $R(s) \geqq 0$. Let $m < i < n$; then by (2.19) and (2.21),

$$s \sum_{j=m}^{n} R_{ij} = \frac{F_i^1}{\lambda_0 U} s \sum_{j=m}^{i} F_j^0 \pi_j + \frac{F^0}{\lambda_0 U} s \sum_{j=i+1}^{n} F_j^1 \pi_j$$

$$= 1 - \frac{1}{\lambda_0 U} [F_i^1 G_m^0 - F_i^0 G_{n+1}^1].$$

Letting $m \to -\infty$, $n \to \infty$, and applying Lemma 2.2,

$$s \sum_{j=-\infty}^{\infty} R_{ij} = 1 - \frac{F^1}{UQ^0_{-\infty}} - \frac{F^0}{UQ^1_{\infty}} \leqq 1. \tag{3.6}$$

The verification of the resolvent equation (3.4) remains. Let $m < i \leqq j < n$; then by (2.20) and (2.21),

$$(v-u) \sum_{k=m}^{n} R_{ik}(u)R_{kj}(v) = R_{ij}(u) - R_{ij}(v) + o_m(1) + o_n(1),$$

the final terms tending to zero as $m \to -\infty$, $n \to \infty$ by Lemma 2.2, because they involve terms of the form $G^0_m(v)F^0_{m-1}(u)$ and $G^1_{n+1}(v)F^1_n(u)$. A similar computation may be made for $i > j$ or the symmetry of $R\pi^{-1}$ may be used to complete the proof.

In the remainder of the paper, the notation $R(s)$ will be used to refer to the matrix of Theorem 3.3; when referring to an arbitrary solution of (3.2), $R^*(s)$ will be used. It will be convenient to write

$$R^*(s) = R(s) + D(s)\pi, \tag{3.7}$$

where $D\pi$ must satisfy the homogeneous version of (3.2) so that

$$D_{ij} = \sum_{\alpha,\beta=0}^{1} D_{\alpha\beta} Q^\alpha_i Q^\beta_j. \tag{3.8}$$

Now, let

$$B_{ij} = \sum_{\alpha=0}^{1} D_{i\alpha} H^\alpha_j, \qquad C_{ij} = \sum_{\alpha=0}^{1} H^\alpha_i D_{\alpha j}.$$

When R^* satisfies (3.3) the sequence $\{D_{ij}\}$ is bounded in i for fixed j, and the sequence $\{B_{ij}\}$ is bounded in j for fixed i since it is essentially the summed form of $D_{ik}\pi_k$.

LEMMA 3.1. *In order that R^* satisfy* (3.3), *it is necessary and sufficient that $D \geqq 0$ and*

$$\frac{F^1_i}{UQ^0_{-\infty}} + \frac{F^0_i}{UQ^1_{\infty}} - \lim_{n\to\infty} B_{in} + \lim_{n\to-\infty} B_{in} \geqq 0.$$

Proof. It is clear that $D \geqq 0$ implies $R^* \geqq 0$. If $R^* \geqq 0$, then for $i \geqq j$, $i \geqq 0$,

$$0 \leqq R^*_{ij} = \pi_j Q^1_i \left\{ \frac{F^1_i F^0_j}{\lambda_i U Q^1_0} + D_{1j} + \frac{Q^0_i}{Q^1_i} D_{0j} \right\}.$$

Dividing by $\pi_j Q^1_i$ and letting $i \to \infty$ yields $D_{1j} - U_1 D_{0j} \geqq 0$. A similar argument at $-\infty$ gives the inequality $-U_0 D_{1j} + D_{0j} \geqq 0$. An application of Lemma 2.1 now shows that $D_{ij} \geqq 0$. If R^* satisfies (3.3), then for $m < n$

$$B_{in} - B_{im} = s\sum_{k=m}^{n-1} D_{ik}\pi_k \geqq 0$$

so that $\{B_{ij}\}$ is a nondecreasing sequence in j. Furthermore, this sequence is bounded for fixed i as pointed out above, so $\lim_{n\to\infty} B_{in}$ and $\lim_{n\to-\infty} B_{in}$ exist. Now, by (3.6),

$$s\sum_{j=-\infty}^{\infty} R^*_{ij} = 1 - \frac{F^1_i}{UQ^0_{-\infty}} - \frac{F^0_i}{UQ^1_\infty} + \lim_{n\to\infty} B_{in} - \lim_{n\to-\infty} B_{in}$$

which completes the proof of the lemma.

THEOREM 3.4. *If $P^*(t)$ satisfies* (1.1)–(1.5), *then $P^*(t) \geqq P(t)$ where $P(t)$ is the martix whose Laplace transform is $R(s)$.*

Proof. Let $R^*(s)$ be the Laplace transform of $P^*(t)$. Then, by Lemma 3.1, $R^* \geqq R$ and by induction $R^{*n} \geqq R^n$. It is a consequence of (3.4) that

$$\frac{d^n}{ds^n}[R^*(s) - R(s)] = (-1)^n n![R^{*n+1}(s) - R^{n+1}(s)]$$

so that $R^*(s) - R(s)$ is a completely monotonic function. Therefore $P^*(t) - P(t) \geqq 0$.

This theorem shows that $P(t)$ is the minimal solution of Feller. The minimal property makes it apparent that the solution of (1.1)–(1.5) will be unique whenever the minimal solution is honest. (3.6) shows that this is the case whenever each boundary is either entrance or natural. However, the minimal solution is the unique solution in other cases, which is the subject of

THEOREM 3.5. *In order that there is one and only one solution of* (1.1)–(1.5), *it is necessary and sufficient that one of the following conditions is satisfied*:

(1) *One boundary is natural and the other is not regular.*

(2) *Both boundaries are exit.*

(3) *Both boundaries are entrance.*

Proof. The sufficiency of the conditions will be proved here. The necessity follows from the construction of the general solution in all other cases which is given in Theorems 4.1–4.5.

Let $P^*(t)$ be a solution of (1.1)–(1.5) and $R^*(s)$ its Laplace transform. When any one of (1), (2), or (3) hold, then either $\{Q^\alpha_n\}$ diverges at both plus and minus infinity or else $\{H^\alpha_n\}$ does. Suppose the $\{Q^\alpha_n\}$ systems are diverging. Then, since $R^*(s)$ is given by (3.7) with $\{D_{ij}\}$ bounded in i,

$$\left| D_{1j} + \frac{Q^0_i}{Q^1_i} D_{0j} \right| = \left| \frac{D_{ij}}{Q^1_i} \right| \leqq \frac{M}{Q^1_i}.$$

Letting $i \to \infty$, $D_{1j} = U_1 D_{0j}$. A similar argument at $-\infty$ gives the relation

$D_{0j} = U_0 D_{1j}$. Combining these results, $D_{1j}U = 0$, and therefore $D_{1j} = 0$, $D_{0j} = 0$, and $D_{ij} = 0$. But this implies $R^*(s) = R(s)$ and $P^*(t) = P(t)$. If, on the other hand, the $\{H_n^\alpha\}$ systems are diverging, then the boundedness of $\{B_{ij}\}$ in j implies $D_{i1} = U_1 D_{i0}$ and $D_{i0} = U_0 D_{i1}$ by (2.16) and (2.17). The remainder of the argument proceeds as above.

The same type of argument shows that $P(t)$ is the unique solution of (1.1), (1.3)–(1.5) when both boundaries are either entrance or natural. Considerations of this type will not be pursued here, but this result is of importance as it is sometimes undesirable to assume the equation (1.2).

4. **The general solutions.** In the event that Q_∞^α and $Q_{-\infty}^\alpha$ exist, D_{ij} is well defined for $i,\ j = \pm\infty$. If, in addition, the limits H_∞^α and $H_{-\infty}^\alpha$ exist, then B_{ij}, C_{ij} are also defined for $i,\ j = \pm\infty$. Whenever the appropriate functions exist, the following identities may be easily established:

$$
\begin{aligned}
A_1 D_{1j} &= Q_{-\infty}^0 D_{\infty j} - Q_\infty^0 D_{-\infty j}, & A_1 D_{0j} &= Q_\infty^1 D_{-\infty j} - Q_{-\infty}^1 D_{\infty j},\\
A_1 D_{i1} &= Q_{-\infty}^0 D_{i\infty} - Q_\infty^0 D_{i-\infty}, & A_1 D_{i0} &= Q_\infty^1 D_{i-\infty} - Q_{-\infty}^1 D_{i\infty}.
\end{aligned} \tag{4.1}
$$

Substitution of these into the definitions of B_{ij}, C_{ij} yields

$$
\begin{aligned}
A_1 C_{\infty j} &= A_3 D_{\infty j} - \lambda_0 D_{-\infty j}, & A_1 C_{-\infty j} &= \lambda_0 D_{\infty j} - A_2 D_{-\infty j},\\
A_1 B_{i\infty} &= A_3 D_{i\infty} - \lambda_0 D_{i-\infty}, & A_1 B_{i-\infty} &= \lambda_0 D_{i\infty} - A_2 D_{i-\infty}.
\end{aligned} \tag{4.2}
$$

These identities will be useful in the proof of Theorem 4.1.

Let

$$
\begin{aligned}
M_{ijn}^{\alpha\beta}(u,v) = {} & \left(C_{nj}(v) + \frac{F_j^\alpha(v)G_n^\beta(v)}{\lambda_0 U(v)} \right) \left(D_{i,n-1}(u) + \frac{F_i^\alpha(u)F_{n-1}^\beta(u)}{\lambda_0 U(u)} \right) \\
& - \left(D_{n-1,j}(v) + \frac{F_j^\alpha(v)F_{n-1}^\beta(v)}{\lambda_0 U(v)} \right) \left(B_{in}(u) + \frac{F_i^\alpha(u)G_n^\beta(u)}{\lambda_0 U(u)} \right)
\end{aligned}
$$

and $M_{ij}(u,v) = \lim_{n\to\infty} M_{ijn}^{01}(u,v) - \lim_{n\to-\infty} M_{ijn}^{10}(u,v)$ whenever the limits exist. This leads to

LEMMA 4.1. *In order for the resolvent equation* (3.4) *to be satisfied by a matrix* $R^*(s)$ *having the form* (3.7), *it is necessary and sufficient that* $M_{ij}(u,v) \equiv 0$.

Proof. It follows from (2.9) and (2.21) that for $m < i,\ j < n$,

$$
\begin{aligned}
(v-u)\sum_{k=m}^{n} R_{ik}^*(u)R_{kj}^*(v) = {} & (v-u)\sum_{k=m}^{n} R_{ik}(u)R_{kj}(v) - \pi_j D_{ij}(v) + \pi_j D_{ij}(u) \\
& + \pi_j\left(M_{ij,n+1}^{01}(u,v) - M_{ijm}^{10}(u,v)\right) + o_m(1) + o_n(1),
\end{aligned}
$$

the final terms tending to zero as $m \to -\infty$, $n \to \infty$ by Lemma 2.2. At the same time the first three terms on the right tend to $R^*_{ij}(u) - R^*_{ij}(v)$ by Theorem 3.3 and this completes the proof.

THEOREM 4.1. *If both plus and minus infinity are regular boundaries, then* $R^*(s)$ *is a solution of* (3.2)–(3.4) *if and only if it is given by* (3.7) *and* (3.8) *with*

$$D_{1-\alpha,1-\beta} = \frac{(-1)^{\alpha+\beta}}{DA_1}\Big[\gamma_2 Q^\alpha_\infty Q^\beta_\infty + \gamma_3 Q^\alpha_{-\infty} Q^\beta_{-\infty} + \gamma_4\{Q^\alpha_\infty H^\beta_\infty - Q^\alpha_{-\infty} H^\beta_{-\infty}\} - \gamma_5 Q^\alpha_{-\infty} Q^\beta_\infty - \gamma_6 Q^\alpha_\infty Q^\beta_{-\infty}\Big],$$

for $\alpha, \beta = 0, 1$, *where*

$$D = \sum_{k=1}^{4} \gamma_k A_k - (\gamma_5 + \gamma_6)\lambda_0,$$

and the γ_i *satisfy*

$$\gamma_2\gamma_3 = \gamma_1\gamma_4 + \gamma_5\gamma_6; \qquad \sum_{k=1}^{6} \gamma_k = 1;$$
$$\gamma_k \geqq 0, \quad k = 1,4,5,6; \qquad \gamma_2 \geqq \gamma_5, \ \gamma_3 \geqq \gamma_6.$$

Proof. For the purpose of the proof the functions

$$D_{\infty\infty} = \frac{\gamma_3 A_1 + \gamma_4 A_2}{D}; \qquad D_{\infty-\infty} = \frac{\gamma_5 A_1 + \gamma_4 \lambda_0}{D};$$

$$D_{-\infty\infty} = \frac{\gamma_6 A_1 + \gamma_4 \lambda_0}{D}; \qquad D_{-\infty-\infty} = \frac{\gamma_2 A_1 + \gamma_4 A_3}{D}$$

will be used. These functions are obtained from those stated in the theorem using (3.8), while (4.1) guarantees that they also determine the D_{ij}.

It has already been seen that the form (3.7)–(3.8) is the general solution of (3.2). An application of Lemma 2.1 to the conditions stated in Lemma 3.1 shows that (3.3) is equivalent to

$$\text{(4.3)} \qquad D_{\alpha\beta} \geqq 0, \ \alpha,\beta = \pm\infty; \qquad 1 - B_{\alpha\infty} + B_{\alpha-\infty} \geqq 0, \ \alpha = \pm\infty.$$

In the same way, the preceding lemma gives the equivalence of (3.4) and $M_{\alpha\beta}(u,v) \equiv 0$ for $\alpha,\beta = \pm\infty$.

Now let $R^*(s)$ satisfy (3.2)–(3.4). The solution $R(s)$ is given by $\gamma_1 = 1$, $\gamma_k = 0$ for $k \neq 1$, so it will be assumed that $R^*(s) \neq R(s)$. Then for some i, j, s_0, $D_{ij}(s_0) > 0$ so that at least one of $D_{\alpha\beta}(s_0)$, $\alpha,\beta = \pm\infty$, must be positive. Let $d_{\alpha\beta} = D_{\alpha\beta}(s_0)$, $\alpha,\beta = \pm\infty$; $c_{\infty\beta} = C_{\infty\beta}(s_0) - \delta_{\infty\beta}$ and $c_{-\infty\beta} = C_{-\infty\beta}(s_0) + \delta_{-\infty\beta}$ for $\beta = \pm\infty$. Then $M_{\alpha\beta}(s,s_0) \equiv 0$ is

$$c_{\infty\beta}D_{\alpha\infty} - d_{\infty\beta}(B_{\alpha\infty} - \delta_{\alpha\infty}) - c_{-\infty\beta}D_{\alpha-\infty} + d_{-\infty\beta}(B_{\alpha-\infty} + \delta_{\alpha-\infty}) \equiv 0,$$

for $\alpha,\beta = \pm\infty$. Substituting the values of the B's given in (4.2), the system becomes

$$\left(c_{\infty\beta} - d_{\infty\beta}\frac{A_3}{A_1} + d_{-\infty\beta}\frac{\lambda_0}{A_1}\right) D_{\alpha\infty} + \left(d_{\infty\beta}\frac{\lambda_0}{A_1} - c_{-\infty\beta} - d_{-\infty\beta}\frac{A_2}{A_1}\right) D_{\alpha-\infty}$$
$$= -\delta_{\alpha\infty}d_{\infty\beta} - \delta_{\alpha-\infty}d_{-\infty\beta}.$$

If

$$\gamma_1 = c_{\infty-\infty}c_{-\infty\infty} - c_{\infty\infty}c_{-\infty-\infty}, \qquad \gamma_4 = d_{\infty\infty}d_{-\infty-\infty} - d_{\infty-\infty}d_{-\infty\infty},$$
$$\gamma_2 = c_{\infty-\infty}d_{-\infty\infty} - c_{\infty\infty}d_{-\infty-\infty}, \qquad \gamma_5 = c_{\infty-\infty}d_{\infty\infty} - c_{\infty\infty}d_{\infty-\infty},$$
$$\gamma_3 = c_{-\infty-\infty}d_{\infty\infty} - c_{-\infty\infty}d_{\infty-\infty}, \qquad \gamma_6 = c_{-\infty-\infty}d_{-\infty\infty} - c_{-\infty\infty}d_{-\infty-\infty},$$

then these equations become

$$DD_{\infty\infty} = \gamma_3 A_1 + \gamma_4 A_2, \qquad DD_{\infty-\infty} = \gamma_5 A_1 + \gamma_4\lambda_0,$$
$$DD_{-\infty\infty} = \gamma_6 A_1 + \gamma_4\lambda_0, \qquad DD_{-\infty-\infty} = \gamma_2 A_1 + \gamma_4 A_3, \tag{4.4}$$

with D as in the statement of the theorem. The identity $A_2A_3 - \lambda_0^2 = A_1A_4$ has been used to simplify the expression for D.

Suppose that $\gamma_4 \neq 0$. Since multiplication of all the γ_i by a constant does not change (4.4), γ_4 may be taken to be positive. Then the left side of each equation in (4.4) is non-negative for sufficiently large s by Lemma 2.5. But $\gamma_5 A_1 + \gamma_4\lambda_0$ is non-negative for large s only if $\gamma_5 \geqq 0$, and this implies $\gamma_5 A_1 + \gamma_4\lambda_0$ never vanishes. Therefore D does not vanish and it only remains to check the conditions on the γ_i. As before, the non-negativity of the $D_{\alpha\beta}$ implies $\gamma_5 \geqq 0$, $\gamma_6 \geqq 0$. Using the relations in (4.2),

$$1 - B_{\infty\infty} + B_{\infty-\infty} = \frac{(\gamma_3 - \gamma_6)\lambda_0 + \gamma_1 A_1 + (\gamma_2 - \gamma_5)A_2}{D}$$
$$1 - B_{-\infty\infty} + B_{-\infty-\infty} = \frac{(\gamma_2 - \gamma_5)\lambda_0 + \gamma_1 A_1 + (\gamma_3 - \gamma_6)A_3}{D} \tag{4.5}$$
$$M_{\infty\infty}(u,v) = \frac{1}{D(u)D(v)}[A_1(u)A_2(v) - A_2(u)A_1(v)](\gamma_1\gamma_4 + \gamma_5\gamma_6 - \gamma_2\gamma_3).$$

An application of Lemma 2.5 here, recalling that the first two expressions are non-negative and the third identically zero, yields $\gamma_2 - \gamma_5 \geqq 0$, $\gamma_3 - \gamma_6 \geqq 0$, and $\gamma_2\gamma_3 = \gamma_1\gamma_4 + \gamma_5\gamma_6$. Writing the last equality as $\gamma_1\gamma_4 = \gamma_2\gamma_3 - \gamma_5\gamma_6$ shows that $\gamma_1 \geqq 0$ since $\gamma_4 > 0$. Finally the sum of the γ_i is positive and may be taken to be one after multiplying all the γ_i by a positive constant.

If $\gamma_4 = 0$ and $D(s)$ vanishes for some $s > 0$, then from (4.4), $\gamma_2 = \gamma_3 = \gamma_5 = \gamma_6 = 0$. But if the $c_{\alpha\beta}$ are expressed in terms of the $d_{\alpha\beta}$ using (4.2) in the definitions of the γ_i,

$$\gamma_2 = -\frac{A_3(s_0)}{A_1(s_0)}\gamma_4 + d_{-\infty-\infty}; \qquad \gamma_3 = -\frac{A_2(s_0)}{A_1(s_0)}\gamma_4 + d_{\infty\infty};$$
$$\gamma_5 = -\frac{\lambda_0}{A_1(s_0)}\gamma_4 + d_{\infty-\infty}; \qquad \gamma_6 = -\frac{\lambda_0}{A_1(s_0)}\gamma_4 + d_{-\infty\infty}.$$

But this results in the contradiction that all the $d_{\alpha\beta}$ are zero. Thus $D(s)$ has no zeroes and the remainder of the argument is as before with the exception that since γ_4 is now zero, the non-negativity of γ_1 must be obtained in a different manner. In the present case

$$\begin{aligned} 0 &= \gamma_2\gamma_3 - \gamma_5\gamma_6 = (\gamma_2 - \gamma_5)(\gamma_3 - \gamma_6) + \gamma_5(\gamma_3 - \gamma_6) + (\gamma_2 - \gamma_5)\gamma_6 \\ &\geqq (\gamma_2 - \gamma_5)(\gamma_3 - \gamma_6) \geqq 0, \end{aligned}$$

so that $(\gamma_2 - \gamma_5)(\gamma_3 - \gamma_6) = 0$. If $\gamma_2 - \gamma_5 = 0$ the first expression in (4.5) will be negative for large s unless $\gamma_1 \geqq 0$, while if $\gamma_3 - \gamma_6 = 0$, a similar argument applies to the second expression.

Suppose $R^*(s)$ is given by the representation stated in the theorem. Then

$$D = \{(\gamma_2 - \gamma_5) + (\gamma_3 - \gamma_6)\}\lambda_0 + \gamma_1 A_1 + \gamma_2(A_2 - \lambda_0) + \gamma_3(A_3 - \lambda_0) + \gamma_4 A_4$$

and thus D is positive for all $s > 0$. The $D_{\alpha\beta}$ and the first two functions in (4.5) are now clearly non-negative and the $M_{\alpha\beta}(u,v)$ vanish identically for $\alpha, \beta = \pm\infty$ since each contains as a factor $\gamma_1\gamma_4 + \gamma_5\gamma_6 - \gamma_2\gamma_3$. These conditions then imply that $R^*(s)$ satisfies (3.2)–(3.4).

The theorems giving the general solution for other combinations of boundaries will now be stated without proof. The proof, in each case, is a combination of the type of proof used in the uniqueness theorem with the type used in Theorem 4.1. Also Lemma 2.3 plays an important role when entrance or exit boundaries are present.

THEOREM 4.2. *If plus infinity is an exit boundary and minus infinity a regular boundary, then $R^*(s)$ is a solution of* (3.2)–(3.4) *if and only if it is given by* (3.7) *and* (3.8) *with*

$$D_{1-\alpha,1-\beta} = \frac{(-1)^{\alpha+\beta}}{DA_1}[\gamma_2 Q^{\alpha}_{\infty} Q^{\beta}_{\infty} - \gamma_5 Q^{\alpha}_{-\infty} Q^{\beta}_{\infty}]$$

for $\alpha, \beta = 0, 1$, where $D = -\gamma_5\lambda_0 + \gamma_1 A_1 + \gamma_2 A_2$ and the γ_i satisfy

$$\gamma_1 + \gamma_2 + \gamma_5 = 1, \qquad \gamma_1 \geqq 0, \qquad \gamma_5 \geqq 0, \qquad \gamma_2 \geqq \gamma_5.$$

THEOREM 4.3. *If plus infinity is an entrance boundary and minus infinity a regular boundary, then $R^*(s)$ is a solution of* (3.2)–(3.4) *if and only if it is given by* (3.7) *and* (3.8) *with*

$$D_{1-\alpha,1-\beta} = \frac{(-1)^{\alpha+\beta}}{DA_3}[\gamma_4 H^{\alpha}_{\infty} H^{\beta}_{\infty} - \gamma_6 H^{\alpha}_{\infty} Q^{\beta}_{-\infty}]$$

for $\alpha, \beta = 0, 1$, where $D = -\gamma_6\lambda_0 + \gamma_3 A_3 + \gamma_4 A_4$ and the γ_i satisfy

$$\gamma_3 + \gamma_4 + \gamma_6 = 1, \qquad \gamma_4 \geqq 0, \qquad \gamma_6 \geqq 0, \qquad \gamma_3 \geqq \gamma_6.$$

THEOREM 4.4. *If plus infinity is a natural boundary and minus infinity a* *regular boundary, then $R^*(s)$ is a solution of* (3.2)–(3.4) *if and only if it is given by* (3.7) *and* (3.8) *with*

$$D_{\alpha\beta} = \gamma_2 D^{-1} U_1^{\alpha+\beta}$$

for α, $\beta = 0, 1$, where $D = UQ^0_{-\infty}(\gamma_1 UQ^0_{-\infty} - \gamma_2 G_{-\infty})$ and

$$\gamma_1 + \gamma_2 = 1, \qquad \gamma_1 \geqq 0, \qquad \gamma_2 \geqq 0.$$

THEOREM 4.5. *If plus infinity is an exit boundary and minus infinity an entrance boundary, then $R^*(s)$ is a solution of* (3.2)–(3.4) *if and only if it is given by* (3.7) *and* (3.8) *with*

$$D_{\alpha\beta} = \gamma_5 D^{-1} U_0^{1-\alpha} U_1^{\beta}$$

for α, $\beta = 0, 1$, where $D = U(-\gamma_5\lambda_0 + \gamma_2 A_2)$ and

$$\gamma_2 + \gamma_5 = 1, \qquad \gamma_5 \geqq 0, \qquad \gamma_2 \geqq \gamma_5.$$

REFERENCES

1. W. Feller, *The parabolic differential equations and the associated semi-groups of transformations*, Ann. of Math. (2) **55** (1952), 468–519.

2. ———, *On boundaries and lateral conditions for the Kolmogorov differential equations* Ann. of Math. (2) **65** (1957), 527–570.

3 ———, *The birth and death processes as diffusion processes*, J. Math. Pures Appl. **38** (1959), 301–345.

4. ———, *An introduction to probability theory and its applications*, Wiley, New York, 1950.

5. S. Karlin and J. McGregor, *The differential equations of birth-and-death processes and the Stieltjes moment problem*, Trans. Amer. Math. Soc. **85** (1957), 489–546.

6. ———, *The classification of birth and death processes*, Trans. Amer. Math. Soc. **86** (1957), 366–400.

7. ———, *Occupation time laws for birth and death processes*, Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. II, pp. 249–272, Univ. California Press, Berkeley, Calif., 1961.

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