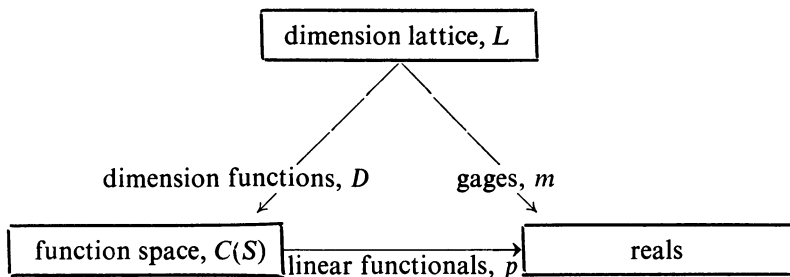# A RADON-NIKODYM THEOREM IN DIMENSION LATTICES[1]

BY

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1. **Introduction.** A dimension lattice is the abstract analogue of the lattice of projection operators in a von Neumann algebra (ring of operators or W^* algebra). For the purposes of this introduction one can imagine it to be such a projection lattice with the usual equivalence relation of equal relative dimension. Our principal results center about three mappings: (1) *gages*, which are real-valued completely additive functions on the dimension lattice invariant under its equivalence relation; (2) *dimension functions*, mapping the lattice into a certain function space; and (3) linear functionals on this function space. A gage is the analogue of a measure and plays the corresponding role of a basic measure in Segal's development of an integration theory in operator algebras [8]. The terminology is his. In the operator algebra case, the analogue of a dimension function is often referred to as a "center-valued" or "function-valued" trace. In Dixmier's book [2] these are called "applications ♮." In the abstract lattice theoretic version the range of the dimension function is the space $C(S)$ of non-negative continuous functions on the Stone space S of a certain complete Boolean sublattice of L—the invariant elements of L (central projections). A diagram is most helpful in visualizing the situation.

It is the primary purpose of this paper to prove that: if either a gage m or a dimension function D is given, then the remaining two mappings can be constructed so that the diagram is commutative; i.e., the equation $m = p \circ D$ holds. Moreo<er, if two of m, p, D are given, then the third satisfying $m = p \circ D$ is *uniquely* determined. This is the general idea; the precise statements and complete definitions are given in §4.

These results are formally identical with those established by Dixmier for operator algebras [2, Chapter III, §4]; in fact, our presentation is modeled after his. The technical machinery is however quite different in this abstract version.

As a by-product of this complex of theorems, we are able to obtain an abstract version of a theorem of Segal on operator algebras [8, Theorem 15]. This is the Radon-Nikodym theorem of the title.

Our theorems have a natural place in a developing pattern of algebraic and lattice-type theorems which have been abstracted from the theory of von Neumann algebras. Von Neumann himself initiated this program with his 1935 theory of Continuous Geometries, developing the notion of relative dimensionality as an intrinsic property of continuous complemented modular lattices (Continuous Geometries). This applied only to the finite case, and Loomis and S. Maeda, taking a slightly different tack, succeeded in characterizing the notion by a few natural lattice theoretic axioms. On this basis they were able to construct a complete and satisfactory lattice-theoretic generalization of the dimension theory, ncluding the proof of the existence of a dimension function. Our results complete this latter investigation by showing that the relationships between dimension functions and gages can also be generalized, and that gages possess an intrinsic Radon-Nikodym theorem.

In §§2 and 3 we develop some preliminary machinery for the main results of §§4 and 5. The elementary exposition of orthomodular lattices presented in §2 has perhaps some independent interest. §4 presents the results on gages and dimension functions, and §5 contains the Radon-Nikodym theorem.

2. **Orthomodular lattices.** A lattice L is *orthocomplemented* if there exists a mapping $a \to a^\perp$ of L onto itself such that $a^{\perp\perp} = a$, $a^\perp \vee a = 1$, $a^\perp \wedge a = 0$, and $a \leq b \to a^\perp \geq b^\perp$. The element $a^\perp$ is called the orthocomplement of a, and we say that a and b are *orthogonal*, written $a \perp b$, in case $a \leq b^\perp$. (This relation is symmetric.) If $a \leq b$ we write $b - a$ for $b \wedge a^\perp$, and, if $a \perp b$, $a \oplus b$ for $a \vee b$. Also if a_α is an orthogonal family ($a_\alpha \perp a_\beta$ for $\alpha \neq \beta$) we write $\oplus a_\alpha$ instead of $\vee a_\alpha$.

DEFINITION 1. *If* $a = (a \wedge b) \oplus (a \wedge b^\perp)$ *for* $a, b \in L$, *we say that* a *commutes with* b *and write* aCb.

For example if $a \perp b$ so that $a \leq b^\perp$, then $a \wedge b \leq b^\perp \wedge b = 0$ so $a \wedge b = 0$. Then $(a \wedge b) \oplus (a \wedge b^\perp) = 0 \oplus a = a$, whence a commutes with b. Similarly, if $a \leq b$, then aCb.

THEOREM 1. *The following three conditions are equivalent in any ortho-complemented lattice L:*

(1) $a \leqq b$ *implies* $b = a \oplus (b - a)$;

(1′) $a \leqq b$ *implies* $a = b - (b - a)$;

(2) *whenever a commutes with b, then b commutes with a.*

Proof. (1) is equivalent to (1′) by application of the dual automorphism $\perp$. We complete the proof by showing that (1) and (2) are equivalent. Suppose that (1) holds and that a commutes with b; that is, $a = (a \wedge b) \oplus (a \wedge b^{\perp})$. Taking the $\perp$ of this last equation we get $a^{\perp} = (a^{\perp} \vee b^{\perp}) \wedge (a^{\perp} \vee b)$ so that $b \wedge a^{\perp} = b \wedge (a^{\perp} \vee b^{\perp}) \wedge (a^{\perp} \vee b) = b \wedge (a^{\perp} \vee b^{\perp}) = b - a \wedge b$. Then

$$(b \wedge a) \oplus (b \wedge a^{\perp}) = (b \wedge a) \oplus (b - a \wedge b) = b$$

by (1), which is exactly the statement that b commutes with a. Finally suppose that (2) holds. If $a \leqq b$, then, as we have already remarked, aCb. Applying (2), we have bCa, which is to say

$$b = (b \wedge a) \oplus (b \wedge a^{\perp}) = a \oplus (b - a).$$

But this is (1), which concludes the proof.

DEFINITION 2. *An orthomodular lattice is an orthocomplemented lattice which satisfies any one (and hence all) of the equivalent conditions of Theorem 1.*

THEOREM 2. *Let L be an orthomodular lattice. Then the following hold:*

(1) aCb *if and only if* $a \wedge b = a \wedge (b \vee a^{\perp})$ $(= a - a \wedge b^{\perp})$;

(2) *if* aCb, *then* $a^{\perp}Cb$ *and* $a^{\perp}Cb^{\perp}$;

(3) *if* a_1Cb *and* a_2Cb, *then* $a_1 \vee a_2Cb$ *and* $a_1 \wedge a_2Cb$;

(4) *if L is in addition complete, and* $a_{\alpha}Cb$ *for all* α([2]), *then* $(\bigvee a_{\alpha})Cb$ *and* $(\bigwedge a_{\alpha})Cb$.

Proof. (1). Suppose that aCb, so that $a = (a \wedge b) \oplus (a \wedge b^{\perp})$. Taking the $\perp$ gives $a^{\perp} = (a^{\perp} \vee b^{\perp}) \wedge (a^{\perp} \vee b)$. Now $a \wedge b \leqq a \wedge (b \vee a^{\perp})$, and we compute

$$a \wedge (b \vee a^{\perp}) - a \wedge b = a \wedge [(b \vee a^{\perp}) \wedge (a^{\perp} \vee b^{\perp})] = a \wedge a^{\perp} = 0$$

using the formula just derived. Then by Theorem 1, (1),

$$a \wedge (b \vee a^{\perp}) = (a \wedge b) \oplus (a \wedge (b \vee a^{\perp}) - a \wedge b) = (a \wedge b) \oplus 0 = a \wedge b,$$

which proves the implication in one direction. Conversely suppose that $a \wedge b = a \wedge (b \vee a^{\perp}) = a - a \wedge b^{\perp}$. Then

$$(a \wedge b) \oplus (a \wedge b^{\perp}) = (a - a \wedge b^{\perp}) \oplus (a \wedge b^{\perp}) = a$$

by Theorem 1, (1). But this says exactly that aCb, completing the proof of (1).

([2]) We will omit reference to the indexing set where convenient.

(2).   The definitions of xCy and $xCy^\perp$ are the same, and using this along with the symmetry of the relation of commutativity, we can put together the chain of equivalences,

$$aCb \to bCa \to bCa^\perp \to a^\perp Cb \to a^\perp Cb^\perp,$$

which completes the proof of (2).

(3) and (4). We suppose L is complete and prove (4). The same argument, with trivial modifications, will prove (3), and this verification is left to the reader.

It is enough to prove that $(\bigvee a_\alpha)Cb$, for if this is established we can use (2) to conclude $(\bigwedge a_\alpha)Cb$ as follows: $a_\alpha Cb$ for all α implies $a_\alpha^\perp Cb^\perp$ for all α, so that $(\bigvee a_\alpha^\perp)Cb^\perp$, and then $(\bigwedge a_\alpha)Cb$ by (2) again.

Always $(\bigvee a_\alpha) \wedge b \geqq \bigvee(a_\alpha \wedge b)$ and $(\bigvee a_\alpha) \wedge b^\perp \geqq \bigvee(a_\alpha \wedge b^\perp)$. Then

$$[(\bigvee a_\alpha) \wedge b] \vee [(\bigvee a_\alpha) \wedge b^\perp] \geqq [\bigvee(a_\alpha \wedge b)] \vee [\bigvee(a_\alpha \wedge b^\perp)] = \bigvee[(a_\alpha \wedge b) \vee (a_\alpha \wedge b^\perp)].$$

Using the fact that $a_\alpha Cb$ for all α, the right-hand side is $\bigvee a_\alpha$, hence

$$[(\bigvee a_\alpha) \wedge b] \vee [(\bigvee a_\alpha) \wedge b^\perp] \geqq \bigvee a_\alpha.$$

It is obvious that the reverse inequality is also true, so we have equality. This proves $(\bigvee a_\alpha)Cb$, and completes the proof of Theorem 2.

The notion of commutativity has an intimate connection with the distributive law which we specify in the next theorem, after some preliminary comments and definitions.

DEFINITION 3.   (See F. Maeda [5, Chapter I, Definition 1.6], and von Neumann [7, Part I, Definition 5.1].) *Let a, b, c be elements of the lattice L. If $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ we write $(a,b,c)D$. If $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ we write $(a,b,c)D^*$. If both $(a,b,c)D$ and $(a,b,c)D^*$ hold for all permutations of a,b,c, then we write $(a,b,c)T$ and say that (a,b,c) is a distributive triple.*

The relation $(a,b,c)D$ is obviously the same as $(b,a,c)D$, and similarly $(a,b,c)D^*$ is the same as $(b,a,c)D^*$. From this it follows easily that there are six different laws contained in the relation $(a,b,c)T$. Now von Neumann has shown [7, Part I, Theorem 5.1] that in a modular lattice the validity of any one of these laws, for a particular triple, entails the validity of the five others. Hence, in a modular lattice $(a,b,c)D$ and $(a,b,c)T$ are the same. It is not difficult to prove that this property characterizes modular lattices; in fact the following result even holds: if the validity of four of the six laws of $(a,b,c)T$ for a particular triple always implies the validity of the other two, then the lattice is already modular. Accordingly in a nonmodular lattice the relations $(a,b,c)D$ and $(a,b,c)T$ are different. More interest attaches to $(a,b,c)T$, and in our next theorem we give a criterion for deciding when a triple is distributive. This theorem is quite useful in applications.

THEOREM 3.   *Let a,b,c be elements of the orthomodular lattice L, and sup-*

pose that some one of these elements commutes with the other two. Then (a, b, c) *is a distributive triple.*

Proof. We can assume without loss of generality that c commutes with a and with b. As we mentioned earlier there are six different laws in $(a,b,c)T$ which we have to verify, three involving D and three involving D^*. These are $(a, b, c)D$, $(a, c, b)D$, $(b, c, a)D$, $(a, b, c)D^*$, $(a, c, b)D^*$, and $(a,b,c)D^*$. Now if we show under the hypothesis that cCa and cCb that $(a, b, c)D$, then, knowing also that $c^\perp Ca^\perp$ and $c^\perp Cb^\perp$ (Theorem 2,(2)), we conclude at the same time that $(a^\perp, b^\perp, c^\perp)D$. If we take the $\perp$ of this last relation we get $(b, c, a)D^*$. Similarly the other two relations involving D^* follow from their counterparts involving D. This leaves just the three D-relations, and by symmetry we need only prove two: $(a, b, c)D$ and $(c, a, b)D$.

$(a, b, c)D$: Always $(a \vee b) \wedge c \geqq (a \wedge c) \vee (b \wedge c)$, so we can write

$$(a \vee b) \wedge c - (a \wedge c) \vee (b \wedge c) = (a \vee b) \wedge [c \wedge (a^\perp \vee c^\perp)] \wedge (b^\perp \vee c^\perp)$$

$$= (a \vee b) \wedge (c \wedge a^\perp) \wedge (b^\perp \vee c^\perp) \text{ using } cCa^\perp \text{and}$$

Theorem 2, (1)

$$= (a \vee b) \wedge a^\perp \wedge [c \wedge (b^\perp \vee c^\perp)]$$

$$= (a \vee b) \wedge a^\perp \wedge (c \wedge b^\perp) \text{ using } cCb^\perp \text{ and}$$

Theorem 2, (1)

$$= (a \vee b) \wedge (a \vee b)^\perp \wedge c = 0 \wedge c = 0.$$

Hence, $(a \vee b) \wedge c - (a \wedge c) \vee (b \wedge c) = 0$ and we conclude as in the proof of Theorem 2,(1), that $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$; that is, $(a, b, c)D$.

$(c, a, b)D$: Always $(c \vee a) \wedge b \geqq (c \wedge b) \vee (a \wedge b)$ so we can write

$$(c \vee a) \wedge b - (c \wedge b) \vee (a \wedge b) = (c \vee a) \wedge [b \wedge (c^\perp \vee b^\perp)] \wedge (a^\perp \vee b^\perp)$$

$$= (c \vee a) \wedge (b \wedge c^\perp) \wedge (a^\perp \vee b^\perp) \text{ using } bCc^\perp \text{ and}$$

Theorem 2,(1)

$$= [(c \vee a) \wedge c^\perp] \wedge b \wedge (a^\perp \vee b^\perp)$$

$$= (c^\perp \wedge a) \wedge b \wedge (a^\perp \vee b^\perp) \text{ using } c^\perp Ca \text{ and}$$

Theorem 2,(1)

$$= c^\perp \wedge (a \wedge b) \wedge (a \wedge b)^\perp = c \wedge 0 = 0,$$

whence $(c \vee a) \wedge b = (c \wedge b) \vee (a \wedge b)$ which shows $(c, a, b)D$ and completes the proof of Theorem 3.

Given an element, a, of a lattice L with 0 and 1, another element, b, is said to be a *complement* of a if $a \vee b = 1$ and $a \wedge b = 0$. In an orthomodular lattice we can give an explicit formula for the complements of an element.

THEOREM 4. *Let L be an orthomodular lattice, with $a \in L$. Then, for any $x \in L$, $b(x) = (x - x \wedge a) \oplus (x \vee a)^{\perp}$ is a complement of a, and every complement of a can be written in this fashion for some $x \in L$. Moreover, $b(x) = a^{\perp}$ if and only if x commutes with a.*

Proof. We will write b for $b(x)$. Using Theorem 3 we verify easily that $(a, x, a \wedge x)T$, and it follows that $a \vee (x - x \wedge a) = a \vee x$. Hence

$$a \vee b = a \vee (x - x \wedge a) \vee (x \vee a)^{\perp} = (a \vee x) \vee (a \vee x)^{\perp} = 1.$$

Since $x - x \wedge a \leqq x \vee a$ we have $(x - x \wedge a)C(x \vee a)^{\perp}$ using the remark following Definition 1 and Theorem 2, (2). Similarly $aC(x \vee a)^{\perp}$, so we can conclude again by Theorem 3 that $(a, x - x \wedge a, (x \vee a)^{\perp})T$. Then

$$a \wedge b = a \wedge [(x - x \wedge a) \vee (x \vee a)^{\perp}] = [a \wedge (x - x \wedge a)] \vee [a \wedge (x \vee a)^{\perp}]$$

$$= [(a \wedge x) \wedge (x \wedge a)^{\perp}] \vee (a \wedge x^{\perp} \wedge a^{\perp})$$

$$= 0 \vee 0 = 0,$$

which shows that b is a complement of a for any $x \in L$. Conversely, if c is a given complement of a, then setting $x = c$ represents c in the form desired.

As for the last sentence of the theorem, suppose that xCa. Then $xCa^{\perp}$ by Theorem 2, (2), and $x - x \wedge a = x \wedge a^{\perp}$ by Theorem 2, (1). Hence

$$b(x) = (x \wedge a^{\perp}) \oplus (x^{\perp} \wedge a^{\perp}) = a^{\perp}.$$

Conversely, if $b(x) = a^{\perp}$, then $x - x \wedge a \leqq a^{\perp}$. Since always $x - x \wedge a \leqq x$, we have $x - x \wedge a \leqq x \wedge a^{\perp}$. But the reverse inequality is immediate, since $x - x \wedge a = x \wedge (x^{\perp} \vee a^{\perp})$. Hence equality holds, $x - x \wedge a = x \wedge a^{\perp}$, and the criterion of Theorem 2, (1), then tells us that $xCa^{\perp}$. Then by Theorem 2, (2), xCa, which completes the proof of Theorem 4.

COROLLARY. *Let x be an element of the orthomodular lattice L. Then x has a unique complement if and only if x commutes with all elements of L.*

This is an immediate consequence of Theorem 4.

We now turn our attention to the characterization of the *center* of an orthomodular lattice. One piece of notation: an element z is called *neutral* if $(a, b, z)T$ for all a, b in the lattice.

THEOREM 5. *Let L be an orthomodular lattice. Then the following conditions on an element $z \in L$ are equivalent:*

(1) *z is neutral;*

(2) *z has a unique complement, namely $z^{\perp}$;*

(3) *z commutes with every $a \in L$.*

Proof. The equivalence of (2) and (3) is the corollary to Theorem 4. And if (3) is true, then by Theorem 3, $(a, b, z)T$ for all $a, b \in L$ so that z is neutral. Hence (3) implies (1). Finally if z is neutral, then $(a, a \overset{\perp}{,} z)T$ for all $a \in L$ which contains as a special case aCz for all $a \in L$. Thus (1) implies (3), and this completes the proof. The set of elements which satisfy the equivalent conditions of Theorem 5 is called the center of L and denoted center (L).

COROLLARY. *The center of an orthomodular lattice L is a Boolean sublattice of L containing the 0 and 1 of L.*

Proof. By Theorem 5, (3), always $0, 1 \in \text{center}(L)$. Using the criterion (3) of Theorem 5 again, together with Theorem 2, (3), we see that center (L) contains, along with $z, w, z \vee w$ and $z \wedge w$, and so is a sublattice of L containing the 0 and 1 of L. By Theorem 5, (1), center (L) is clearly distributive. It is evident that $z^{\perp}$ has a unique complement along with z, so that by (2) of Theorem 5, center (L) contains with every z, $z^{\perp}$, and so is complemented. Hence center (L) is distributive and complemented, thus a Boolean lattice. This completes the proof of the corollary.

For some other characterizations of the center see Birkhoff [1] and F. Maeda [5], and S. Maeda [6, Lemma 1.2].

Our commutativity relation allows us to generalize the concept of the center of an orthomodular lattice. If L is orthomodular with M a nonempty subset of L, then we define the commutor of M, written M' by $M' = (x \in L; xCa$ for all $a \in M)$.

THEOREM 6. *Let M be a nonempty subset of the orthomodular lattice L. Then M' is an orthomodular sublattice of L under the same $\perp$ and with the same 0 and 1. If L is complete, so is M'. Moreover, always center $(L) \subseteq M'$, L' $= \text{center } (L)$, and $(\text{center } (L))' = L$.*

Proof. Always $0Ca$ and $1Ca$ for all $a \in L$. Hence $0, 1 \in M'$. If $x \in M'$, then by Theorem 2, (2), $x^{\perp} \in M'$. If $x, y \in M'$, then $x \vee y$ and $x \wedge y \in M'$ by Theorem 2, (3). Thus M' is always a sublattice of L with the same 0 and 1, and accordingly is also orthomodular. If $x_{\alpha} \in M'$ for all α, and $\bigvee x_{\alpha}$ and $\bigwedge x_{\alpha}$ exist, then by Theorem 2, (4), $\bigvee x_{\alpha}$ and $\bigwedge x_{\alpha}$ are also in M'. Consequently M' is complete if L is. The remaining statements follow from Theorem 5.

The following corollary is an immediate application of Theorem 6.

COROLLARY. *The center of a complete orthomodular lattice is complete.*

This was proved by S. Maeda [6, Theorem 1.3].

On the basis of the commutativity relation, we are able to give a handy criterion for deciding when the lattice operations are continuous.

THEOREM 7. *Let L be a complete orthomodular lattice with (a_{α}) a family*

of elements of L and $b \in L$. If $a_\alpha C b$ for every α, then $\bigvee(a_\alpha \wedge b) = (\bigvee a_\alpha) \wedge b$ and $\bigwedge(a_\alpha \vee b) = (\bigwedge a_\alpha) \vee b$.

Proof. It is always true that $\bigvee(a_\alpha \wedge b) \leqq (\bigvee a_\alpha) \wedge b$. Hence we can write, with $a = \bigvee a_\alpha$,

$$a \wedge b - \bigvee(a_\alpha \wedge b) = a \wedge b \wedge (a_\alpha^\perp \vee b^\perp) = \bigwedge(a \wedge b \wedge (a_\alpha^\perp \vee b^\perp)).$$

Since $a_\alpha C b$ for every α, $b C a_\alpha^\perp$ for every α, and by Theorem 2,(1), $b \wedge (a_\alpha^\perp \wedge b^\perp) = b \wedge a_\alpha^\perp$. Hence

$$a \wedge b - \bigvee(a_\alpha \wedge b) = \bigwedge(a \wedge b \wedge a_\alpha^\perp) = (a \wedge b) \wedge \bigwedge a_\alpha^\perp = b \wedge a \wedge a^\perp = 0.$$

Then by Theorem 1,(1), $a \wedge b = \bigvee(a_\alpha \wedge b)$, and by the usual trick the dual also holds. This proves the theorem. We draw an immediate

COROLLARY. *If, for each α, one of the relations $a_\alpha \leqq b$ or $a_\alpha \geqq b$ holds; or if $a_\alpha \in$ center(L) for all α; or if $b \in$ center(L), then we have $\bigvee(a_\alpha \wedge b) = (\bigvee a_\alpha) \wedge b$ and $\bigwedge(a_\alpha \vee b) = (\bigwedge a_\alpha) \vee b$.*

Part of the corollary was proved by S. Maeda [6, Lemma 1.3].

3. **Measures on complete orthomodular lattices.** Several theorems of standard measure theory do not use the distributivity of the underlying measure algebra. In this section we shall establish some of these results which will be useful in the study of gages. The equivalence relation does not yet play any role, so we work in the gene al context of measures on orthomodular lattices.

DEFINITION 4. *A measure on the complete orthomodular lattice L is a function m on L such that:*
  (1) $0 \leqq m(a) \leqq \infty$ for all $a \in L$;
  (2) $m(0) = 0$;
  (3) $m(\oplus a_\alpha) = \sum m(a_\alpha)$ (complete additivity).
A measure m is semi-finite if every nonzero element of L majorizes a nonzero element b with $m(b) < \infty$; finite if $m(a) < \infty$ for all a in L, and faithful if $m(a) = 0$ implies $a = 0$.

A measure automatically has the property: $a \leqq b \to m(a) \leqq m(b)$. For if $a \leqq b$ then $b = a \oplus (b - a)$ so $m(b) = m(a) + m(b - a) \geqq m(a)$.

THEOREM 8. *Let m be a function on the complete orthomodular lattice L which has the following properties:*
  (1) $0 \leqq m(a) \leqq \infty$ for all $a \in L$;
  (2) $m(0) = 0$;
  (3) $m(\oplus_{i=1}^n a_i) = \sum_{i=1}^n m(a_i)$ for any finite orthogonal family (a_i).
Then the following are equivalent:
  (i) *m is completely additive; that is, is a measure[3];*

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[3] It was pointed out by the referee that a countably additive measure is completely additive if and only if it is completely additive on null elements.

(ii) *if (a_α) is an ascending directed set, then $m(\bigvee a_\alpha) = \mathrm{LUB}\, m(a_\alpha)$;*

(iii) *if (a_α) is an increasing family, then $m(\bigvee a_\alpha) = \mathrm{LUB}\, m(a_\alpha)$;*

(iv) *if (a_α) is a well-ordered increasing family, then $m(\bigvee a_\alpha) = \mathrm{LUB}\, m(a_\alpha)$.*

Proof. That (iv) → (iii) follows from the fact that every increasing family has a well-ordered sub-family with the same LUB. We complete the proof by showing (iii) → (ii) → (i) → (iv).

(iii) → (ii). Denote the directed set (a_α) by D. A simple construction shows that every *countable* directed set has a linearly ordered subset with the same LUB. So if (iii) is true, then (ii) is true for every countable directed set. Now we use transfinite induction on the cardinality of D. The following ingenious lemma of Iwamura is crucial (F. Maeda [5, Appendix 2]).

LEMMA. *Let D be an infinite directed set. Then there exists in D a transfinite sequence (D_ρ), $\rho < \Omega$ (Ω some transfinite ordinal), of directed subsets with the following properties:*

(1) $\mathrm{card}(D_\rho) < \mathrm{card}(D)$;

(2) *if $\rho < \sigma < \Omega$, then $D_\rho \subseteq D_\sigma$;*

(3) $D = \bigcup_{\rho < \Omega} D_\rho$.

Let $a_\rho = \bigvee(x; x \in D_\rho)$; then, applying the lemma and the induction hypothesis, $m(a_\rho) = \mathrm{LUB}(m(x); x \in D_\rho) \leqq \mathrm{LUB}(m(a_\alpha); a_\alpha \in D)$. Whence $\mathrm{LUB}(m(a_\rho); \rho < \Omega)$ $\leqq \mathrm{LUB}(m(a_\alpha); a_\alpha \in D)$. By (2) and (3) of the lemma, (a_ρ) is an increasing family with $\bigvee a_\rho = \bigvee (a_\alpha; a_\alpha \in D) = a$ so $m(a) = m(\bigvee a_\rho) = \mathrm{LUB}(m(a_\rho); \rho < \Omega) \leqq$ $\mathrm{LUB}(m(a_\alpha); a_\alpha \in D)$. But the reverse inequality, $m(a) \geqq \mathrm{LUB}(m(a_\alpha); a_\alpha \in D)$ is obvious, so we have $m(a) = \mathrm{LUB}(m(a_\alpha); a_\alpha \in D)$.

(ii) → (i). If (a_α) is an orthogonal family, then $\oplus a_\alpha$ is the LUB of the directed set of spans of finitely many a_α. By (ii), $m(\oplus a_\alpha)$ is the LUB of the m's of finite spans, which, by finite additivity, is the sum of the m's. The sum of a (possibly uncountable) series of positive terms, whether $+\infty$ or not, is the LUB of finite partial sums, which gives the result.

(i) → (iv). We may assume that the well-ordered increasing family (a_α) contains 0 and contains all $\bigvee(a_\beta; a_\beta < a_\alpha)$ (i.e., contains all limit numbers). Setting $a_\alpha' = $ successor of a_α, a transfinite induction applied to Theorem 1,(1), gives $a = \oplus(a_\alpha' - a_\alpha)$, and $a_\alpha = \oplus_{a_\beta < a_\alpha}(a_\beta' - a_\beta)$. So $m(a_\alpha) = \sum_{a_\beta < a_\alpha} m(a_\beta' - a_\beta)$, and $\mathrm{LUB}\, m(a_\alpha) = \sum m(a_\beta' - a_\beta) = m(a)$, and this completes the proof.

COROLLARY. *Let m be a measure on the complete orthomodular lattice L, and (a_α) a decreasing family of elements of L with $\bigwedge a_\alpha = a$. If, for some index β, $m(a_\beta) < \infty$, then $\mathrm{GLB}\, m(a_\alpha) = m(a)$.*

This is an immediate application of the theorem.

THEOREM 9. *Let m be a measure on a complete orthomodular lattice which*

has the property $m(a) = 0, m(b) = 0 \rightarrow m(a \vee b) = 0$, and let $q = \bigvee(x \in L; m(x) = 0)$. Then $m(q) = 0$. We call q the maximal null element for m.

Proof. Zorn's lemma assures us of the existence of a maximal orthogonal family (x_α) of elements of L such that $m(x_\alpha) = 0$ for all α. Setting $p = \oplus x_\alpha$, $m(p) = \sum m(x_\alpha) = 0$. Hence $p \leq q, q$ being the span of all elements of measure 0. Now we show $m(a) = 0 \rightarrow a \leq p$. This will establish $q \leq p$, and so $p = q$.

We assume $m(a) = 0$, $a \not\leq p$, and derive a contradiction. If $a \not\leq p$, then $p \vee a - p \neq 0$. Since $p \oplus (p \vee a - p) = p \vee a$, we have $m(p) + m(p \vee a - p) = m(p \vee a)$. Because $m(p) = 0$ and $m(a) = 0$, it follows that $m(p \vee a) = 0$, so $m(p \vee a - p) = 0$. But $p \vee a - p \leq p^\perp, \neq 0$, and has measure zero, so $p \vee a - p$ can be added to the orthogonal family (x_α), contradicting the maximality of this family. Hence $m(a) = 0 \rightarrow a \leq p$ and the proof is complete.

The following theorem dates back in principle to the second Murray–von Neumann paper where the unrestricted additivity of the trace was first established. The key step in the proof of that result was a local approximation of the trace by functionals of the form (Tx, x). The subsequent work of Kadison and Dixmier (Dixmier [2, Lemma 8, p. 314]) clarified the role played by this fundamental lemma, and Kadison emphasized its significance by coining the apt phrase "local approximate additivity" to describe it. With little more than some formal changes, this key argument, especially as expounded by Dixmier [loc. cit.], proves our theorem. This theorem, asserting that two measures in an orthomodular lattice, under certain conditions, are locally approximately proportional, contains all its ancestors as special cases and is, we feel, by virtue of its symmetry and generality, the most natural and satisfactory consequence of the basic ideas of Murray and von Neumann. While we do not make use of this result in the sequel, it fits most naturally at this point.

THEOREM 10. *Let m and n be two semi-finite measures on the complete orthomodular lattice L such that $m(a) = 0$ if and only if $n(a) = 0$ (i.e., m and n are absolutely continuous with respect to each other). Let the nonzero element b of L be given, and let $\varepsilon > 0$ be given. Then we can find $0 \neq c \leq b$ and $\theta > 0$ such that $x \leq c \rightarrow \theta m(x) \leq n(x) \leq \theta(1 + \varepsilon)m(x)$.*

Since the basic ideas for the proof of this theorem are already contained in the proof of Lemma 8 in Dixmier [2, p. 314], we shall not repeat the details. The theorem is still valid if the lattice L is only countably complete and the measures m and n countably additive.

Concerning the assumption of semi-finiteness: by taking m to be a faithful finite measure, and setting $n(a) = 0$ if $a = 0$, and $= \infty$ otherwise, we define two measures on the lattice, absolutely continuous with respect to each other, for which the theorem clearly cannot hold. We assume semi-finiteness to exclude this possibility. A weaker hypothesis might serve as well.

If every nonzero element of L majorizes a minimal element, or atom, as, for example, when L is the lattice of *all* projections on a Hilbert space, the theorem is clearly trivial. The only interesting application, then, is to the continuous or nonatomic case. Notice also, that, having obtained this estimate within the element c of the lattice, clearly the same argument can be applied within $c^\perp$. A transfinite induction then establishes the existence of an orthogonal family (c_α) with $\oplus c_\alpha = 1$, and a family (θ_α) of numbers, $0 < \theta_\alpha < \infty$, such that $x \leq c_\alpha \rightarrow \theta_\alpha m(x) \leq n(x) \leq \theta_\alpha (1 + \varepsilon) m(x)$. However with the lack of a distributive law, if nothing more is known about the measure, these piecewise estimates cannot in general be put together to form a global estimate.

4. Gages and dimension functions. In this section we prove our results connecting dimension functions and gages. We begin by giving a summary of the theory of dimension lattices and the definitions of S. Maeda's dimension function and of a gage.

Following Loomis, we define a *dimension lattice* to be a complete orthomodular lattice L which carries an equivalence relation $\sim$ satisfying the following axioms:
   (A)   if $a \sim 0$, then $a = 0$;
   (B)   if $a_1 \perp a_2$ and $b \sim a_1 \oplus a_2$, then $b = b_1 \oplus b_2$ with $b_1 \sim a_1$, $b_2 \sim a_2$;
   (C)   if (a_α) and (b_α) are orthogonal families with $a_\alpha \sim b_\alpha$ for all α, then $\oplus a_\alpha \sim \oplus b_\alpha$;
   (D') if a and b have a common complement, then $a \sim b$.

All the results we shall establish can be proved using the following two axioms in place of (D'):
   (D)   if *not* $a \perp b$, then there exist $0 \neq a_1 \leq a$ and $0 \neq b_1 \leq b$ with $a_1 \sim b_1$;
   (E)   if $a_1 \oplus a_2 = b_1 \oplus b_2$, $a_1 \sim a_2$, $b_1 \sim b_2$, then $a_1 \sim b_1$.

Axiom (D') implies both (D) and (E) (Loomis [4, Lemma 39, Lemma 44 and Corollary]), while probably (D) and (E) together do not imply (D'), although this is apparently not known. However a considerable simplification in the exposition is possible using (D') so we shall assume it, eschewing the added generality in favour of simplicity of exposition.

An element a of L is called *finite* if $b \leq a$, $b \sim a \rightarrow b = a$; otherwise a is *infinite*. Two elements a, b of L are called *related* if there exists $0 \neq a_1 \leq a$, $0 \neq b_1 \leq b$ with $a_1 \sim b_1$. Thus axiom (D) says that any two nonorthogonal elements are related. An element z of L is called *invariant* if z is not related to $z^\perp$. Every invariant element is in the center of L and the set B of all invariant elements is a complete Boolean sublattice of the center of L (Loomis [4, Theorem 2 and its proof]). If $a \sim x \leq c$ we write $a \precsim c$, and if $a \sim x < c$, $a \prec c$. If a is in L then the following two constructions, $\bigvee (x \in L; x \precsim a)$ and $\bigwedge (z \in B; z \geq a)$, yield the same invariant element called the *hull* of a and denoted by $|a|$ (Loomis [4, Theorem 3]). An element $a \neq 0$ is called *properly infinite*[4] if $a \wedge z$ is either

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   [4] Loomis says purely infinite.

0 or is infinite for every $z \in B$ and is called *purely infinite* if $x \leqq a$, $x \neq 0 \rightarrow x$ infinite. Finally an element a is called *simple* if b is not related to $a - b$ for every $b \leqq a$.

All of these objects have their counterparts in von Neumann algebras. The invariant elements are the central projections, the simple elements, the abelian projections, and the hull of a projection is its central support.

A dimension lattice is said to be *finite* if 1 is finite and said to be *properly infinite* if all nonzero invariant elements are infinite. It is called of *Type* I if it has a simple element a such that $|a| = 1$, or, equivalently, if 1 is the union of simple elements. It is called of *Type* II if it has no simple elements and if it has a finite element b such that $|b| = 1$, or, equivalently, if 1 is the union of finite elements. A dimension lattice is called of *Type* III if all nonzero elements are infinite. We further classify as follows: of Type I_1 (resp. II_1) if of Type I (resp. II) and finite; and of Type I_∞ (resp. II_∞) if of Type I (resp. II) and properly infinite. Then we have the following theorem; in a dimension lattice L, 1 can be written uniquely as the orthogonal span of five invariant elements $z_I^{(1)}$, $z_I^{(\infty)}$, $z_{II}^{(1)}$, $z_{II}^{(\infty)}$, and z_{III} such that the lattice $L(0, z_I^{(1)})$ (resp. $L(0, z_I^{(\infty)})$, $L(0, z_{II}^{(1)})$, $L(0, z_{II}^{(\infty)})$, $L(0, z_{III})$) is of Type I, (resp. I_∞, II_1, II_∞, III). We shall write z_I for $z_I^{(1)} \oplus z_I^{(\infty)}$ and z_{II} for $z_{II}^{(1)} \oplus z_{II}^{(\infty)}$. Finally we call a dimension lattice a *factor* lattice if 0 and 1 are the only invariant elements.

As we have already noted, the set B of invariant elements is a complete Boolean sublattice of the center. By a classical theorem of Stone, B is isomorphic to the lattice of all compact open subsets of a compact Hausdorff space S which we shall call the Stone space of B. Let $C(S)$ be the space of all non-negative (finite or infinite) continuous functions on S. The space S and $C(S)$ have the following properties (see Stone [9] and Dixmier [3]): (1) In S the closure of any open set is open, and the interior of any closed set, closed; (2) If $E(z)$ represents the compact open set corresponding to $z \in B$, then the $(E(z); z \in B)$ are a basis for the topology of S; (3) If y is the span in B of the family of elements (y_α), then $E(y)$ is the closure of the set-theoretic union of the $E(y_\alpha)$ and dually; (4) $C(S)$ is a complete lattice. If we define the product of two functions in $C(S)$ to be the unique continuous function which is the "upper semi-continuous regularization" (replace the value of $f(x)$ by $\limsup_{y \rightarrow x} f(y)$ for every x) of their pointwise product (computed with the convention that $0 \infty = 0$) then all the usual algebraic rules in $C(S)$ are valid.

S. Maeda [6] defines a *dimension function* to be a mapping D of L into $C(S)$ with the following properties:

(1) if $a \sim b$, then $D(a) = D(b)$;

(2) if $a \perp b$, then $D(a \oplus b) = D(a) + D(b)$;

(3) if $z \in B$, then $D(z \wedge a) = \chi(z) D(a)$, where $\chi(z)$ is the characteristic function of $E(z)$;

(4) if $a > 0$, $D(a) > 0$;

(5) if a is finite, then $D(a)$ is finite valued a.e., where a.e. means "except on a set of the first category."

We shall make frequent use of S. Maeda's results on dimension functions and shall refer to his theorems as needed.

A *gage* [Segal, [8]] on a dimension lattice L is a semi-finite measure on L which takes the same value on equivalent elements([5]). It follows from axiom (D′) that a gage m has the further property that $m(a) = 0$ and $m(b) = 0$ together imply that $m(a \vee b) = 0$. In our alternate treatment using the weaker axioms (D) and (E) this fact was an added assumption.

The maximal null element of a gage (Theorem 9) is invariant. For let $q = \bigvee (x \in L; \ m(x) = 0)$, and suppose $a \precsim q$. Then $m(a) \leq m(q) = 0 \rightarrow a \leq q$ so q is invariant by Loomis [4, Lemma 21]. In particular if L is a factor then every gage is either faithful or identically zero. Notice that this argument does not use the semi-finiteness of m. Also, using Lemmas 26 and 28 in Loomis [4], one checks easily that if m is faithful and $a \in L$ is infinite, then $m(a) = \infty$.

A measure p (in the sense of §3) on the Boolean lattice B of invariant elements of a dimension lattice can be transplanted to the compact open subsets of its Stone space S, and determines, by standard integration theory methods, an extended real-valued functional (integral) on $C(S)$, which we also denote([6]) by p, with the properties $0 \leq p(f) \leq \infty$, $p(f + g) = p(f) + p(g)$ and $p(\alpha f) = \alpha p(f)$ for all f, g in $C(S)$ and $0 \leq \alpha \leq \infty$. By virtue of the complete additivity of the measure p, the functional it determines is *normal*; that is, if (f_α) is a directed set (up) of functions in $C(S)$ with $\mathrm{LUB} f_\alpha = f \in C(S)$, then $\mathrm{LUB}\, p(f_\alpha) = p(f)$. Also the functional is semi-finite (resp. faithful) if and only if the parent measure is semi-finite (resp. faithful).

If p is such a functional, and D is a dimension function, we denote by $p \circ D$ the composition of these two mappings: $[p \circ D](a) = p(D(a))$ for all $a \in L$.

THEOREM 11. *Let D be a dimension function on the dimension lattice L, and p a measure on B, the Boolean lattice of invariant elements of L. Then:*

(1) *$m = p \circ D$ is a measure on L which takes the same value on equivalent elements;*

(2) *m is semi-finite (and so is a gage) if and only if both $z_{\mathrm{III}} \leq$ maximal null element of p, and p is semi-finite;*

(3) *m and p have the same maximal null element. In particular m is faithful if and only if p is faithful.*

([5]) Segal's definition states only unitary equivalence, but this is the same as assuming invariance under the equivalence relation of the von Neumann algebra. See, for example, Dixmier, Ann. Sci. École Norm. Sup. **68** (1951), 185.

([6]) The same symbol is used for the measure and its associated functional to avoid a confusing multiplication of notation. It is always clear from the context which is meant.

Proof. We prove these results in the order (1), (3), (2).

(1) The equation $m(0) = 0$ is immediate If $a = \oplus a_\alpha$, then by (S. Maeda [6, Theorem 5.5]) $D(a) = \sum D(a_\alpha)$, so $m(\oplus a_\alpha) = p(D(\oplus a_\alpha)) = p(\sum D(a_\alpha))$ $= \sum p(D(a_\alpha)) = \sum m(a_\alpha)$ which proves the complete additivity. If $a \sim b$, then $m(a) = p(D(a)) = p(D(b)) = m(b)$.

(3) We show first that $m(a) = 0 \leftrightarrow p(|a|) = 0$. If $p(|a|) = 0$, then it follows immediately from [6, Theorem 5.2] that $m(a) = p(D(a)) = 0$. On the other hand if $p(|a|) \neq 0$, then, since $[D(a)](x) > 0$ a.e. in $E(|a|)$ [ibid.] we easily reach a contradiction unless $m(a) = p(D(a)) \neq 0$.

Now let s be the maximal null element of m and w that of p. We know that s is invariant, and that $m(s) = 0$, whence $p(|s|) = p(s) = 0$. Hence $s \leqq w$, and similarly $w \leqq s$, so that $w = s$.

(2) Suppose m is semi-finite. If *not* $z_{III} \leqq s \; (= w)$, then on $z_{III} - z_{III} \wedge s$, m is faithful and identically infinite on nonzero elements, a contradiction. If p is not semi-finite, then there exists $z \in B$ with the property that $0 \neq w \leqq z \to p(w) = \infty$. It follows from [6, Theorem 5.2] that $0 \neq a \leqq z \to m(a) = p(D(a)) = \infty$, contradicting the semi-finiteness of m. Thus both $z_{III} \leqq w$ and p semi-finite follow from the semi-finiteness of m. Suppose now, conversely, that both these conditions are satisfied. If $a \in L$ is given we wish to prove that there exists $0 \neq b \leqq a$ with $m(b) < \infty$. If $a \leqq z_{III} \leqq w = s$, then $m(a) = 0$ and there is nothing to prove. So we can assume $0 \neq a \wedge z_{III}^\perp = a \wedge (z_I \oplus z_{II})$, or, by changing notation, that $0 \neq a \leqq z_I \oplus z_{II}$. Then there is a finite element c with $0 \neq c \leqq a$. By [6, Theorem 5.2], $f = D(c)$ is $< \infty$ a.e. on $E(|c|)$. By the semi-finiteness of p we may choose $0 \neq z \leqq |c|$ with $p(z) < \infty$. Set $b = z \wedge c \; (\leqq a)$, then $|b| = z \wedge |c| = z \neq 0$, so $b \neq 0$ and one easily checks then that $m(b) = p(D(b)) < \infty$. Thus m is semi-finite and the theorem is proved.

By definition there is a simple element $h \leqq z_I$ with $|h| = z_I$. Any other simple element with this property is equivalent to h [4, Lemma 30]. Since any simple element is finite [4, Lemma 8], h is finite. Also a finite element d can be found with $|d| = z_{II}$. We suppose these constructions are carried out and h and d are fixed in the next lemma.

LEMMA. *Let L be a dimension lattice, and B its complete Boolean lattice of invariant elements. Let m be a gage on L, and c an element of L. Then*:

(1) *$p_c(z) = m(c \wedge z)$, considered as a function of $z \in B$, is a measure on B;*

(2) *If c is finite, p_c is semi-finite;*

(3) *If $c = h \oplus d$ (h, d as above) and m is faithful, then p_c is faithful and semi-finite.*

Proof. (1) We have $p_c(0) = m(0) = 0$, and $p_c(\oplus z_\alpha) = m(c \wedge \oplus z_\alpha) = m(\oplus(c \wedge z_\alpha))$ (corollary to Theorem 7) which in turn equals $\sum m(c \wedge z_\alpha) = \sum p_c(z_\alpha)$ so that p_c is a measure on B.

(2) Given $0 \neq z \in B$ we must find $w \in B$, $0 \neq w \leqq z$ such that $p_c(w) < \infty$. If $z \wedge c = 0$ then $p_c(z) = m(c \wedge z) = m(0) = 0 < \infty$ and we are finished already with $w = z$. So suppose $z \wedge c \neq 0$. Set $a = c \wedge z$, so that a is finite, $\neq 0$. By the semi-finiteness of m, there exists $b \in L$, $0 \neq b \leqq a$ with $m(b) < \infty$. Since $b \leqq a$, b is finite also. Now we quote Theorem 4.1 in S. Maeda [6]. He proves the existence of a sequence $r_n(b, a)$, $n = 0, 1, 2, \cdots$, with the following properties: (i) $r_n \in B$, $n = 0, 1, 2 \cdots$; (ii) $\oplus_{n=0}^{\infty} r_n = \lfloor b \rfloor$; and (iii) $r_0 \wedge a \ll r_0 \wedge b$, and for $n = 1, 2, 3, \cdots$, $r_n \wedge a \sim \oplus_{k=1}^{n} u_k^{(n)} \oplus p$, $u_k^{(n)} \sim r_n \wedge b$ and $p \ll r_n \wedge b$ where $x \ll y$ means that for any $z \in B$ either $z \wedge x = z \wedge y = 0$ or $z \wedge x \prec z \wedge y$. Now $b = a \wedge b \leqq a \wedge \lfloor b \rfloor = a \wedge \oplus_{n=0}^{\infty} r_n = \oplus_{n=0}^{\infty} (a \wedge r_n)$. Since $b \neq 0$, $a \wedge r_k \neq 0$ for some $k = 0, 1, 2, \cdots$. If $k = 0$, then $a \wedge r_0 \ll b \wedge r_0 \rightarrow m(a \wedge r_0) \leqq m(b \wedge r_0)$ which is $< \infty$ by the choice of b. If $k \neq 0$, then $m(r_k \wedge a) = km(r_k \wedge b) + m(p)$ by the second part of (iii) above and $p \ll r_k \wedge b \rightarrow m(p) \leqq m(r_k \wedge b)$ so $m(r_k \wedge a) \leqq (k+1)m(r_k \wedge b) < \infty$. So in any case we have $m(a \wedge r_k) < \infty$. Now let $w = r_k \wedge z$. Clearly $w \leqq z$ and $a \wedge r_k \neq 0 \rightarrow (c \wedge z) \wedge r_k \neq 0 \rightarrow w = z \wedge r_k \neq 0$. Also $w \in B$. Finally

$$p_c(w) = m(c \wedge w) = m(c \wedge z \wedge r_k) \leqq m(a \wedge r_k) < \infty$$

which shows p_c is semi-finite.

(3) Both h and d are finite by the remarks directly before the lemma, and it follows from axiom (D′) that $h \oplus d$ is finite [4, Theorem 7]. Hence if $c = h \oplus d$, p_c is semi-finite by (2). Suppose now that m is faithful and $c = h \oplus d$. Suppose $p_c(z) = 0 = m((h \oplus d) \wedge z) = m(h \wedge z) + m(d \wedge z)$. Then $m(h \wedge z) = m(d \wedge z) = 0$ which implies, by the faithfulness of m, that $h \wedge z = 0$ and $d \wedge z = 0$. Whence $0 = \lfloor h \wedge z \rfloor = \lfloor h \rfloor \wedge z = z_{\mathrm{I}} \wedge z$, and similarly $z_{\mathrm{II}} \wedge z = 0$. Since L admits a faithful gage, $z_{\mathrm{III}} = 0$, so that $1 = z_{\mathrm{I}} \oplus z_{\mathrm{II}}$, and $z = z \wedge 1 = z \wedge (z_{\mathrm{I}} \oplus z_{\mathrm{II}}) = (z \wedge z_{\mathrm{I}}) \oplus (z \wedge z_{\mathrm{II}}) = 0 \oplus 0 = 0$. This shows p_c is faithful and the lemma is proved.

THEOREM 12. *If m is a faithful gage on L, then there exists a faithful semi-finite measure p on B and a dimension function D on L so that $m = p \circ D$.*

Proof. Setting $c = h \oplus d$ and $p(z) = p_c(z) = m(c \wedge z)$, then p is a faithful semi-finite measure on B by the lemma. It determines a faithful semi-finite normal functional on $C(S)$. Likewise, for any $a \in L$, $p_a(z) = m(a \wedge z)$ is a measure on B which determines a normal functional or integral on $C(S)$. The appropriate form of the Radon-Nikodym theorem for this situation is given by Dixmier [3, Proposition 8]. According to this theorem, there is a one-to-one correspondence between functions $f_a \in C(S)$ and (functionals determined by) measures p_a such that $p_a(g) = p(f_a g)$ for all $g \in C(S)$. Denote the well-defined mapping $a \rightarrow f_a$ by D. We proceed to show that D is a dimension function on L.

(1) If $a \sim b$ then [4, Lemma 22] $a \wedge z \sim b \wedge z$, $z \in B \rightarrow m(a \wedge z) = m(b \wedge z)$, $z \in B \rightarrow p_a = p_b \rightarrow f_a = f_b \rightarrow D(a) = D(b)$.

(2) If $a \perp b$, then $m((a \oplus b) \wedge z) = m(a \wedge z) + m(b \wedge z) \to p_{a \oplus b} = p_a + p_b$ $\to f_{a \oplus b} = f_a + f_b \to D(a \oplus b) = D(a) + D(b)$.

(3) Clearly $p_{a \wedge z}(g)$ and $p_a(\chi(z)g)$ agree for $g = \chi(w)$, $w \in B$, since both reduce to the measure $m((a \wedge z) \wedge w)$, $w \in B$. Hence they agree for all $g \in C(S)$, so that $p(f_{a \wedge z} g) = p_{a \wedge z}(g) = p_a(\chi(z)g) = p(f_a \chi(z)g)$, $g \in C(S)$. It follows that $f_{a \wedge z} = \chi(z) f_a$, or $D(a \wedge z) = \chi(z) D(a)$.

(4) If $a > 0$, then $p_a(1) = p(a) \neq 0$ (p is faithful), so that $D(a) \neq 0$.

(5) If a is finite, p_a is semi-finite by the lemma. Then by Dixmier's theorem [3, Proposition 8] $D(a) < \infty$ in an open dense set, hence $D(a) < \infty$ a.e.

Thus for every $a \in L$, $p_a(g) = p(gD(a))$, $g \in C(S)$, where $D(a)$ is a dimension function on L. In particular, $m(a) = p_a(1) = p(D(a))$ which proves the theorem.

THEOREM 13. *Let m be a faithful gage on L and p a faithful semi-finite measure on B. Then there is a dimension function D, uniquely determined by m and p, so that $m = p \circ D$.*

Proof. By Theorem 12 there is a dimension function D and a faithful semi-finite measure p_1 on B so that $m = p_1 \circ D$. By the classical Radon-Nikodym theorem (Stone space version [3, Proposition 8]), $p_1(f) = p(gf)$ for all $f \in C(S)$ where the derivative g is in $C(S)$, $0 < g(x) < \infty$ a.e. Since L admits a faithful gage, $z_{111} = 0$. Then by [6, Corollary to Theorem 5.3], $D = gD_1$ is a dimension function on L. Then $m(a) = p_1(D_1(a)) = p(gD_1(a)) = p(D(a))$ which proves the existence. To show that D is uniquely determined, suppose that $p(D(a)) = p(D_1(a))$ for all $a \in L$ where D and D_1 are dimension functions. By [6, Theorem 5.3] there exists a function $f \in C(S)$, $0 < f(x) < \infty$ a.e. such that $D_1(a) = fD(a)$ for all $a \in L$. Let h and d be as in the Lemma. Then $c = h \oplus d$ is finite and $|c| = |h| \oplus |d| = z_1 \oplus z_{11} = 1$. Setting $g = D(c)$, it follows from [6, Theorem 5.2] that $0 < g(x) < \infty$ a.e. Set $p_1(h) = p(gh)$ and $p_2(h) = p(fgh)$, $h \in C(S)$. Both p_1 and p_2 are faithful and semi-finite [3, Proposition 8], and they are the same for $h = \chi(z)$, $z \in B$, as is easily checked. It follows that $p_1 = p_2$, whence [3, Proposition 8] $g = fg$. Since $0 < g(x) < \infty$ a.e., $f(x) = 1$ a.e. $\to f \equiv 1 \to D_1 = D$ which proves the uniqueness.

THEOREM 14. *Let m be a faithful gage and D a dimension function on L. Then there is a uniquely determined faithful semi-finite measure p on B so that $m = p \circ D$.*

Proof. By Theorem 12 there is a dimension function D_1 and a faithful semi-finite measure p_1 on B so that $m = p_1 \circ D_1$. By [6, Theorem 5.3] there is an $f \in C(S)$, $0 < f(x) < \infty$ a.e. such that $D_1 = fD$. Then setting $p(g) = p_1(fg)$, all $g \in C(S)$, p is faithful and semi-finite. Then $m(a) = p_1(D_1(a)) = p_1(fD(a)) = p(D(a))$ which proves the existence. The uniqueness follows as in Theorem 13.

5. **The Radon-Nikodym theorem.** Segal's Radon-Nikodym theorem [8,

Theorem 15], which we generalize in this section to the abstract case of a dimension lattice, compares two gages m and n on a von Neumann algebra A, and asserts that, if m is faithful, then there exists a unique positive self-adjoint operator T (in general, unbounded) affiliated with the center of A such that $n(P) = m(PT)^c$ for all projections P in A. (Segal's theorem actually states a little more.) The operator T is the Radon-Nikodym derivative. There are two problems which must be surmounted in generalizing this result: a substitute must be found for the self-adjoint operator, and one has to make sense out of the product of such an "operator" with a lattice element. As for the construction of the formal self-adjoint operator, the clue is furnished by the spectral representation of self-adjoint operators on a Hilbert space. According to the spectral theorem there is a one-to-one correspondence between self-adjoint operators and certain one-parameter families of projections. Following this lead, we define a formal selfadjoint operator to be a real-indexed, increasing family of lattice elements, and label it central if every member of its spectral family is an invariant element. We can define a product of such formal operators, all of whose spectral projections commute, by mimicking the formula for the spectral family for a product of functions. There remains the problem of extending the gage m to these objects, but this is easily accomplished. Based on these constructions then, one can state and prove the exact analogue of Segal's theorem.

It should be pointed out however that this theorem of Segal is by no means the most significant theorem of this type in his paper [8]. In this paper, and an earlier paper by Dye, the comparison of a general linear functional (not invariant under the equivalence) with a gage is the principal problem, and the corresponding Radon-Nikodym theorems are considerably deeper. To generalize these theorems to dimension lattices in the face of the serious difficulties occasioned by the noncommutativity is a problem of a higher order of difficulty.

We begin the proof by constructing the space of formal self-adjoint operators mentioned above. If L is a complete orthomodular lattice (we do not need the full set of axioms of a dimension lattice for this) denote by $e(L)$ (the e suggesting "extension") the set of all one-parameter families (a_λ) of elements from L indexed by non-negative reals, with the following properties:

  (1) $\lambda \leq \mu \rightarrow a_\lambda \leq a_\mu$;
  (2) $a_\lambda = \bigwedge (a_\mu; \mu > \lambda)$.

Set $a_\infty = \bigvee (a_\lambda; 0 \leq \lambda < \infty)$. Equality in $e(L)$ is defined by $(a_\lambda) = (b_\lambda) \leftrightarrow a_\lambda = b_\lambda$, $0 \leq \lambda < \infty$, and a relation $\leq$ (which one easily checks is a partial order) by $(a_\lambda) \leq (b_\lambda) \leftrightarrow a_\lambda \geq b_\lambda$, $0 \leq \lambda < \infty$. By identifying an element $a \in L$ with the one-parameter family $(x_\lambda) \in e(L)$, where $x_\lambda = a^\perp$ for $0 \leq \lambda < 1$, and $x_\lambda = 1$ for $\lambda \geq 1$, we embed L in a one-to-one fashion in $e(L)$. Since the equality and partial order in $e(L)$ are clearly consistent with those in L, we can regard L as a subpartially ordered set of $e(L)$, L being a complete lattice in the shared order. For convenience

the letters $a, b, c, \cdots$ will be used to denote elements of the extension-set $e(L)$ as well as elements of the original lattice L, whenever this will not cause confusion.

We shall say that two elements $a = (a_\lambda)$, $b = (b_\mu)$ of $e(L)$ *commute* if each a_λ commutes with each b_μ as elements of the lattice L. It is possible to define the product of commuting elements of $e(L)$ in terms of their spectral families which agrees with the operator product in the special case where the elements are commuting self-adjoint operators. However, we will only need a definition of the product of an element of L and one of $e(L)$, and so we restrict ourselves to this case. If $a \in L$ and $b = (b_\lambda) \in e(L)$ commute we define the product ab (easily seen to be an element of $e(L)$) by

$$ab = (a^\perp \vee b_\lambda).$$

(Note that a and (b_λ) commute if and only if a commutes with each b_λ.) It is then straightforward to check that if $a, b \in e(L)$, $a \leqq b$, and $c \in L$ commutes with both a and b, then $ac \leqq bc$.

The next step of these preliminary constructions is the extension of a measure m from the base lattice L to the extension-set $e(L)$. The definition we will give is motivated by the standard formula for the integral of a non-negative function

$$\int f d\mu = \int_0^\infty \lambda d\mu(A_\lambda) + \infty \mu(A_\infty^\perp),$$

where A is the set on which $f < \lambda$. If one bears this parent formula in mind, many of the properties of the extended measure discussed below become quite obvious.

Suppose, then, that m is a measure on the complete orthomodular lattice L. If $a = (a_\lambda) \in e(L)$ we define $m(a)$ by

(1) if $m(a_\infty^\perp) \neq 0$ or $m(a_\lambda^\perp) = \infty$ for any λ, $0 < \lambda < \infty$, then set $m(a) = \infty$;

(2) if neither condition in (1) obtains, set $m(a) = \lim_{\varepsilon \to 0^+} \int_0^\infty \lambda dm(a_\lambda \wedge a_\varepsilon^\perp)$.

The limit in (2) always exists (it may be $+\infty$), so that the definition is effective. Moreover a direct computation shows that the extension is *consistent*; that is, when applied to an element a of L, the definition gives the original value, $m(a)$. We have anticipated this fact in using the same notation for the extension as for the original measure.

Much as in standard measure theory, the following properties of the extended measure then follow.

THEOREM 15. *If $a = (a_\lambda)$, $b = (b_\lambda)$ are elements of $e(L)$, L complete orthomodular, and the measure m on L is extended to $e(L)$, then:*

(1) $m(a) < \infty$ *implies* $\lambda m(a_\lambda^\perp) \leqq m(a)$ *for* $0 < \lambda < \infty$ *and* $\lambda m(a_\lambda^\perp) \to 0$ *as* $\lambda \to \infty$;

(2) $a \leqq b$ *implies* $m(a) \leqq m(b)$.

With this lattice extension and corresponding measure extension at hand, we have already the requisite structure to make sense out of the statement of Segal's Radon-Nikodym theorem. A central self-adjoint operator is now simply an element $z = (z_\lambda)$ of $e(L)$ such that every z_λ, $0 \leq \lambda < \infty$, belongs to the center of L. The product az, for $a \in L$, then is an element of $e(L)$. If m is a measure extended to $e(L)$, then $m(az)$ makes sense. Supposing now the element z is fixed, we prove in the next theorem that $n(a) = m(az)$ is always a measure on L.

THEOREM 16([7]). $n(a) = m(az)$ is a measure on L.

Proof.   The finite additivity of n and the fact that $n(0) = 0$ are immediate. Then, by Theorem 8, to complete the proof we need only show that, if (a_t) is an increasing family of elements of L, t in some linearly ordered indexing set T, then $\text{LUB}\, n(a_t) = n(\bigvee a_t)$. Let $a = \bigvee a_t$. In the order of $e(L)$, $a_t z \leq az$ for all t, so $m(a_t z) \leq m(az)$. So it is enough to prove that $\text{LUB}\, m(a_t z) \geq m(az)$.

Accordingly we can assume $\text{LUB}\, m(a_t z) < \infty$, so that the second part of the definition of the extended measure ·s to be used in computing $m(a_t z)$. Using Theorem 15, one checks easily that the same is true for $m(az)$. If, now $\alpha > \varepsilon > 0$, the following inequality for the integrals which occur in this definition is easily established:

$$0 \leqq \int_\varepsilon^\alpha \lambda\, dm(a \wedge (z_\lambda - z_\varepsilon)) - \int_\varepsilon^\alpha \lambda\, dm(a_t \wedge (z_\lambda - z_\varepsilon)) \leqq \alpha m((a - a_t) \wedge (z_\alpha - z_\varepsilon)).$$

We have $m((a - a_t) \wedge (z_\alpha - z_\varepsilon)) < \infty$, and since $\bigwedge_t((a - a_t) \wedge (z_\alpha - z_\varepsilon)) \leq \bigwedge_t(a - a_t)$ $= 0$, the corollary to Theorem 8 gives $\text{GLB}_t\, m((a - a_t) \wedge (z_\alpha - z_\varepsilon)) = 0$. As a function of a_t, $m((a - a_t) \wedge (z_\alpha - z_\varepsilon))$ decreases as a_t increases, and we conclude the following: if α and ε are fixed, and $\delta > 0$ is given, then there is an a_{t_0} such that $a_t \geqq a_{t_0}$ implies

$$0 \leqq \int_\varepsilon^\alpha \lambda\, dm(a \wedge (z_\lambda - z_\varepsilon)) - \int_\varepsilon^\alpha \lambda\, dm(a_t \wedge (z_\lambda - z_\varepsilon)) \leqq \delta.$$

The inequality $\text{LUB}\, m(a_t z) \geqq m(az) - \delta$ now follows for any $\delta > 0$, and we conclude that $\text{LUB}\, m(a_t z) \geqq m(az)$ which completes the proof.

We now turn our attention to the case of maximum interest, where the base lattice L is a dimension lattice, and the measure m is a gage. We shall denote by B the complete Boolean lattice of invariant elements of L.

THEOREM 17.   *Let m be a gage on the dimension lattice L with maximal null element w_0, and let $z = (z_\lambda)$ be an element of $e(B)$. Then:*

([7]) As pointed out by the referee this theorem becomes clearer if one observes that for a fixed, $m(a \wedge z)$ is a measure on the Stone space of the center of L, that (z_λ) is the spectral family (essentially) of a continuous function f on S, and that $n(a)$ is simply $\int f\, dm_a$. A proof can be given along these lines.

(1) $n(a) = m(az)$, $a \in L$, is a measure on L which has the property that $a \sim b \to n(a) = n(b)$ so that n lacks only the property of semi-finiteness to be a gage;

(2) if $z_\infty^\perp \leqq w_0$, then n is semi-finite, and so is a gage on L;

(3) the maximal null element for n is $z_0 \vee w_0$, so that, in particular, n is faithful if and only if both $z_0 = 0$ and m is faithful.

Proof. (1) It follows from the previous theorem that $n(a)$ is a measure. The invariance is proved in a straightforward manner using Lemma 22 of Loomis [4].

(2) By the definition of a gage, m is semi-finite. So if $a \in L$ is given, $a \neq 0$, there is a $b_0 \leqq a$, $b_0 \neq 0$ with $m(b_0) < \infty$. Since $b_0 \neq 0$ it follows that there is a μ, $0 \leqq \mu < \infty$, with $b_0 \wedge (z_\mu \vee z_\infty^\perp) \neq 0$. Let this element be b, so $0 \neq b \leqq b_0 \leqq a$. By assumption $z_\infty^\perp \leqq w_0$, so that $m(b \wedge z_\infty^\perp) = 0$. Also $m(b \wedge z_\lambda^\perp) \leqq m(b) < \infty$ for $0 < \lambda < \infty$, so that the second part of the definition of the extended measure is to be used in computing $n(b) = m(bz)$ according to the formula

$$n(b) = \lim_{\varepsilon \to 0+} \int_\varepsilon^\infty \lambda \, dm(b \wedge (z_\lambda - z_\varepsilon)).$$

It is easy to see that $m(b \wedge (z_\lambda - z_\varepsilon))$ is constant for $\lambda \geqq \mu$, so that the integral cuts off at μ. Also, since $m(b \wedge (z_\lambda - z_\varepsilon)) \leqq m(b_0) < \infty$, the limit as $\varepsilon \to 0$ is finite. Hence $n(b) < \infty$ with $b \leqq a$, $b \neq 0$ and this shows that n is semi-finite.

(3) Let r be the maximal null element of n. One checks easily that $n(z_0 \vee w_0) = 0$ so that $z_0 \vee w_0 \leqq r$. Conversely, if $n(a) = m(az) = 0$, then by Theorem 15, $m(a \wedge z_\lambda^\perp) = 0$, $0 < \lambda < \infty$. Now $z_0 = \bigwedge_{\mu > 0} z_\mu \to z_0^\perp = \bigvee_{\mu > 0} z_\mu^\perp$ and by the corollary to Theorem 7, $\bigvee_{\mu > 0}(a \wedge z_\mu^\perp) = a \wedge \bigvee_{\mu > 0} z_\mu^\perp = a \wedge z_0^\perp$ Then by Theorem 8, $m(a \wedge z_0^\perp) = \text{LUB } m(a \wedge z_\mu^\perp) = 0$ so that $a \wedge z_0^\perp \leqq w_0$. Then $a = (a \wedge z_0) \oplus (a \wedge z_0^\perp) \leqq z_0 \vee w_0$, so that $r \leqq z_0 \vee w_0$. This completes the proof. The converse of this result is our Radon-Nikodym theorem.

THEOREM 18.   Let m be a faithful gage on the dimension lattice L. Then there is a one-to-one correspondence between gages n on L and elements $z = (z_\lambda)$ of $e(B)$ with $z_\infty = 1$ such that

$$n(a) = m(az), \qquad a \in L.$$

Proof.   If $z = (z_\lambda)$ is given, with $z_\infty = 1$, then, by Theorem 17, $n(a) = m(az)$ is a gage.

To prove the converse part, consider n given and suppose, to begin with, that it is faithful. By Theorem 12, the faithful gage m factors in the form $m = p \circ D$ where p is a faithful semi-finite measure on B and D is a dimension function. Then by Theorem 14, $n = p_1 \circ D$ where p_1 is a faithful semi-finite measure on B. The existence of an $f \in C(S)$, $0 < f(x) < \infty$ a.e. such that $p_1(g) = p(fg)$, all $g \in C(S)$, now follows from [3, Proposition 8]. Then $n(a) = p_1(D(a)) = p(fD(a))$.

For a non-negative real number λ, the set

$$F(\lambda) = \text{interior (closure } (x \in S; \, f(x) \leqq \lambda))$$

is both open and closed. Accordingly, there is a unique $z_\lambda \in B$ with $E(z_\lambda) = F(\lambda)$. A straightforward calculation shows that the one-parameter family (z_λ) satisfies the two conditions stated at the beginning of this section and so belongs to $e(B)$. The condition $f(x) < \infty$ a.e. implies that $z_\infty = 1$, so $z = (z_\lambda)$ satisfies the requirements of the theorem.

It remains now to show that $n(a) = m(az)$. The function $m(az)$ is obtained by extending the measure m from the base lattice L to the element $az = (a^\perp \vee z_\lambda)$ of $e(L)$. Suppose now a is fixed, and consider $m(az)$ as a function of $z \in e(B)$. For those z in $e(B)$ which are also in B, $az = a \wedge z$, so $m(az) = m(a \wedge z)$. This is a measure on B by the lemma preceding Theorem 12; call it p_a. This measure has its own extension to $e(B)$, also denoted p_a, and it is clear from our definition of measure extension that $p_a(z) = m(az)$, all $z \in e(B)$. Keeping a fixed, suppose now that z is that particular one-parameter family arising from the function f as described in the last paragraph (essentially the spectral family of f). Now p_a can be considered interchangeably as a functional on $C(S)$ and an extended measure on $e(B)$, and the methods of standard integration theory show that $p_a(f) = p_a(z)$.

Finally, referring to the proof of Theorem 12 (second-last sentence), the equation $m = p \circ D$ was deduced from the stronger result $p_a(g) = p(gD(a))$, all $g \in C(S)$. In particular this is true for $g = f$, f as above. Putting together our assembled equalities,

$$m(az) = p_a(z) = p_a(f) = p(fD(a)) = p_1(D(a)) = n(a),$$

which proves the existence in the case that n is faithful.

Maintaining this assumption, we prove that z is unique. Suppose that $n(a) = m(ax)$, $x = (x_\lambda) \in e(B)$, $x_\infty = 1$. (Actually $x_\infty = 1$ is a consequence of the assumed semi-finiteness of n.) By Theorem 17, (3), the maximal null element of $m(ax)$ is x_0, so that our assumption requires that $x_0 = 0$.

Consider the family of subsets $F(\lambda)$, $0 \leqq \lambda < \infty$ of S defined by

$$F(\lambda) = \bigcap_{\mu > \lambda} E(x_\mu),$$

and define a function h on S by

$$(x \in S; \, h(x) = \lambda) = F(\lambda) \cap \bigcap_{\mu > \lambda} F(\mu)^\perp.$$

The function h is defined in all of S and $h \in C(S)$; this is (essentially) the function whose spectral family is (x_λ). The conditions $x_0 = 0$, $x_\infty = 1$ imply that $0 < h(x) < \infty$ a.e. It follows as in our previous remarks that $m(ax) = p(hD(a))$

so that $n = p_2 \circ D$ where p_2 is the measure $p_2(g) = p(hg)$, all $g \in C(S)$. Whence by the uniqueness portion of Theorem 14, $p_1 = p_2$. It follows that $h = f$, $x = z$.

If n is not faithful, then, if z_0 is its maximal null element, we reduce to the dimension lattice $L(0, z_0^\perp)$, apply the previous result to get a unique one-parameter family $w = (w_\lambda)$, then set $z = (z_0 \vee w_\lambda)$. This z will satisfy $n(a) = m(az)$ and is clearly unique. This completes the proof of Theorem 18.

COROLLARY. *Let m and n be gages on the dimension lattice L such that $m(a) = 0$ implies $n(a) = 0$. Then there is a $z \in e(B)$ such that $n(a) = m(az)$. If m is faithful, z is unique.*

Proof. This is an immediate consequence of the theorem.

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