SCATTERING FOR HYPERBOLIC EQUATIONS[1]

BY

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Introduction. Consider the equations of evolution $(-\infty < t < +\infty)$

$$\text{(0)} \qquad\qquad d^r u/dt^r = Au,$$

$$\text{(1)} \qquad\qquad d^r u/dt^r = Au + Tu,$$

where u has values in a topological linear space K, and A and T are (possibly nonlinear) operators acting on a class of functions with values in K. In a general way, assume that the Cauchy problem for these equations is well-posed. Consider (0) as a 'known' equation and (1) as a perturbation of it. Then a natural problem is this: For each u_0 in a given class H_0 of solutions of (0) with a given topology, can we find a solution u_1 of (1) with the following property (P)?

(P): If $u_0^{(s)}$ is the solution of equation (0) with

$$d^j u_0^{(s)}/dt^j\big|_{t=s} = d^j u_1/dt^j\big|_{t=s} \qquad (j = 0, 1, \cdots, r-1),$$

then $u_0^{(s)}$ converges to u_0 in the topology of H_0 as $s \to +\infty$.

This might be called the Cauchy problem at $+\infty$ for (1). The mapping $u_0 \to u_1$ is called the wave operator W_+, and similarly one has the wave operator W_- by requiring the same condition except $s \to -\infty$. The scattering operator is then $S = W_+^{-1} W_-$ (where defined) and roughly describes the 'scattering' by (1) of the solutions of (0) from the early past to the late future.

We shall discuss scattering in the case when: K is a Hilbert space, A is a fixed non-negative self-adjoint operator on K, T satisfies a Lipschitz condition and is sufficiently small at infinity, and $r = 2$. As H_0 we take the completion, with respect to one inner product of a quite natural class of inner products, of the collection of solutions of (0) with 'smooth' initial conditions. The two wave operators then have equal ranges and S is a nonsingular operator on H_0.

Our primary application is to perturbations

$$\text{(1*)} \qquad \Box u = m^2 u + Tu, \qquad \Box = -\partial^2/\partial t^2 + \sum_{j=2}^{n} \partial^2/\partial x_j^2 ,$$

of the Klein-Gordon equation. First, the space of solutions of the Klein-Gordon

equation ((1*) with $Tu = 0$) is characterized in terms of the initial conditions of its elements on spacelike hyperplanes. In quantum field theory this space is relevant to a neutral scalar meson field (cf. [1], for example). §2 treats the scattering problem in the framework indicated above, which is immediately applicable to equation (1*). All solutions considered are global. Related considerations for (1*) are also made in §3 concerning the wave operators in cases not covered by the abstract treatment.

Nonrelativistic scattering theory, which is concerned with the case when $r = 1$ and A and T are fixed linear operators on K, has recently been treated with considerable success [4]. Niznik has considered a linear relativistic scattering problem from a more restrictive point of view [5].

In §5 the common range H_T of the two wave operators is considered in its own right. Although the scattering operator is not in general unitary, there does exist a uniquely determined *skew*-symmetric form on H_T derived from the one on H_0. This form, analogous to the fundamental form of classical mechanics, is basic to the quantization of equations such as (1*) [6]. Following a general suggestion of Segal [6], another way of obtaining a Hilbert space of solutions of (1*) in the linear case is by invoking the theory of eigenfunction expansions ([2], for example). For (0) an 'infinitesimal eigenspace' can be obtained relatively easily (§4) and the result is H_0 itself.

It is a pleasure to take this opportunity to thank Professor I. E. Segal for having introduced me to this research and taken a continuing interest in its progress, and for the many suggestions of his that have been realized in this work.

1. **The Klein-Gordon Hilbert space.** We shall consistently use the following notation. E^1 is the real line ('time') with points denoted by either t or x_1; E^{n-1} is $(n-1)$-dimensional Euclidean space ('space') with points $x = (x_2, \cdots, x_n)$; $n \geq 2$; the n-tuples $\mathbf{x} = (t, x) = (x_1, x_2, \cdots, x_n)$ are in E^n. For $\mathbf{x}$ and $\mathbf{y}$ in E^n, denote $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 - x \cdot y$, where $x \cdot y = \sum_{j=2}^n x_j y_j$.

The Laplacian $\sum_{j=2}^n \partial^2/\partial x_j^2$ acts on the space of distributions on E^{n-1} and $-\partial^2/\partial t^2 + \Delta = \square$ on E^n, Δ being the Laplacian. $C_c^\infty(E^n)$ denotes the C^∞ functions with compact support on E^n and $S'(E^n)$ the tempered distributions. The points of the dual space R^n of E^n are denoted by $\mathbf{k} = (k_1, k) = (k_1, k_2, \cdots, k_n)$; $\mathbf{k}^2 = \mathbf{k} \cdot \mathbf{k} = k_1^2 - \sum_{j=2}^n k_j^2$; $dk = dk_2 \cdots dk_n$, $d\mathbf{k} = dk_1 dk$, $dx = dx_2 \cdots dx_n$, $d\mathbf{x} = dt dx$. The superscripts $\hat{\ }$ and $\tilde{\ }$ will always mean Fourier transform with respect to n and $n-1$ variables, respectively:

$$\hat{u}(\mathbf{k}) = (2\pi)^{n/2} \int \exp(-i\mathbf{x} \cdot \mathbf{k}) u(\mathbf{x}) d\mathbf{x}$$

where $\mathbf{x} \cdot \mathbf{k}$ is the Lorentz inner product; $\tilde{u}$ is defined as usual.

M_m is the hyperboloid $[\mathbf{k} | \mathbf{k} \in R^n, \mathbf{k} \cdot \mathbf{k} = m^2]$, a C^∞-manifold with a local coordinate system for each sheet given by the last $n-1$ coordinates $k_2, \cdots, k_n$; fix $m \geq 0$.

LEMMA 1.1. *The measure* $(k^2 + m^2)^{-1/2} dk$ *is the unique measure (up to a constant factor) on* M_m *which is invariant under the Lorentz group of* R^n.

Proof. We may consider just the positive sheet of M_m on which $k_1 = (k^2 + m^2)^{1/2}$. Let p be the measure with 'volume' element $dp(k) = \phi(k_1) dk$, where ϕ is some measurable function of k_1, and assume that p is Lorentz-invariant. This implies that for any $\alpha > 0$, $\phi(k_1) dk_2 \cdots dk_n$ equals

$$\phi(k_1 \cosh \alpha + k_2 \sinh \alpha)\,(dk_1 \sinh \alpha + dk_2 \cosh \alpha)\,dk_3 \cdots dk_n.$$

Since $k_1 dk_1 = \sum_{j=2}^{n} k_j dk_j$ on M_m, it follows that $k_1 \phi(k_1)$ is a constant and p is as required; as the computation indicates, p is Lorentz-invariant. Now let q be an arbitrary Lorentz-invariant measure on the positive sheet of M_m. By a general theorem concerning quasi-invariant measures on homogeneous spaces, q is unique up to absolute continuity; hence, $dq(k) = \theta(k) k_1^{-1} dk$ for some measurable θ. But since θ is invariant under rotations in R^{n-1}, it must be just a function of k_1. Hence, we are reduced to the case considered above.

We shall be concerned with the Klein-Gordon (KG) equation $\Box u_0 = m^2 u_0$. Any solution u_0 has the property that its Fourier transform has support on M_m, because $(\mathbf{k}^2 - m^2)\hat{u}_0 = 0$. The *real (complex) KG Hilbert space* K_m of mass m consists of all real-valued (complex-valued) tempered distributions u_0 on E^n of the form

$$(2) \qquad u_0(\mathbf{x}) = 2^{-1}(2\pi)^{n/2} \int_{M_m} e^{-i\mathbf{x} \cdot \mathbf{k}} \phi(\mathbf{k}) dp$$

where p is the measure of Lemma 1.1, and $\phi \in L_2(M_m, p)$. K_m is considered as a Hilbert space by transference of the Hilbert space structure of $L_2(M_m, p)$ to K_m via (2) (with a normalization factor π): $\|u_0\|_{K_m}^2 = \pi^{-1} \int_{M_m} |\phi|^2 dp$. Equation (2) can also be written as

$$(3) \qquad F^{-1} u_0(\mathbf{k}) = \phi(\mathbf{k}) \delta(\mathbf{k}^2 - m^2)$$

where F is the Fourier transformation in E^n and where

$$2\delta(\mathbf{k}^2 - m^2) = k'^{-1}[\delta(k_1 - k') + \delta(k_1 + k')], \quad k' = (k^2 + m^2)^{1/2}.$$

Later we shall show how in a slightly more general situation an equation such as (3) expresses u_0 as an eigenfunction of ϕ. Using the notation $k_1 = (k^2 + m^2)^{1/2}$ from now on and putting $\phi_{\pm}(k) = \phi(\pm k_1, k)$, we can write this more explicitly:

$$(4) \qquad u_0(t, x) = 2^{-1}(2\pi)^{-n/2} \int [e^{-itk_1} k_1^{-1} \phi_+(k) + e^{itk_1} k_1^{-1} \phi_-(k)]\, e^{i\mathbf{x} \cdot \mathbf{k}} dk,$$

$$(5) \qquad 4\pi \|u_0\|_{K_m}^2 = \int |\phi_+(k)|^2 k_1^{-1} dk + \int |\phi_-(k)|^2 k_1^{-1} dk.$$

If L is a Lorentz transformation on E^n and $\mathbf{z} \in E^n$ and if $u_0 \in K_m$, define $U(L, \mathbf{z}) u_0(\mathbf{x}) = u_0(L^{-1}\mathbf{x} + \mathbf{z})$. Because the measure p is Lorentz-invariant,

$U(L,\mathbf{z})$ is an orthogonal operator on the real KG space and U is an irreducible orthogonal representation of the inhomogeneous Lorentz group.

Now equations (4) and (5) show that, if $m > 0$, u_0 is a square-integrable function on E^{n-1} for every t. On the other hand, if $m = 0$, we must and do assume that $n > 2$ in order that the function $k_1^{-1}\phi(k)$ be locally integrable; (4) is then interpreted by means of Fourier transformation on the tempered distributions on R^{n-1}. The following explicit characterization of K_m expresses the KG inner product in terms of the values of the functions on any hyperplane $t =$ constant. (For any function or distribution f, f' shall denote $\partial f/\partial t$.)

THEOREM 1.1. *Let u_0 be a function of t with values in $S'(E^{n-1})$ and assume it is twice differentiable. Then: $u_0 \in K_m$ if and only if u_0 satisfies the KG equation and*

$$(6) \qquad \int |\tilde{u}_0(t,k)|^2 k_1 dk + \int |\tilde{u}_0'(t,k)|^2 k_1^{-1} dk$$

is finite for some t (for all t if $m = 0$). If $u_0, v_0 \in K_m$,

$$(7) \qquad (u_0, v_0)_{K_m} = \int \left[\tilde{u}_0(t,k)(\tilde{v}_0)^-(t,k)k_1 + \tilde{u}_0'(t,k)(\tilde{v}_0')^-(t,k)k_1^{-1} \right] dk,$$

the right-hand side being independent of t.

Proof. First assume $u_0 \in K_m$. From (4) u_0' is obtained by formal differentiation under the integral sign. Using the Fourier inversion formula on this result and on (4), and solving the result for ϕ_+ and ϕ_-, we obtain

$$(8) \qquad \phi_\pm(k) = (2\pi)^{1/2} \exp(\pm itk_1)[k_1 \tilde{u}_0(t,k) \pm i\tilde{u}_0'(t,k)].$$

Putting (8) into (5), we conclude that (6) is equal to $\|u_0\|_{K_m}^2$ for all t and that it is finite. Equation (7) is obtained by polarization. We have the converse left to prove. Assume that u_0 satisfies the KG equation and (6) is finite for a fixed $t = t_0$ (for all t if $m = 0$).

Define $\phi_+(t,k)$ as the right-hand side of equation (8) for all t. Strictly speaking, define its action on testing functions in the obvious way (no problem here for $m = 0$ because of the additional assumption). Then

$$\partial \phi_+/\partial t = (2\pi)^{1/2} i \exp(itk_1)[\tilde{u}_0'' + k_1^2 \tilde{u}_0] = 0$$

and, therefore, ϕ_+ is independent of t; similarly for ϕ_-. Now put ϕ equal to ϕ_+ on the positive sheet of M_m and ϕ_- on the negative sheet. Then ϕ satisfies (2) and is square-integrable on M_m, so that $u_0 \in K_m$.

COROLLARY 1.1. *Let $m \geqq 0$. The collection of all elements u_0 of K_m with $u_0(t_0, \cdot) \in C_c^\infty(E^{n-1})$ and $u_0'(t_0, \cdot) \in C_c^\infty(E^{n-1})$ for a given t_0, is dense in K_m.*

Proof. This is equivalent to showing that $C_c^\infty(E^{n-1})$ is dense in the Hilbert space $N = [f \mid k_1^{1/2} \tilde{f}(k)$ is square-integrable on $R^{n-1}]$ as well as the same space

with $k_1^{1/2}$ replaced by $k_1^{-1/2}$. Let us take the first case and $m = 0$, the other cases being simpler. Let $f \in N$. Let g_j be a sequence of elements of $C_c^\infty(R^{n-1})$ converging to the square-integrable function $|k|^{1/2}\tilde{f}(k)$ in the mean square sense as $j \to \infty$. Let h_i be a C^∞-function on R^{n-1} equal to 1 for $i^{-1} < |k| < i$, between 0 and 1, and vanishing outside the set $(i+1)^{-1} < |k| < i+1$. Then f_{ij} converges to f in N, where $\tilde{f}_{ij}(k) = |k|^{-1/2}h_i(k)g_j(k) \in C_c^\infty(R^{n-1})$, as $i,j \to \infty$. Thus the space $S(E^{n-1})$ of rapidly decreasing functions is dense in N, and, therefore, so is $C_c^\infty(E^{n-1})$, since the latter is dense in $S(E^{n-1})$.

Denote the complex KG Hilbert space temporarily by K_m^c. It has two distinguished subspaces: the real KG Hilbert space K_m^r, and the complex Hilbert space K_m^p consisting of all elements of K_m^c whose Fourier transforms as tempered distributions on E^n have support on the positive sheet of M_m. An element of the latter space is said to have positive frequencies. These spaces and their inner products are Lorentz-invariant [7].

K_m^p is naturally isomorphic to K_m^r when the latter is endowed with a certain complex structure. In fact, let J_0 be the Hilbert transform with respect to time; J_0 is the operation of convolution by the Fourier transform of the function $-i\,\mathrm{sgn}(k_1)$ on R^n, k_1 here being the variable dual to t. It is easy to see that J_0 is an isometry on K_m^r which commutes with Lorentz transformations and that $J_0^2 = -I$ on K_m^r. Now define multiplication by i on K_m^r to be the operator J_0, and the imaginary part of the inner product of u_0 and v_0 to be $(J_0 u_0, v_0)_{K_m^r}$ $(u_0, v_0 \in K_m^r)$. Now K_m^p is mapped into K_m^r by $u_0 \to \mathrm{Re}(u_0)$. If we consider K_m^r as a complex Hilbert space in this way, then this mapping is an isomorphism of complex Hilbert spaces which commutes with Lorentz transformations, as is easily seen by Fourier transformation.

Alternatively, $\beta_0(u_0, v_0) = (J_0 u_0, v_0)$ may be considered as a certain skew-symmetric bilinear form on the real KG space. This real space, together with the form β_0, is relevant to a neutral scalar meson field.

Let u_0, v_0 be in K_m^p. Then $\phi_- = 0$ in the notation used earlier. By equation (8), $\tilde{u}_0' = +ik_1\tilde{u}_0$ and the same for v_0; therefore,

$$(u_0, v_0)_{K_m^p} = -i\int \left[\tilde{u}_0(t,k)(\tilde{v}_0')^-(t,k) - \tilde{u}_0'(t,k)(\tilde{v}_0)^-(t,k)\right] dk$$

(9)

$$= -i\int_t \left[u_0\bar{v}_0' - u_0'\bar{v}_0\right] dx.$$

This is a standard formula [1].

We conclude this section with a brief survey of certain Green's functions for the KG equation. The *Riemann function* D is the tempered distribution given by

(10) $$\tilde{D}(t,k) = (2\pi)^{-1/2}(k^2 + m^2)^{-1/2}\sin\left[(k^2 + m^2)^{1/2}t\right].$$

It is the unique solution of the KG equation with the initial conditions: $D(0, x) = 0$, $D'(0, x) = \delta(x)$. Let $Y(t)$ be 1 for $t > 0$ and 0 for $t < 0$. Let $D_{\text{ret}}(t, x) = - Y(t)D(t, x)$ and $D_{\text{adv}}(t, x) = Y(-t)D(t, x)$. It is then immediate that D_{ret} and D_{adv} are elementary solutions of the KG equation; that is, solutions of $(\Box - m^2)u = \delta$. D_{ret} is the only elementary solution vanishing for negative t. From the definition of D one may calculate that

$$\hat{D}(\mathbf{k}) = - i \, \text{sgn}(k_1) \, \delta(\mathbf{k}^2 - m^2) \quad \text{and} \quad \hat{E}(\mathbf{k}) = \delta(\mathbf{k}^2 - m^2)$$

where $E = \partial D / \partial t$. For specific values of n, these functions may be calculated without the intervention of Fourier transforms. For instance, if $n = 4$, we have [1]:

$$(11) \qquad D_{\text{ret}}(\mathbf{x}) = (2r)^{-1} \delta(r - t) + (m/2) Y(t) Y(t^2 - r^2) \alpha(\mathbf{x})^{-1} J_1(m\alpha(\mathbf{x}))$$

where $r = |x|$, $\alpha(\mathbf{x}) = (t^2 - r^2)^{1/2}$, and J_1 is the Bessel function of order one. For every $n \geq 2$, D has support in the solid light cone $[\mathbf{x} \mid \mathbf{x} \in E^n, |t| \geq |x|]$. For even $n \geq 4$ and $m = 0$, the support of D is on the surface of the cone (Huygens' Principle).

2. Nonlinear scattering. In this section K is a fixed Hilbert space (real or complex, separable or inseparable) and its inner product and norm are denoted simply by $(\ ,\ )$ and $|\ |$, respectively. Also a non-negative self-adjoint operator A with bounded inverse is given on K. For convenience, let B denote $A^{1/2}$.

Most of the functions considered in this section will be functions of t $(-\infty < t < +\infty)$ with values in K. The terminology of Hille and Phillips, [3, Chapter 3], on vector-valued functions will be followed. In general, u' denotes du/dt, $D(A)$ denotes the domain of A, etc.

First we set up the Hilbert space of solutions of the 'free' equation:

$$(12) \qquad\qquad\qquad -d^2 u_0 / dt^2 = A u_0(t)$$

DEFINITION 2.1. $H_0 = H_0(K, A)$ is the set of all K-valued functions u_0 of t satisfying: (a) u_0 is strongly differentiable, and its derivative u_0' is absolutely continuous and strongly differentiable a.e.; (b) $u_0(0) \in D(A)$ and $u_0'(0) \in D(B)$; (c) u_0 satisfies (12) a.e.

LEMMA 2.1. *For any $u_0, v_0 \in H_0$, the function*

$$\alpha(t) = (u_0(t), v_0(t)) + (B^{-1} u_0'(t), B^{-1} v_0'(t))$$

is constant.

Proof. α is differentiable a.e. and a direct calculation shows that $\alpha'(t) = 0$ a.e. by using the fact that u_0 and v_0 satisfy (12). But since α is absolutely continuous, it is constant.

LEMMA 2.2. *Every element $u_0 \in H_0$ is represented uniquely as*

$$(13) \qquad\qquad u_0(t) = \cos(tB) u_0(0) + \sin(tB) B^{-1} u_0'(0)$$

and therefore, $u_0(t) \in D(A)$, $u_0'(t) \in D(B)$ for all t, u_0'' is strongly continuous, and (12) is satisfied everywhere.

Proof. Let $v_0(t)$ be the right-hand side of (13) and let $w_0 = u_0 - v_0$. Then $w_0 \in H_0$ and $w_0(0) = 0$, $w_0'(0) = 0$. By the preceding lemma, w_0 vanishes identically. Therefore, (13) holds and the rest is immediate.

PROPOSITION 2.1. *For any $u_0, v_0 \in H_0$, the function*

$$\beta(t) = (B^2 u_0(t), B^2 v_0(t)) + (B u_0'(t), B v_0'(t))$$

is constant. If we define $(u_0, v_0)_{H_0} = \beta(t)$, then H_0 becomes a Hilbert space.

Proof. Let M be the collection of $u_0 \in H_0$ with $u_0(0) \in D(B^6)$ and $u_0'(0) \in D(B^6)$. By Lemma 2.1 applied to $B^4 u_0 \in H_0$ and v_0, $\beta(t)$ is constant for any $u_0 \in M$ and $v_0 \in H_0$. Since $[u_0(0) \mid u_0 \in M] = D(B^6)$ is dense in K, we have for any t $\sup_{u_0 \in M} (B^2 u_0(t), B^2 v_0(t)) = |B^2 v_0(t)|^2$ and hence, $\sup_{u_0 \in M} \beta(t) = |B^2 v_0(t)|^2 + |B v_0'(t)|^2$, the latter independent of t. Since v_0 is an arbitrary element of H_0, $\beta(t)$ is constant for any two elements of H_0 by polarization. Obviously, H_0 is a pre-Hilbert space. To show completeness of H_0, note that by Lemma 2.2 it is isomorphic to the direct sum of $D(A)$ and $D(B)$ furnished with the norm $(|Af|^2 + |Bg|^2)^{1/2}$ for $f \in D(A)$, $g \in D(B)$. The latter is a Hilbert space.

DEFINITION 2.2. X will denote the collection of all K-valued functions f of $t, -\infty < t < +\infty$, such that: f is strongly differentiable with a strongly continuous derivative f', $f(t) \in D(A)$, $f'(t) \in D(B)$ for all t, and

$$\|f\|_X = \sup_t [|Af(t)|^2 + |Bf'(t)|^2]^{1/2}$$

s finite. Then X is a Banach space with the above norm. To show that X is complete, assume that $f_1, f_2, \cdots$ is a Cauchy sequence in X. Then both $f_n(t)$ and $f_n'(t)$ are convergent uniformly in t, strongly in K. By standard reasoning, applying the mean value theorem to the numerically-valued function $(f_n(t), h)$ where $h \in K$, f_n converges in X to an element of X.

DEFINITION 2.3. Let T be any mapping defined on X whose range consists of functions defined for a.e. t with values in K. We shall call T an *admissible perturbation* if it satisfies the following conditions:

(a) $T0 = 0$ a.e.; $Tu(t) \in D(B)$ a.e. (all $u \in X$).

(b) $Tu(t)$ is a strongly measurable function of t ($u \in X$).

(c) If $u, v \in X$ then we have

(14) $$|B[Tu(t) - Tv(t)]| \leq \theta(t) |A[u(t) - v(t)]| \quad \text{a.e.}$$

where θ is a fixed integrable function.

LEMMA 2.3. *For any $u \in X$, let Lu be defined by*

(15) $$(Lu)(t) = -\int_{-\infty}^{t} \sin[B(t-s)] B^{-1} (Tu)(s) ds.$$

Then $(Lu)' = (d/dt)(Lu)$ exists strongly, $(Lu)''$ exists strongly a.e. and both of these are obtained by formal differentiation of the integral. All the integrals exist in the sense of Bochner ('are B-integrable').

Proof. We show the last statement first. To show that the integrand in (15) is B-integrable, we need two things: that it is absolutely integrable and a strongly measurable function of s. The measurability has been relegated to Lemma 2.4 below, which applies because $\sin[B(t-s)]$ is certainly strongly continuous, being just a linear combination of exponentials. On the other hand,

$$\int_{-\infty}^{t} |\sin[B(t-s)] B^{-1} Tu(s)| ds \leq \int_{-\infty}^{+\infty} \|A^{-1}\| \cdot \theta(s) \cdot |Au(s)| ds$$

$$\leq \|A^{-1}\| \cdot \int \theta(s) ds \cdot \|u\|_X < \infty.$$

Similarly, the formal derivatives are given by B-integrals (see below). Now to show that Lu is differentiable, it suffices to consider an integral such as $e^{iBt} \int_{-\infty}^{t} g(s) ds = v(t)$, where $g(s) = e^{-iBs} B^{-1} Tu(s)$. For any t, $\int_{-\infty}^{t} g(s) ds$ is in $D(B)$. Combining the fact that $e^{iBt}x$ is strongly differentiable if $x \in D(B)$ with the theorem on differentiation of indefinite B-integrals [3, p. 88], it follows by a standard argument that $v'(t)$ exists a.e. and equals the formal derivative. Therefore, $(Lu)'$ exists a.e. and

(16) $$(Lu)'(t) = -\int_{-\infty}^{t} \cos[B(t-s)] Tu(s) ds$$

a.e. Since Lu is strongly absolutely continuous, it is the indefinite integral of the right-hand side of (16) (a strongly continuous function) and therefore, $(Lu)'$ exists strongly everywhere and (16) holds for all t. The rest is similar. Indeed,

(17) $$(Lu)''(t) = B\int_{-\infty}^{t} \sin[B(t-s)] Tu(s) ds - Tu(t) \quad a.e.$$

LEMMA 2.4. *Let $x(t)$ be a strongly measurable function of t with values in K and $U(t)$ a strongly measurable function whose values are bounded self-adjoint operators on K. Then $t \to U(t)x(t)$ is also strongly measurable.*

Proof. If $h \in K$, then $x(\cdot) + U(\cdot)h$ is strongly measurable and $|x(\cdot) + U(\cdot)h|$ is measurable, and so is $(U(\cdot)x(\cdot), h)$. It remains to show that $U(\cdot)x(\cdot)$ is almost separably-valued [3], but this is easily seen to be the case because $U(\cdot)$ and $x(\cdot)$ are.

Finally, we have the following standard lemma about Volterra operators.

LEMMA 2.5. *Let Y be $L_\infty(E^1)$ with the sup norm. Define $(Gf)(t) = \int_{-\infty}^t g(t,s)f(s)ds$ for all $f \in Y$, where g is a given function satisfying $|g(t,s)| \leqq \alpha(s)$ (all s, t), $\alpha(\cdot)$ being an integrable function. Then the iterated operator G^n is bounded in norm by $[\int \alpha(s)ds]^n/(n-1)!$.*

Proof. We assert that if g_n denotes the kernel of the iterated operator G^n, then

$$(18) \qquad |g_n(t,s)| \leqq \alpha(s)\left[\int_t^s \alpha(r)dr\right]^{n-1}\bigg/(n-1)!.$$

Assuming inductively that (18) is true for $n = m - 1$,

$$|g_m(t,s)| \leqq \int_t^s |g_{m-1}(t,r)|\,|g(r,s)|\,dr$$

$$\leqq [(m-2)!]^{-1}\alpha(s)\int_t^s \alpha(r)\left[\int_r^t \alpha(p)dp\right]^{m-2}dr.$$

Upon integrating this by parts, (18) is obtained for $n = m$. This proves (18). Using this result,

$$\|G^n f\|_Y \leqq \sup_t \int_{-\infty}^\infty |g_n(t,s)|\,|f(s)|\,ds$$

$$\leqq \left[\int \alpha(r)dr\right]^n\bigg/(n-1)! \cdot \|f\|_Y,$$

proving the lemma.

We make the following two conventions. First, for $w \in X$, $\|w\|_{H_0} = \|w\|_{H_0}(t) = [\,|Aw(t)|^2 + |Bw'(t)|^2]^{1/2}$ even if w is not an element of H_0. Second, if $N : X \to X$ and $\|Nv - Nw\|_X \leqq k\|v-w\|_X$ for all v, $w \in X$, then $\|N\|$ will denote the smallest such constant k.

THEOREM 2.1. *For any $u_0 \in X$, the equation $u = u_0 + Lu$ has a unique solution $u \in X$. If $u_0 \in H_0$ then u satisfies*

$$(19) \qquad\qquad -u'' = Au + Tu$$

a.e.; this equation is satisfied everywhere in case $Tv(\cdot)$ is strongly continuous for all $v \in X$.

Proof. Let u, $v \in X$. For each t we have $Lu(t) \in D(A)$ and $(Lu)'(t) \in D(B)$ and $|A[Lu(t) - Lv(t)]| \leqq \int_{-\infty}^t |B[Tu(s) - Tv(s)]|\,ds$; $|B[(Lu)'(t) - (Lv)'(t)]|$ is bounded by the same quantity. Therefore, for all t

$$\|Lu - Lv\|_{H_0}(t) \leq 2^{1/2} \int_{-\infty}^{t} |B(Tu(s) - Tv(s))| \, ds$$

(20)

$$\leq \int_{-\infty}^{t} g(t,s) \|u - v\|_{H_0}(s) \, ds$$

by (14), where $g(t,s) = 2^{1/2}\theta(s)$. Now let G be the Volterra operator with kernel g. Then $\|L^n\| \leq \|G^n\|$ $(n \geq 0)$ and $\|G^m\| < 1$ for some m, by Lemma 2.5. Let $u_0 \in X$ and $Mu = Lu + u_0$ $(u \in X)$. Then L and M are bounded operators on X and M^m is a contraction mapping. Therefore, M has a unique fixed point u in X and indeed, u may be obtained as the limit in X as $p \to \infty$ of $M^{mp}u_0$. Now if $u_0 \in H_0$ then (12) is satisfied. But (17) tells us that $(Lu)'' = -ALu - Tu$ a.e. (everywhere under the additional assumption on T). This, together with the fact that u is a fixed point of M, shows that (19) is satisfied.

The solution u of $u = u_0 + Lu$ satisfies exactly the desired property that it is asymptotically equal to u_0 as $t \to -\infty$ (see the following theorem). The correspondence $u_0 \to u$ is called the *wave operator W_-* for the pair of equations (12) and (19); that is, W_- is the restriction of the operator $(I - L)^{-1}$ to H_0. The other wave operator W_+ is strictly analogous: simply substitute

$$(L_{\text{adv}}v)(t) = \int_{t}^{+\infty} \sin[B(t-s)]B^{-1}(Tv)(s) \, ds$$

for $L = L_{\text{ret}}$, and define W_+u_0 as the unique solution v of the equation $v = u_0 + L_{\text{adv}}v$. Then W_+ is asymptotically equal to u_0 as $t \to +\infty$. The next theorem shows in particular that any solution of (19) satisfies $u = u_0 + Lu$ for some $u_0 \in H_0$.

THEOREM 2.2.(a) $W_-(H_0)$ *consists precisely of those elements u of X such that u'' exists strongly a.e. and u satisfies* (19) *a.e.*

(b) *Let $u_0 \in H_0$ and $u = W_-u_0$. Define*

$$u_0^{(s)}(t) = \cos[B(t-s)]u(s) + \sin[B(t-s)]B^{-1}u'(s)$$

$(-\infty < t < +\infty)$. *Then $u_0^{(s)}$ converges to u_0 strongly in H_0 as $s \to -\infty$. (Property (P) of the introduction.)*

(c) *If u_0 is a given element of H_0, then W_-u_0 is the unique element of $W_-(H_0)$ satisfying property* (P).

Proof. (a) Let $u \in X$ with second derivarive existing a.e. and satisfying (19) a.e. Define $u_0 = u - Lu \in X$. Then $-u_0''$ exists and equals $Au + Tu - ALu - Tu = Au_0$ a.e. Hence $u_0 \in H_0$ and obviously $W_-u_0 = u$. (b) Let $u, u_0, u_0^{(s)}$ be as defined. Then $\|Lu\|_{H_0}(t) \leq \int_{-\infty}^{t} g(t,s)\|u\|_{H_0}(s) \, ds < \infty$ for all t by (20). Hence,

(21) $\|u - u_0\|_{H_0}(t) \to 0$ as $t \to -\infty$.

However, $u_0^{(t)}(t) = u(t)$ and $u_0^{(t)'}(t) = u(t)$, so that (P) holds. (c) Suppose that $u_0 \in H_0$ and v is in the range of W_- and v satisfies (P). Then (21) holds with u replaced by v. If $v_0 = W_-^{-1}v$ then for any t

$$\| u_0 - v_0 \|_{H_0}(t) \leqq \| v - u_0 \|_{H_0}(t) + \| v - v_0 \|_{H_0}(t),$$

which converges to 0 as $t \to -\infty$. But as the left-hand side of this inequality does not depend on t, $v = W_- u_0$.

COROLLARY 2.1. $W_- H_0 = W_+ H_0$.

Proof.   Exactly the same theorem holds for W_+ with the obvious changes.

DEFINITION 2.4. The *scattering operator* is the one-to-one mapping of H_0 onto itself defined by $S = W_+^{-1}W_-$.

LEMMA 2.6. $(I - L)^{-1}$ *is bounded on X with a bound not more than* $(1-\alpha)^{-1}\beta$, *where* $\alpha = \| L^m \| < 1$ *and* $\beta = 1 + \| L \| + \cdots + \| L^{m-1} \|$.

Proof.   Since $I - L$ is a one-to-one mapping of X onto itself, the closed graph theorem could be applied in the linear case, but to find a specific bound, consider two elements u_0, v_0 of X and let $u = (I - L)^{-1}u_0$ and $v = (I - L)^{-1}v_0$. Then, as we observed previously, $u = \lim_{p \to \infty} M^{mp}u_0 = \sum_{q=0}^{\infty}L^q u_0$ and similarly for v. Hence

$$\| u - v \|_X \leqq \lim_{p \to \infty} \sum_{q=0}^{p} \left\| \left[L^{mq}(I + L + \cdots + L^{m-1}) + L^{mp} \right] [u_0 - v_0] \right\|_X$$

$$\leqq \sum_{q=0}^{\infty} \alpha^q \beta \| u_0 - v_0 \|_X = (1-\alpha)^{-1}\beta \| u_0 - v_0 \|_X.$$

Thus is the required inequality.

COROLLARY 2.2. *There are positive constants* c_1, c_2 *such that for every* $u_0 \in H_0$

$$c_1 \| u_0 \|_{H_0} \leqq \| S u_0 \|_{H_0} \leqq c_2 \| u_0 \|_{H_0}.$$

Proof. S is the restriction of $(I - L_{\mathrm{adv}})(I - L)^{-1}$ to H_0. The inequalities follow by the preceding lemma applied to L and L_{adv}.

THEOREM 2.3. *Let* T_n, T *be admissible perturbations with wave operators* $W_-^{(n)}, W_-$ *respectively* $(n = 1, 2, \cdots)$. *Assume that there is a sequence* θ'_n *of integrable functions with the property that* $\int \theta'_n(t)dt \to 0$ *as* $n \to \infty$ *and* $\theta'_n \geqq 0$ *and that*

$$\left| B[T_n(v-w)(t) - T(v-w)(t)] \right|_K \leqq \theta'_n(t) \left| A[v(t) - w(t)] \right|_K$$

for a.e. t, for all v, $w \in X$ $(n = 1, 2, \cdots)$. *Then* $W_-^{(n)}$ *converges to* W_- *strongly in X as* $n \to \infty$.

Proof. Let θ_n be an integrable function associated with T_n in the sense of (14), and θ such a function for T. Let L_n and L be the associated integral operators (15). The hypothesis implies that we may choose $\theta_n = \theta + \theta_n'$. It may be assumed that $\int \theta_n'(t)dt < 1$ for all n. For any positive integer m, let L_n^m denote L_n iterated m times. Then, by Lemma 2.5 and (20) with appropriately added subscripts, we have $\|L_n^m\| \leq [2^{1/2} \int \theta_n(s)ds]^m/[m-1]! \leq 2^{m/2}[1+ \int \theta(s)ds]^m/[m-1]!$ for all $m > 0$, $n > 0$. Fix m so large that the right-hand side of this inequality is less than 1. Lemma 2.6 then shows that $\|(I - L_n)^{-1}\| \leq C$, C independent of n. Now if $u_0 \in H_0$, $u = W_- u_0$, and $u_n = W_-^{(n)} u_0$, then $u_n - u = (I - L_n)^{-1}(L - L_n)(I - L)^{-1} u_0$, so that $\|u_n - u\|_X \leq C \|L - L_n\| \|u\|_X$. But by Lemma 2.5 and (20) applied to the perturbation $T_n - T$, we have

$$\|L_n - L\| \leq 2^{1/2} \int^t \theta_n'(t)dt.$$

Hence $\|u_n - u\|_X \to 0$ as $n \to \infty$, proving the theorem.

THEOREM 2.4 (FINITE CAUCHY PROBLEM). *Let t_0 be real and let $f \in D(A)$, $g \in D(B)$. Then there is a unique u in the range of the wave operators with $u(t_0) = f$ and $u'(t_0) = g$.*

Proof. Let v_0 be the (unique) element of H_0 with $v_0(t_0) = f$ and $v_0'(t_0) = g$. Instead of defining L by (15), define it by the same formula with $\int_{-\infty}^t$ replaced by $\int_{t_0}^t$. The equation $u = v_0 + Lu$ can again be solved in the same way and we immediately get $u(t_0) = f$ and $u'(t_0) = g$. This demolishes the present theorem. Let $W(t_0)$ be the operator $v_0 \to u$ defined in this way, let $W(-\infty) = W_-$, let $W(+\infty) = W_+$, and let $S(t_0, t_1) = W(t_1)^{-1}W(t_0)$. Then $S = S(-\infty, \infty)$. All these wave operators $W(t)$ have equal ranges and each $S(t,s)$, $-\infty \leq t, s \leq +\infty$, is nonsingular by the same reasoning.

Now let us not assume that A has a bounded inverse, but still assume that A is one-to-one. Here is a summary of the changes needed to make most of the foregoing treatment apply in this case. Let $D = D(A^3) \cap D(A^{-1})$. Define H_0' exactly as H_0 except to require that $u_0(0)$ and $u_0'(0)$ are in D. Lemmas 2.1–2.2 and Proposition 2.1 hold for all $u_0, v_0 \in H_0'$ except that H_0' is no longer complete. Define X as before, and to the definition of admissible perturbation add the condition: (d) For a.e. t, $Tu(t) \in D(B^{-1})$, and for all $u \in X$ we have (for $j = 0$, $j = -1$) $\int \|B^j Tu(s)\| ds < \infty$. Assume K separable. Then Theorems 2.1–2.3 and Corollary 2.2 hold provided that H_0 is replaced by H_0' and Theorem 2.2(a) is replaced by: (a') $W_-(H_0')$ contains all $u \in X$ such that u'' exists strongly a.e., u satisfies (19) a.e. and $u(t), u'(t) \in D$ for all t.

3. Scattering for the KG equation. We apply the results of §2. For this purpose, let $m \geq 0$ and choose K to be the Hilbert space of all tempered distributions f on E^{n-1} such that $(k^2 + m^2)^{-3/4}\tilde{f}(k)$ is a square-integrable function of k, with $(f,g)_K = \int_{R^{n-1}}(k^2 + m^2)^{-3/2}\tilde{f}(k)\tilde{g}(k)dk$. The differential operator $-\Delta + m^2$

operating on $C_c^\infty(E^{n-1})$ has a unique self-adjoint extension on K, which is our choice for A. $D(A)$ consists of all functions such that $\int(k^2+m^2)^{1/2}|\tilde{f}(k)|^2 dk < \infty$, and on this domain we have $(Af)^\sim(k) = (k^2+m^2)\tilde{f}(k)$.

Let $m > 0$. Then A has a bounded inverse and the KG space coincides with the space $H_0(K, A)$ by Theorem 1.1. Indeed, it is straightforward that the derivatives of elements of K_m exist in the appropriate sense. We remark that K was chosen in just such a way that the elements of K_m as well as their first two derivatives with respect to time would all have values in K. From §2 we get

COROLLARY 3.1. *Let $K_m = H_0$ be the complex or real KG space with $m > 0$. Let T be an admissible perturbation in the sense of Definition 2.3. Then for every u_0 in H_0 there is a function $u = W_- u_0$ satisfying:*

(a) *For each t, $u(t, \cdot)$ is square-integrable and*

$$(22) \qquad \sup_t \left[\int |\tilde{u}(t,k)|^2 k_1 dk \right]^{1/2} + \left[\int\int |\tilde{u}'(t,k)|^2 k_1^{-1} dk \right]^{1/2} < \infty$$

(where $k_1 = (k^2+m^2)^{1/2}$).

(23)(b) $$\Box u = m^2 u + Tu \quad a.e.$$

(c) *If $u_0^{(s)}$ is the element of H_0 with the same Cauchy conditions as u on the hyperplane $t = s$, then $u_0^{(s)} \to u_0$ in H_0 as $t \to -\infty$.*

Furthermore, properties (a)–(c) characterize u uniquely in the sense that any two such functions agree a.e. For convenience, all derivatives are taken in the sense of distributions.

Proof. This is just a restatement of parts of Theorems 2.1 and 2.2. Everything is immediate but the uniqueness; for this, mimic the proof of Theorem 2.2, keeping in mind the weaker sense in which derivatives are taken.

COROLLARY 3.2. *In the same situation, the wave operators have equal ranges and S is a one-to-one mapping of H_0 onto itself, bounded with bounded inverse, and linear if T is linear.*

PROPOSITION 3.1. *In the KG situation with $m > 0$, a sufficient condition for T to be an admissible perturbation is*

(a) *For every function satisfying (22), Tu is a measurable function on E^n. If $u = 0$ a.e. then $Tu = 0$ a.e.*

(b) *There is an integrable function θ such that*

$$\| Tu(t) - Tv(t) \|_{L_2(E^{n-1})} \leq \theta(t) \| u(t) - v(t) \|_{L_2(E^{n-1})}$$

a.e. if u and v satisfy (22).

Proof. For such u and v,

$$\left\| A^{1/2}[Tu(t) - Tv(t)] \right\|_K = \left\| A^{-1/4}[Tu(t) - Tv(t)] \right\|_{L_2}$$

$$\leqq m^{-1/2} \left\| Tu(t) - Tv(t) \right\|_{L_2}$$

$$\leqq (1/m)\theta(t) \left\| A^{1/4}(u - v)(t) \right\|_{L_2}$$

$$= (1/m)\theta(t) \left\| A(u - v)(t) \right\|_K.$$

To show that T is admissible, it remains to show that Tu is a weakly measurable K-valued function of t. But $\int Tu(t,x)\overline{g(x)}dx$ is in fact an integrable function for $g \in L_2(E^{n-1})$, u satisfying (22).

For instance, T is admissible if it is multiplication by a function $V = V(t,x)$, measurable, such that $\left\| V \right\|_{L_\infty(E^{n-1})}$ is integrable. Or, if T is of the form $Tu(t,x) = \int V_1(t, x - y)u(t,y)dy$, where $\left\| \tilde{V}_1(t, \cdot) \right\|_{L_\infty(R^{n-1})}$ is integrable.

We remark that if we wish to prove smoothness of $u = W_-u_0$ (if u_0 and T are sufficiently smooth), the same successive approximations technique can be used by varying the sense in which the approximations are required to converge.

Now let $m \geqq 0$ and K and A chosen as above. Then

$$(24) \qquad (Lu)\tilde{\ }(t,k) = - \int_{-\infty}^t k_1^{-1}\sin[(t - s)k_1](Tu)\tilde{\ }(s,k)ds$$

and by (10) the integral equation we solved was

$$(25) \qquad\qquad u = u_0 + D_{\mathrm{ret}} * Tu.$$

With D_{ret} replaced by an arbitrary elementary solution of the KG equation, (25) is still formally equivalent to (23).

Consider perturbations which are not admissible as previously defined. If we are not concerned with the existence of solutions of (23) but actually have a particular solution at hand, when is it a solution of the Cauchy problem at $t = \pm \infty$? We say it is, if there is a $u_0 \in H_0$ such that property (P) holds, i.e., (21) holds. In case u satisfies (25) and for some t_0

$$(26) \qquad\qquad \int_{-\infty}^{t_0} \left\| k_1^{-1/2}(Tu)\tilde{\ }(s,k) \right\|_{L_2} ds < \infty,$$

(24) implies that for $t < t_0$

$$\left\| u - u_0 \right\|_{H_0}(t) \leqq 2^{1/2} \int_{-\infty}^t \left\| k_1^{1/2}(Tu)\tilde{\ }(s,k) \right\|_{L_2} ds$$

$(k_1 = (k^2 + m^2)^{1/2})$, so that (21) does hold. Replacing (26) by a condition independent of u, we have the following criterion (to be applied below).

PROPOSITION 3.2. *Let* $m \geqq 0$. *Let* $u_0 \in H_0$ *and assume that* u *satisfies* (25) *and that indeed* $\sum_{j=0}^\infty L^{qj}u_0$ *converges to* u *in the sense of distributions for*

some positive integer q. Let X' be a Banach space containing v_0 (see below) with the following properties: (a) X' consists of functions defined on E^n with supports contained in the half space $t < t_0$ (t_0 fixed), (b) L maps X' into itself and for all $w \in X'$, $\|w\|_{X'} = \sup_{t < t_0} \|w\|_{H_0}(t)$. Assume that T maps X' into measurable functions, that $T0 = 0$ and that

$$(27) \qquad \int_{-\infty}^{t_0} \| k_1^{-1/2}[(Tw)^{\sim}(t,k) - (Tv)^{\sim}(t,k)] \|_{L_2} dt \leqq c \|w - v\|_{X'}$$

for all $w, v \in X'$, c being constant. Then $\|u - u_0\|_{H_0}(t) \to 0$ as $t \to -\infty$.

Proof. Let v_0 be equal to u_0 for $t < t_0$ and zero otherwise. Using (27), (25) can be solved in the space X' by successive approximations, obtaining a function $w \in X'$ such that $w = \sum_{j=0}^{\infty} L^{qj} v_0$ converges in X'. Therefore, $w = u$ on the set $[\mathbf{x} \,|\, t < t_0]$ as distributions. Hence, (26) is satisfied for u and the conclusion follows.

Similar considerations apply of course for W_+.

Now consider the vanishing mass case. The remarks at the end of §2 could be applied here, but the results are not as sharp as those obtained directly, as follows. Let H_0 be the KG space with $m = 0$. Suppose that $T0 = 0$, Tu is measurable and

$$\| k^{-1/4}[(Tu)^{\sim}(t,k) - (Tv)^{\sim}(t,k)] \| \leqq \theta(t) \| k^{1/4}[\tilde{u}(t,k) - \tilde{v}(t,k)] \|$$

a.e., for all u, v satisfying (22), where θ is integrable and $\| \; \|$ is the norm in $L_2(R^{n-1})$. Then Corollaries 3.1 and 3.2 hold.

We show specifically that the wave operators can sometimes be defined even if T does not satisfy these conditions. For instance, T can be multiplication by a function independent of t in the following case. Take $m = 0$, $n = 4$. By (25) and (11) the equation to be solved is

$$(28) \qquad u(t,x) = u_0(t,x) + \int_{E^3} (Tu)(t - |y|, x - y) |2y|^{-1} dy.$$

This is solvable in the following way. Let Y be the Banach space of all functions u on E^n with $\|u\|_Y = \sup_x \int |u(t,x)| \, dt < \infty$.

LEMMA 3.1. $H_0 \cap Y$ *is dense in* H_0.

Proof. The Riemann function for the wave equation is

$$D(t,x) = (2r)^{-1}[\delta(r-t) - \delta(r+t)] \qquad (r = |x|, \; n = 4).$$

If g is in $C_c^\infty(E^{n-1})$, let $u_g = D *' g$ where $*'$ is convolution in E^{n-1}. Then $u_g(t,x) = (2t)^{-1} \int_{|y|=|t|} g(x-y) d\Sigma(y)$, where $d\Sigma$ is the euclidean surface element on the sphere; and

$$\int_{-\infty}^{\infty} |u_g(t,x)| \, dt \;\leq\; \int_{E^3} |g(x-y)| \, |y|^{-1} \, dy$$

is uniformly bounded. Hence, $u_g \in Y$ and similarly, $u_g' \in Y$. But a dense subset of H_0 consists of the functions of the form $u_g + u_f'$ where $f, g \in C_c^\infty$. All of these are in Y.

THEOREM 3.1.   *Let* $m = 0$, $n = 4$, $u_0 \in Y$. *Assume* $T0 = 0$ *and*

$$\int |(Tu - Tv)(t,x)| \, dt \;\leq\; q(x) \, \|u - v\|_Y$$

for all $x \in E^3$ *and all* $u, v \in Y$, *where* $\sup_x \int |q(y)| \, |x-y|^{-1} dy = c_T$ *is finite. Then:* (a) $[u \in Y$ *and* $\square u = Tu]$ *if and only if* u *satisfies* (28) *for some* $u_0 \in Y$ *satisfying the wave equation;* (b) *if* $c_T < 1$ *then there is a unique* $u \in Y$ *satisfying* (28).

Proof. (a) Let (,) denote the distribution pairing and let $\phi \in C_c^\infty(E^4)$ and $v \in Y$. Then, by using the explicit formula for D, $(Lv, \phi) = (D_{ret} * Tv, \phi) = (Tv, D_{adv} * \phi)$. Hence, $(\square Lv, \phi) = (Tv, \square D_{adv} * \phi) = (Tv, \phi)$, from which (a) follows immediately. Using (28) it is easy to see that $\|Lu - Lv\|_Y \leq c_T \|u - v\|_Y$. By successive approximations (28) can, therefore, be solved if $c_T < 1$.

If T is multiplication by a function of x, say V, then

$$(29) \qquad\qquad c_T = \sup_x \int_{E^3} |V(y)| \, |x - y|^{-1} dy;$$

and c_T is finite if V is integrable and square-integrable, by Schwartz's inequality.

If $m = 0$ and n is even, then Proposition 3.2 can be applied relatively easily, as we shall now show. For in that case there is a class of functions dense in the KG space H_0 with supports contained in a set of the form

$$[\mathbf{x} \,|\, \mathbf{x} \in E^n, \; |t| - d < |x| < |t| + d]$$

for some $d > 0$ (d depending on the function). To show this, let $u_0 \in H_0$ with $u_0(0, \cdot) = f$ and $u_0'(0, \cdot) = g$ both in $C_c^\infty(E^{n-1})$. Then (if $u_g = D *' g$) $u_0 = u_g + u_f'$. The assertion follows because D has its support on the set $[\mathbf{x} \,|\, |t| = |x|]$ for even n, and the set of such u_0 is dense in H_0.

For each u_0 in this dense set H_0', suppose we have a solution u of (25) with the property stated at the beginning of Proposition 3.2. Assume that T is causal in the sense that if v vanishes on a backward (solid) light cone C, then Tv also vanishes on C. Since D_{ret} has support on a forward light cone, the operator L given by $D_{ret} * Tv$ is causal in this sense if T is. Therefore, u vanishes on a backward light cone if $u_0 \in H_0'$. To apply Proposition 3.2, we choose X' to consist of functions which vanish on backward light cones. Putting this together with Theorem 3.1, we have

THEOREM 3.2. *Let $m = 0$, $n = 4$. Let T be multiplication by a function V independent of t. Assume $c_T < 1$ (cf. (29)) and also that $|V(x)| = O(|x|^{-2-\varepsilon})$ as $|x| \to \infty$ for some $\varepsilon > 0$. Then there are maps W_- and W_+ defined on H'_0 such that if $u = W_- u_0$ then:* (a) $\Box u = Tu$, (b) $\|u - u_0\|_{H_0} \to 0$ as $t \to \infty$, (c) $u \in Y$, (d) u *vanishes on the same backward light cone as u_0; and similarly for W_+.*

Proof. By rough estimates, (27) (or a slight variant thereof) is seen to hold for w, $v \in X'$ provided

$$\int_{-\infty}^{t_0} \operatorname*{ess\,sup}_{|x| \geq |t|} |V(x)| \, dt$$

and

$$\int_{-\infty}^{t_0} \left[\int_{|x| \geq |t|} |V(x)|^2 dx \right]^{1/2} dt$$

are both finite for some t_0. But they are finite under the growth hypothesis on V. Hence, the existence of W_- is assured by Theorem 3.1 and the asymptotic condition by Proposition 3.2.

Finally it should be mentioned that the case when T is multiplication by a function $V \geq 0$ of x (and not t) can be treated differently provided we restrict our attention to positive-frequency solutions of the KG equation and equation (23); i.e., complex-valued solutions u such that $\hat{u}(\mathbf{k}) = 0$ for $k_1 < 0$. The key to this is the observation that a positive-frequency element $u_0 \in K_m^p$ satisfies the equation $u'_0(t) = +i(-\Delta + m^2)^{1/2} u_0(t)$, and hence the theorem of Kuroda [4] can be applied. Under certain conditions on V, a solution of $u'(t) = iHu(t)$ is obtained which satisfies the asymptotic property (P); here the Hilbert space in question consists of all complex-valued functions f with $\int |(k^2 + m^2)^{1/4} \hat{f}(k)|^2 dk$ finite, and H^2 is a self-adjoint realization of $-\Delta + m^2 + V$. Then u is necessarily positive-frequency, as can be shown to follow (not surprisingly) from the fact that u is obtained as $u(t) = \exp(+itH)u(0)$, $H \geq 0$. It seems difficult, however, to find convenient criteria for the conditions on V to hold. Nor does it seem possible to use Kuroda's theorem for general real solutions of the equations, either by using the positive-frequency result or by reducing the equations to a system of differential equations of the first order in $\partial/\partial t$.

4. **Infinitesimal eigenspaces of** $-d^2/dt^2 - A$. Let R be any self-adjoint operator on a separable Hilbert space $\mathscr{H}$ with associated spectral measure $E(\cdot)$. There is a measure $p(\cdot)$ on the real line (with support on the spectrum of R) relative to which all the countably additive set functions $(E(\cdot)f, g)$, f, g in $\mathscr{H}$, are absolutely continuous. To fix ideas, let us assume that $\mathscr{H} = L_2(E^n)$ and $R : C_c^\infty(E^n) \to C_c^\infty(E^n)$. Suppose that for each $f \in \mathscr{H}$ there is a one-parameter family of distributions f_α (depending on f) such that

(30) $(f_\alpha, g) = \dfrac{d}{dp(\cdot)}(E(\cdot)f, g)_{\mathscr{H}}$ a.e. $[p]$

for all testing functions g, where (,) is the distribution pairing (cf. [2]). Then we have the expansion $(f, g) = \int (f_\alpha, g) dp(\alpha)$, and it is easy to see that for a.e. α the equation $Rf_\alpha = \alpha f_\alpha$ holds weakly. The question arises: for fixed α, how does the class of generalized functions f_α describable in this way, with f ranging throughout $\mathscr{H}$, compare with the spaces H_0 and H_T ($=$ the common range of the wave operators) when R is appropriately chosen? In this section we shall consider the case of H_0.

Now consider any non-negative self-adjoint operator A on a Hilbert space K. For instance, K is $L_2(E^{n-1})$ and $-A$ is the Laplacian. (This notation conflicts with that of §§2, 3 and 5.) Let $\mathscr{H}_1$ be the Kronecker product of $L_2(E^1)$ with K. It may be regarded as consisting of strongly measurable K-valued functions of t; $f \in \mathscr{H}_1$ if and only if $\int |f(t)|^2 dt$ is finite. Let D_2 be the self-adjoint realization of $-d^2/dt^2$ on $L_2(E^1)$. Let R_1 be the infinitesimal generator of the Kronecker product of the one-parameter unitary groups generated by D_2 and $-A$, by Stone's theorem. Formally, $R_1 = -d^2/dt^2 - A$ operating on $\mathscr{H}_1$. In case $f \in \mathscr{H}_1$ of the form $f(t) = F(t)v$, with $v \in K$ and $F \in L_2(E^1)$, f is in the domain of R_1 if and only if $v \in D(A)$ and F is in the domain of D_2, and in this case $(R_1 f)(t) = (D_2 F)(t)v - F(t)Av$.

We intend to show that the infinitesimal eigenspaces of R_1 for positive eigenvalues correspond to the free Hilbert spaces H_0 (with the appropriate change of K and A to agree with the usage in §2). Let $E(\cdot)$ be the spectral resolution of R_1. Let $\mathscr{H}_1'$ be the set of all $f \in \mathscr{H}_1$ with $\int |f(t)|_K dt$ finite. Let $\mathscr{H}$ be the range of the projection $E([0, \infty))$, R the restriction of R_1 to $\mathscr{H} \cap D(R_1)$, and $\mathscr{H}' = \mathscr{H} \cap \mathscr{H}_1'$.

THEOREM 4.1. *The operator R is absolutely continuous. For $\alpha > 0$, let* $G_\alpha(s, t) = (A + \alpha)^{-1/2} \cos[(t - s)(A + \alpha)^{1/2}]$. *Then*

(31) $\dfrac{d}{d\alpha}(E(\alpha)f, g)_{\mathscr{H}_1} = \displaystyle\int\!\!\int (G_\alpha(s, t)f(s), g(t))_K \, ds \, dt$

for all f and g in $\mathscr{H}_1'$ and all $\alpha > 0$.

Proof. The first statement means that $(E(\cdot)f, g)_{\mathscr{H}}$ is absolutely continuous with respect to Lebesgue measure for all $f, g \in \mathscr{H}$. Note that (31) is invariant under unitary transformations of K. Therefore, we may assume that K is the Hilbert space $L_2(M, dk)$, where (M, dk) is a measure space, and that A is multiplication by a real-valued measurable non-negative function $a(\cdot)$ on M. Then $\mathscr{H}_1$ may be identified with $L_2(E^1 \times M)$. Denote Fourier transform with respect to t by F, the dual variable of t by k_1, and a Borel set contained in the interval $[0, \infty)$ of the real line by Q. Then $F^{-1}E(Q)F$ is multiplication by the charac-

teristic function of the set $S(Q) = [(k_1, k) | (k_1, k) \in R^1 \times M$ and $k_1^2 - a(k) \in Q]$. Therefore, for $f, g \in \mathscr{H}_1'$,

$$(E(Q)f, g)_{\mathscr{H}_1} = \iint_{S(Q)} Ff(k_1, k) \overline{Fg}(k_1, k) \, dk_1 dk.$$

Breaking this up into two parts, corresponding to k_1 positive or negative, and in each part substituting $\alpha = k_1^2 - a(k)$, the double integral becomes

$$\int_M dk \int_Q d\alpha (2k^*)^{-1} \left[Ff(k^*, k) \overline{Fg}(k^*, k) + Ff(-k^*, k) \overline{Fg}(-k^*, k) \right]$$

where $k^* = +[a(k) + \alpha]^{1/2}$. The Fubini theorem may now be applied, provided the integrand is a measurable function of (k, α) in the product space $M \times Q$. This is clear in case f and g are simple K-valued functions of t. On the other hand, an arbitrary B-integrable function f can be approximated in the space of B-integrable functions by a sequence of simple functions f_j [3]. This implies that, for fixed k_1, $Ff_j(k_1, k) \to Ff(k_1, k)$ as $j \to \infty$ for k outside of a set of measure zero. If these Fourier transforms are redefined to be zero on this null set, the convergence holds everywhere. Approximating g similarly, we conclude that the integrand is measurable in the required sense and the integrals may be interchanged. In particular, it follows that the operator $R_1 E([0, \infty))$ is absolutely continuous and hence so is R.

Writing out the Fourier transforms explicitly in the iterated integral and combining terms, we obtain

$$\int_Q d\alpha \int_M dk \iint ds \, dt \, k^{*-1} \cos[(t - s)k^*] f(s, k) \bar{g}(t, k).$$

The integral over M can be exchanged with the two inner ones and therefore,

$$(E(Q)f, g)_{\mathscr{H}_1} = \int_Q d\alpha \iint ds \, dt \, (G_\alpha(s, t) f(s), g(t))_K$$

if Q is a Borel set contained in the interval (ε, ∞) for some $\varepsilon > 0$. This gives equation (31) for a.e. $\alpha > 0$; since the right-hand side of (31) is continuous in α for $\alpha > 0$, the equation may be said to hold for all $\alpha > 0$ by regarding $E(\cdot)$ as a function of a real variable in the usual way. This completes the proof.

Now let α be a fixed positive number. For each t, for $f \in \mathscr{H}'$ and $v \in K$, consider $\int (G_\alpha(s, t) f(s), v)_K ds$. This is a continuous linear functional of v and therefore there exists $f_\alpha(t) \in K$ such that

$$(32) \qquad\qquad (f_\alpha(t), v)_K = \int (G_\alpha(s, t) f(s), v)_K ds.$$

From the preceding theorem we have

(33) $$\frac{d}{d\alpha}(E(\alpha)f,g)_{\mathscr{H}} = \int (f_\alpha(t), g(t))_K dt = \int (f(t), g_\alpha(t))_K dt$$

for all $g \in \mathscr{H}'$ (g_α being defined as f_α was), the last equality holding because the operators $E(\cdot)$ are symmetric. We shall say that f_α is the α-component of f in the eigenfunction expansion of R. This terminology will be justified shortly. Equation (33) shows that the left-hand side actually depends only on f_α and g_α, not on f or g. DEFINE $\mathscr{H}'_\alpha$ as the set of all α-components of elements of $\mathscr{H}'$.

LEMMA 4.1. $\mathscr{H}'_\alpha$ is a pre-Hilbert space when furnished with the inner product

(34) $$(f_\alpha, g_\alpha)_\alpha = \frac{d}{d\alpha}(E(\alpha)f, g)_{\mathscr{H}}.$$

Proof.   We have just observed that this is well defined, and it is easily a bilinear hermitian non-negative form. Now assume $f \in \mathscr{H}'$ with $(f_\alpha, f_\alpha)_\alpha = 0$. By Schwarz's inequality, $(f_\alpha, g_\alpha)_\alpha = \int (f_\alpha(t), g(t))_K dt = 0$ if $g \in \mathscr{H}'$. On the other hand, by the preceding theorem the last equality holds if g is in the range of $E((-\infty, 0))$, and therefore for all $g \in \mathscr{H}'_1$. Now choose $g(t) = \pi(t)v$, where $v \in K$ and $\pi \in L_1(E^1) \cap L_2(E^1)$. Then $\int \pi(t)(f_\alpha(t), v)_K dt = 0$ for all such π. But (32) implies that $(f_\alpha(t), v)_K$ is a bounded continuous function of t, hence, identically zero. So $f_\alpha(t) = 0$ for all t.

DEFINE $\mathscr{H}_\alpha$ to be the completion of $\mathscr{H}'_\alpha$. Then $\mathscr{H}$ is the *direct integral* (continuous direct sum) of the spaces $\mathscr{H}_\alpha$ ($\alpha > 0$) with respect to Lebesgue measure. This means, in particular, that

$$(f, g)_{\mathscr{H}} = \int_0^\infty (f_\alpha, g_\alpha)_\alpha d\alpha$$

for f and g in $\mathscr{H}'$. This follows directly from (34).

Now we shall show that, for fixed $\alpha > 0$, $\mathscr{H}_\alpha$ is the same as $H_0(K', A')$ for a certain choice of A' and K' (cf. Definition 2.1). For convenience denote $B = (A + \alpha)^{1/2}$. If $u, v \in K$, define $(u, v)' = (B^{-3}u, v)_K$. With the inner product $(\,,\,)'$, K becomes a pre-Hilbert space. Let K' denote its completion.

LEMMA 4.2.   *The operator $B = (A + \alpha)^{1/2}$ has an extension B' which is a non-negative self-adjoint operator on K' with bounded inverse. Define $A' = B'^2$. Also $D[B'^{(n+3)/2}] = D[B^{n/2}]$ for $n =$ integer ≥ 0.*

Proof. B has a bounded inverse. Since $(B^{-1}u, B^{-1}u)' \leq c(B^{-3}u, u)_K = c(u, u)'$ for $u \in K$ ($c =$ constant), B^{-1} has a unique extension to a self-adjoint bounded operator C on all of K'. It is not difficult to see that C is one-to-one, and if we define $B' = C^{-1}$ with domain equal to the range of C, then B' is self-adjoint. The other assertions are easily proved.

THEOREM 4.2. $\mathcal{H}_\alpha'$ *is a dense subset of* $H_0(K', A')$ *and the Hilbert space structures agree. Indeed,* $u_0 \in \mathcal{H}_\alpha'$ *if* $u_0 \in H_0(K', A')$, $u_0(0) \in D(A'^{3/2})$ *and* $u_0'(0) \in D(A')$. *Therefore,* $\mathcal{H}_\alpha$ *may be identified with* $H_0(K', A')$.

Proof. We shall first show $f_\alpha \in H_0 = H_0(K', A')$ for $f \in \mathcal{H}'$. Note that the kernel G_α can be written as

(35) $$G_\alpha(s, t) = B^{-1}[\cos(sB)\cos(tB) - \sin(sB)\sin(tB)].$$

Let $w \in K$ be defined by $(w, v)_K = -\int(\sin(sB)f(s), v)_K ds$ for all $v \in K$. Then by (32) and (35)

(36) $$f_\alpha(t) = \cos(tB)f_\alpha(0) + \sin(tB)B^{-1}w.$$

Because $w \in K \subset D(B')$ and $f_\alpha(0) \in D(B) \subset D(A')$, Lemma 2.2 shows that $f_\alpha \in H_0$ and that $w = f_\alpha'(0)$. Note:

$$f_\alpha'(0) = -\int \sin(tB)f(t)dt$$

(37)

$$f_\alpha(0) = \int B^{-1}\cos(tB)f(t)dt.$$

for future reference. Now if $f, g \in \mathcal{H}'$ we obtain

$$(f_\alpha, g_\alpha)_\alpha = (Bf_\alpha(0), g_\alpha(0))_K + (B^{-1}f_\alpha'(0), g_\alpha'(0))_K = (f_\alpha, g_\alpha)_{H_0},$$

using (34), (35), (37).

It remains to show the density of $\mathcal{H}_\alpha'$ in H_0. Let $v_1 \in D(A'^{3/2}) = D(B^{3/2})$ and $v_2 \in D(A') = D(B^{1/2})$. We shall construct $f \in \mathcal{H}'$ such that f_α has the initial conditions v_1 and v_2. We may again assume that $K = L_2(M, dk)$, that A is multiplication by a non-negative measurable function $a(\cdot)$ on M and that $\mathcal{H}_1 = L_2(E^1 \times M)$. Let $w_2 = Bv_1 + iv_2$ and $w_1 = Bv_1 - iv_2$, both regarded as square-integrable functions on M. For any positive integer n, let $M_n = [k \mid k \in M, (n-1)\alpha/2 \leq a(k) < n\alpha/2]$ and $z_n = (\alpha/2)^{1/2}[(n+1)^{1/2} - n^{1/2}]$. Le $j_n \in C_c^\infty(E^1)$ with the following properties: $j_n(0) = 1$, $j_n(t) = 0$ for $|t| \geq z_n$, $0 \leq j_n \leq 1$, and $|j_n'(t)| \leq 2z_n^{-1}$ for all t. Define $h(k_1, k)$ to be equal to $j_n(k_1 - b(k))w_1(k)$ if $k_1 \geq 0$, $k \in M_n$; and equal to $j_n(k_1 + b(k))w_2(k)$ if $k_1 \leq 0$, $k \in M_n$ $(n = 1, 2, \cdots)$; where $b(k) = (a(k) + \alpha)^{1/2}$. Finally, let $f = F^{-1}h$, F the Fourier transform with respect to t, and k_1 the variable dual to t.

Then f is measurable on $E^1 \times M$; f is in $\mathcal{H}$ because

$$\int_{R^1 \times M} |h|^2 \leq \sum_{n=1}^\infty 2z_n \int_{M_n} [|w_1(k)|^2 + |w_2(k)|^2]dk < \infty$$

and because $h(k_1, k)$ vanishes for $k_1^2 - a(k) < 0$ by construction (as is easily verified). Now

$$\int_{R^1 \times M} |\partial h/\partial k_1|^2 \le \sum_{n=1}^{\infty} \int_{M_n} \left[|w_1(k)|^2 + |w_2(k)|^2 \right] 16 z_n^{-1} dk$$

$$\le c \sum_{n=1}^{\infty} \int_{M_n} \left[|w_1(k)|^2 + |w_2(k)|^2 \right] b(k) dk$$

(c independent of n), using the bound for j_n' and the fact that $z_n = O(n^{-1/2})$ as $n \to \infty$. Since $w_1, w_2 \in D(B^{1/2})$, $\partial h/\partial k_1$ is square-integrable. The product of the two square-integrable functions $(1 + |t|)^{-1}$ and $(1 + |t|)|f(t)|_K$ is integrable, so that $f \in \mathscr{H}'$. Because $j_n(0) = 1$, we have, for $k \in M$,

$$\int \exp(-itb(k)) f(t,k) dt = h(b(k), k) = w_1(k)$$

and $\int \exp(itb(k)) f(t,k) dt = w_2(k)$. From this follows $\int B^{-1} \cos(tB) f(t) dt = v_1$ and $-\int \sin(tB) f(t) dt = v_2$. Thus $f_\alpha(0) = v_1$ and $f_\alpha'(0) = v_2$ by (37), proving the theorem.

Had we completed the set of α-components (defined in the same way) of elements of $\mathscr{H}_1'$ with respect to the inner product (34), we would have obtained no more than the Hilbert space $H_0(K', A')$. This is proved in exactly the same way.

Consider the case when $K = L_2(E^{n-1})$ and A is the self-adjoint realization of $-\Delta$ on K. Then $\mathscr{H}_1 = L_2(E^n)$, $R = \square$, and $\mathscr{H}$ consist of all elements $f \in \mathscr{H}_1$ with $\hat{f}(k_1, k) = 0$ for $\mathbf{k}^2 = k_1^2 - k \cdot k < 0$. Let $E = \partial D/\partial t$, D the Riemann function for the KG equation of mass $m > 0$, and let $f \in \mathscr{H}'$. From (32) follows that $f_{m^2} = E * f$ and hence $\hat{f}_{m^2}(\mathbf{k}) = \delta(\mathbf{k}^2 - m^2) \hat{f}(\mathbf{k})$, so that f_{m^2} depends only on the restriction of $\hat{f}$ to the mass hyperboloid M_m. Intuitively, f is a result of 'extrapolating f_{m^2} off the mass hyperboloid'. The decomposition in this case is well known.

Recall that the real KG Hilbert space has the natural skew-symmetric bilinear form $\beta_0(u_0, v_0) = (J_0 u_0, v_0)_{H_0}$ associated with it. In the context of the present section we shall show how β_0 is the infinitesimal form of a form on $\mathscr{H}$. Let J_0 be the Hilbert transform with respect to time defined on $L_2(E^1)$ and therefore, also on $\mathscr{H}_1$ and $\mathscr{H}$. (The spaces may be real or complex.) Let $\alpha > 0$ and let J_0 also denote the operator on $\mathscr{H}_\alpha = H_0(K', A')$ which acts formally as the Hilbert transform; i.e., writing any element of $\mathscr{H}_\alpha$ as in (36), $J_0(\sin) = \cos$ and $J_0(\cos) = -\sin$. Define the bilinear form β_0 on $\mathscr{H}_\alpha$ exactly as in the real KG case. Let $F_\alpha(s, t) = -B^{-1} \sin[(t - s)B]$ (essentially the Riemann function in the KG case).

PROPOSITION 4.1. *Let* $f, g \in \mathscr{H}'$ *and* $\alpha > 0$. *Then for all* t

(a) $(J_0 f)_\alpha(t) = J_0(f_\alpha)(t) = \int F_\alpha(s, t) f(s) ds$ (*the first equality holding only if* $J_0 f \in \mathscr{H}'$).

(b) $\int \int (F_\alpha(s, t) f(s), g(t))_K \, ds \, dt = \beta_0(f_\alpha, g_\alpha) = (f_\alpha'(t), g_\alpha(t))_K - (f_\alpha(t), g_\alpha'(t))_K$ *for all* t, *the second equality holding for all* $f_\alpha, g_\alpha \in \mathscr{H}_\alpha$.

Proof. The second equality in (a) follows from the definition of f_α and the formal properties of J_0. As for the first one, J_0 commutes with A and with $D_2 = -d^2/dt^2$, hence, with R_1, with $E(\cdot)$, and with the operation $f \to f_\alpha$ (for $J_0 f \in \mathscr{H}'$). The first equality in (b) follows from (a) and (33); and the final one by a simple calculation, after expanding f_α and g_α as in (36).

5. Forms on the space of solutions. Let K, A (with bounded inverse), $B = A^{1/2}$, Banach space X, and admissible perturbation T be as in §2. Recall that, for $u_0 \in H_0$, $\| u_0 - W_+ u_0 \|_{H_0}(t) \to 0$ as $t \to \pm \infty$, that H_T denotes the range of W_+, and that $S = W_+^{-1} W_-$ maps H_0 onto itself in a one-to-one manner.

We shall need $H_T' = [v | v \in H_T, v(t) \in D(B^3), v'(t) \in D(A)$ for all $t]$ to be dense in H_T. To assure this, substitute for X the Banach space X' which has the norm $\| w \|_{X'} = \sup_t (|B^3 u(t)|^2 + |Au'(t)|^2)^{1/2}$; with this change, and with the assumption that $|A[Tu(t) - Tv(t)]| \leqq \theta(t) |B^3[u(t) - v(t)]|$ a.e. $(u, v \in X')$ (cf. (14)), §2 carries through as before with the obvious changes of domain, and it follows that H_T' is dense in H_T. Assume this additional condition on T, which is assured by the assumptions of Proposition 3.1, throughout this section.

For $u, v \in H_T$, *define* the inner products $(u, v)_\pm = (W_\pm^{-1} u, W_\pm^{-1} v)_{H_0}$. Corollary 2.2 states exactly that the two metrics on H_T given by $\text{dist}_\pm(u, v) = \| W_\pm^{-1} u - W_\pm^{-1} v \|_{H_0}$ are equivalent. When we speak of topological properties of H_T, we are referring to the topology induced by these two metrics, which is the same as the relative topology of H_T as a subset of X. In case T is linear, so is H_T and each of the inner products defines H_T as a Hilbert space. The two metrics are equal if and only if the scattering operator is isometric; in the linear case, if and only if H_T is defined uniquely as a Hilbert space in this way.

If $\pi_u(t) = |Au(t)|^2 + |Bu'(t)|^2$, where $u \in H_T'$, then π_u is everywhere differentiable and $\pi_u'(t) = -2 \operatorname{Re}(Au'(t), Tu(t))$ a.e. The latter equation holds everywhere in case $(Tu)(\cdot)$ is strongly continuous. Since $\| u \|_\pm^2 = \lim_{t \to \pm \infty} \pi_u(t)$, S is isometric if and only if, for u in a dense subset of H_T', we have $\operatorname{Re} \int_{-\infty}^\infty (Au'(t), Tu(t)) dt = 0$.

PROPOSITION 5.1. *Assume that the strongly continuous perturbation T is of the form $(Tu)(t) = V(t)u(t)$, $u \in X$, where $V(t)$ is a transformation on K such that $[g | g \in K, V(t)g = 0]$ is not dense in K and (in the complex case) such that $V(t)$ commutes with multiplication by the scalar $1 + i$, for each t. Let $-\infty < t_1 \leqq +\infty$. For any $s < t_1$ and $u \in X$, let $(T_s u)(t)$ be equal to $(Tu)(t)$ if $s < t < t_1$ and equal to zero otherwise. Let S_s be the scattering operator associated with the perturbation T_s. Then $[s | s < t_1, S_s$ is isometric$]$ has no finite accumulation point.*

Proof. Let $u \in H_T'$. There is a unique $v \in H_{T_s}'$ which coincides with u in the interval $s \leqq t \leqq t_1$ (s fixed). This follows from Theorem 2.4 (with $t_0 = s$) and the formulas for u and v obtainable from that theorem. Then, $\pi_v'(t) = 0$ for $t < s$ and for

$t_1 < t$. Letting $\pi_v(\pm \infty) = \lim_{t \to \pm \infty} \pi_v(t)$, we then have $\pi_v(-\infty) = \pi_v(s) = \pi_u(s)$ and $\pi_v(+\infty) = \pi_v(t_1) = \pi_u(t_1)$. Hence, $\pi_u(s) = \pi_u(t_1)$ if S_s is isometric. Now suppose that S_s is isometric, for s in a set of reals with an accumulation point c. It follows that $0 = \pi_u'(c) = -2 \, \mathrm{Re}\,(Au'(c), V(c)u(c))$. Therefore, $0 = (Au'(c), V(c)u(c))$ for all $u \in H'_T$. Hence $V(c)g = 0$ for all $g \in D(B^3)$ by Theorem 2.4. This contradiction shows our supposition to be impossible.

As an example, let a and d be positive constants, take $A = a$ and $Tu(t) = du(t)$ for $t_2 < t < t_1$, $Tu(t) = 0$ otherwise. Then H_T can be explicitly calculated, and the scattering operator is unitary if $t_1 - t_2 = 2\pi n(a + d)^{-1/2}$, $n = $ integer.

Now let β_0 be the bilinear skew form on H_0 (cf. Proposition 4.1). We shall show that the two forms on H_T which are obtainable from β_0 via the wave operators (see below) are identical. In fact, the following theorem gives both forms as a time-independent expression β which is formally the same as the one obtained for β_0 in Proposition 4.1.

In the present notation (with K, A equal to the K', A' of §4), we have

$$\beta_0(u_0, v_0) = (J_0 u_0, v_0)_{H_0} = (Bu_0'(t), Av_0'(t)) - (Au_0(t), Bv_0'(t)).$$

Define two operators J_+ and J_-, each mapping H_T onto itself, by $J_\pm = W_\pm J_0 W_\pm^{-1}$. They are anti-involutions: $J_\pm^2 = -I$.

Theorem 5.1. *Assume that T is symmetric: for u, $v \in H_T$ and a.e. t, $(u(t), Tv(t)) = (Tu(t), v(t))$. Define*

$$(38) \qquad \beta(u, v) = (Bu'(t), Av(t)) - (Au(t), Bv'(t))$$

for $u, v \in H_T$. Then $\beta(u, v)$ is independent of t and

$$\beta(u, v) = (J_+ u, v)_+ = (J_- u, v)_-.$$

Proof. For the latter notation, see the beginning of this section. Let $u, v \in H'_T$. Let $\beta(t)$ be the right-hand side of (38). Writing $\beta(t_2) - \beta(t_1)$ as an integral of its derivative, we calculate

$$\beta(t_2) - \beta(t_1) = \int_{t_1}^{t_2} [(B^3 u(t), v''(t)) - (u''(t), B^3 v(t))]\,dt = 0$$

because u and v satisfy (19) and T is symmetric. Therefore β is independent of t. Since H'_T is dense in H_T, the same is true for all $u, v \in H_T$. Now let $u, v \in H_T$, $u_0 = W_-^{-1} u$ and $v_0 = W_-^{-1} v$. Then we have

$$(J_- u, v)_- = (W_-^{-1} J_- u, W_-^{-1} v)_{H_0} = (J_0 u_0, v_0)_{H_0}.$$

Expanding the time-independent expression for the latter term using the equations $u_0 = u + Lu$ and $v_0 = v + Lv$, where L is the integral operator (15), we obtain eight terms of which two give us $\beta(u, v)$. We claim that the other six terms all converge to zero as $t \to -\infty$. Indeed, a typical such term is bounded by

$$\left|(Bu_0'(t), ALv(t))_K\right| \leq \|u_0\|_{H_0} \left|ALv(t)\right|_K \to 0 \text{ as } t \to -\infty.$$

The two time-independent expressions $\beta(u,v)$ and $\beta_0(u_0,v_0)$ are therefore equal modulo terms which vanish at $-\infty$. So they are always equal. Similarly $\beta(u,v) = (J_+u,v)_+$.

COROLLARY. 5.1.   *S is symplectic with respect to β_0; that is, $\beta_0(Su_0, Sv_0) = \beta_0(u_0, v_0)$ for $u_0, v_0 \in H_0$.*

Proof. By the preceding theorem we have $(J_-W_-u_0, W_-v_0) = (J_+W_+Su_0, W_+Sv_0)$. Therefore, $(J_0u_0, v_0)_{H_0} = (J_0Su_0, Sv_0)_{H_0}$.

Consider the case when H_0 is the real KG space for some positive mass. Let L be any Lorentz transformation on E^n. If u is a function on E^n, then u^L is defined by $u^L(\mathbf{x}) = u(L^{-1}\mathbf{x})$. T^L is the admissible perturbation defined by $T^Lu = [T(u^L)]^{L-1}$. Let $W_\pm(T)$ denote the wave operators for a perturbation T. Then it is easy to ascertain that: (a) $[W_\pm(T)u_0]^L = W_\pm(T^L)(u_0^L)$ if L is orthochronous, the signs on the right-hand side being reversed if L is not; (b) $\beta(u^L, v^L) = \pm \beta(u, v)$ for all L, with a plus if L is orthochronous and a minus otherwise.

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