SEPARATION OF THE n-SPHERE BY AN $(n-1)$-SPHERE

BY

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1. **Introduction.** Let A be the closed spherical ball in E^n centered at the origin O, and with radius one, B the closed ball centered at O with radius one-half, and C the closed ball centered at O with radius two. The Generalized Schoenflies Theorem states that, if h is a homeomorphism of $\mathrm{Cl}\,(C-B)$ into S^n, then $h(\mathrm{Bd}A)$ is tame in S^n (the closure of either component of $S^n - h(\mathrm{Bd}A)$ is a closed n-cell) [5]. One is naturally led to the following question: if h is a homeomorphism of $\mathrm{Cl}(A-B)$ into S^n, is the closure of the component of $S^n - h(\mathrm{Bd}A)$ which contains $h(\mathrm{Bd}B)$ a closed n-cell? This question is answered affirmatively by Theorem 1 and should be listed as a corollary to the Generalized Schoenflies Theorem.

Let D be the closed ball in E^n, centered at $(0,0,\cdots,0,-1)$ with radius two. Two other types of embeddings of $\mathrm{Bd}A$ in S^n, $n>3$, are considered in §2, (1) the embedding homeomorphism h can be extended to a homeomorphism of $\mathrm{Cl}(D-B)$ into S^n such that the extension is semi-linear on each finite polyhedron in the open annulus $\mathrm{Int}\,(A-B)$, and (2) h can be extended to a homeomorphism of $\mathrm{Cl}(D-A)$ into S^n such that the extension is semi-linear in a deleted neighborhood of $(0,0,\cdots,0,1)$ (see Definition 1). Theorem 4 strongly suggests that, for an embedding of type (1), $h(\mathrm{Bd}A)$ is tame in S^n. An embedding of this type corresponds to the three dimensional case in which $h(\mathrm{Bd}A)$ is locally polyhedral except at one point.

In §3, three methods of constructing 3-spheres in S^4 from 2-spheres in S^3 are considered: (1) suspension of a 2-sphere in S^3, (2) rotation of a 2-cell in S^3 about the plane of its boundary, and (3) capping a cylinder over a 2-sphere in S^3. The construction methods in cases (1) and (2) were introduced by Artin [2] and have been used by him and by Andrews and Curtis [1] to construct 2-spheres in S^4 from 1-spheres in S^3. Their techniques may be applied directly to establish isomorphism theorems relating the fundamental groups of the complements of the constructed 3-spheres and the fundamental groups of the corresponding complements of the given 2-spheres. Thus, methods (1) and (2) may be used to construct wild (nontame) 3-spheres in S^4. Method (2) is also used to construct

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a 3-sphere in S^4, one complementary domain of which is simply connected but is not an open 4-cell. The third method is used to construct a 3-sphere in S^4 such that one complementary domain has a closure which is a closed 4-cell, and the other complementary domain is an open 4-cell but its closure is not a closed 4-cell.

2. **Some embeddings of S^{n-1} in S^n.** The reader is referred to [5] for the definitions of inverse set and cellular set.

THEOREM 1. *Let h be a homeomorphism of* $\mathrm{Cl}(A - B)$ *into S^n and let G be the component of $S^n - h(\mathrm{Bd}A)$ which contains $h(\mathrm{Bd}B)$. Then $\mathrm{Cl}G$ is a closed n-cell*

Proof. Let G' be the component of $S^n - h(\mathrm{Bd}B)$ which does not contain $h(\mathrm{Bd}A)$. We first observe that $\mathrm{Cl}G'$ is a cellular subset of G. For, if B_i is the closed ball in E^n, centered at O with radius $1/2 + 1/(i+2)$, $i = 1,2,\cdots,$ and G_i is the component of $S^n - h(\mathrm{Bd}B_i)$ which contains G', then, by the Generalized Schoenflies Theorem, $\mathrm{Cl}G_i$ is a closed n-cell. Furthermore $\mathrm{Cl}G_{i+1} \subset G_i$ and $\bigcap_{i=1}^{\infty} \mathrm{Cl}G_i = \mathrm{Cl}G'$.

Let g be a continuous mapping of $\mathrm{Cl}(A - B)$ onto A such that $\mathrm{Bd}B$ is the only inverse set. Define a mapping f of $\mathrm{Cl}G$ onto A by the equations

$$f(x) = gh^{-1}(x), \text{ if } x \in \mathrm{Cl}G - G',$$

$$f(x) = g(\mathrm{Bd}B), \text{ if } x \in G'.$$

The mapping f carries $\mathrm{Cl}G$ continuously onto A such that the only inverse set is the cellular subset $\mathrm{Cl}G'$ of G. Thus, by Theorem 2 of [5], $\mathrm{Cl}G$ is a closed n-cell.

THEOREM 2. *Let h be a homeomorphism of* $\mathrm{Cl}(D - B)$ *into S^n* and *G be let the component of $S^n - h(\mathrm{Bd}A)$ which intersects $h(\mathrm{Bd}D)$. Then G is an open n-cell.*

Proof. Let H be the component of $S^n - h(\mathrm{Bd}A)$ which contains $h(\mathrm{Bd}B)$. By Theorem 1, $\mathrm{Cl}H$ is a closed n-cell and, hence, there is a homeomorphism f of A onto $\mathrm{Cl}H$ such that f and h agree on $\mathrm{Bd}A$. Define a homeomorphism ϕ of D into S^n by the equations

$$\phi(x) = h(x), \text{ if } x \in D - A,$$

$$\phi(x) = f(x), \text{ if } x \in A.$$

Let $\phi[(0,0,\cdots,0,1)] = p$ and let g be a continuous mapping of D onto D such that, (1) g is fixed on $\mathrm{Bd}D$, (2) g is a homeomorphism of $D - A$ onto $D - (0,0,\cdots,0,1)$, and (3) $g(A) = (0,0,\cdots,0,1)$. Now define a continuous mapping ψ of S^n onto S^n by the equations

$$\psi(x) = x, \qquad \text{if } x \in S^n - \phi(D),$$

$$\psi(x) = \phi g \phi^{-1}(x), \text{ if } x \in \phi(D).$$

The mapping ψ carries S^n onto S^n, leaves p fixed, and has $\mathrm{Cl}H$ as the only inverse set. Hence, G is carried homeomorphically onto $S^n - p$, and is an open n-cell.

Let B_1 be the closed ball in E^n which is centered at O and has radius three-fourths, and let L' be the closed segment of the x_n-axis from $(0,0,\cdots,0,3/4)$ to $(0,0,\cdots,0,1)$.

THEOREM 3. *Let h be a homeomorphism of* $\mathrm{Cl}(D-B)$ *into S^n and denote $h(L')$ by L and $h(0,0,\cdots,0,1)$ by p. Let G be the component of $S^n - h(\mathrm{Bd}A)$ which intersects $h(\mathrm{Bd}D)$ and let H be the component of $S^n - h(\mathrm{Bd}B_1)$ which contains $h(\mathrm{Bd}A)$. Then* $\mathrm{Cl}H$ *is a closed n-cell and* $(\mathrm{Cl}G) - p$ *is topologically equivalent to* $\mathrm{Cl}H - L$.

Proof. That $\mathrm{Cl}H$ is a closed n-cell follows immediately from Theorem 1.

Let K be the component of $S^n - h(\mathrm{Bd}D)$ which does not intersect $h(\mathrm{Bd}A)$ and let g be a continuous mapping of $\mathrm{Cl}(D-B_1)$ onto $\mathrm{Cl}(D-A)$ such that (1) g is fixed on $\mathrm{Bd}D$, (2) $g(\mathrm{Bd}B_1) = \mathrm{Bd}A$, and (3) L' is the only inverse set under g. The mapping f of $\mathrm{Cl}H$ onto $\mathrm{Cl}G$ defined by

$$f(x) = x, \qquad \text{if } x \in K,$$
$$f(x) = hgh^{-1}(x), \text{ if } x \in \mathrm{Cl}H - K,$$

is a continuous mapping of $\mathrm{Cl}H$ onto $\mathrm{Cl}G$ such that the only inverse set is L and $f(L) = p$. Hence, f is a homeomorphism of $\mathrm{Cl}H - L$ onto $\mathrm{Cl}G - p$.

If in Theorem 3 there exists a continuous mapping k of $\mathrm{Cl}H$ onto $\mathrm{Cl}H$ such that L is the only inverse set, then we can state that $\mathrm{Cl}G$ is a closed n-cell. In fact, the product mapping kf^{-1} is a homeomorphism of $\mathrm{Cl}G$ onto $\mathrm{Cl}H$.

Let us now suppose that $n > 3$ and that h is semi-linear on each finite polyhedron of Int $(A - B)$ (we assume a curved decomposition of E^n in which A, B, B_1, and L' are polyhedra). Then $h(\mathrm{Bd}B_1)$ is a polyhedron and L is locally polyhedral except at p. Let $\varepsilon > 0$ be such that $S(\varepsilon,p) \subset H$ and use Lemma 2 of [6] to obtain a homeomorphism ϕ of S^n onto S^n such that ϕ is fixed outside $S(\varepsilon,p)$ and $\phi(L)$ is polyhedral. Let q be the endpoint of L which lies on $\mathrm{Bd}H$ and let Q be a polyhedral n-cell in $\mathrm{Cl}H$ such that $q \in \mathrm{Bd}Q$, $\phi(L) - q \subset \mathrm{Int}\, Q$, and Q has a subdivision isomorphic to a subdivision of a simplex (see [7, Lemma 5.3]). Let ψ be a semi-linear homeomorphism of Q onto a simplex R. The arc $\psi\phi(L)$ is then polyhedral in R and, together with the linear segment $\overline{\psi\phi(q)\psi\phi(p)}$, from $\psi\phi(q)$ to $\psi\phi(p)$, bounds a polyhedral 2-cell which, except for $\psi\phi(q)$, lies in the interior of R. Lemma 3 of [9] is then applied to obtain a homeomorphism η of R onto R such that η is fixed on $\mathrm{Bd}R$ and carries $\psi\phi(L)$ onto $\overline{\psi\phi(q)\psi\phi(p)}$. It is then easy to find a continuous mapping θ of R onto R such that θ is fixed on $\mathrm{Bd}R$, $\theta(\overline{\psi\phi(q)\psi\phi(p)}) = \psi\phi(q)$, and $\overline{\psi\phi(q)\psi\phi(p)}$ is the only inverse set. The mapping k, defined by

$$k(x) = \phi(x), \qquad \text{if } x \notin \phi^{-1}(Q),$$
$$k(x) = \psi^{-1}\theta\eta\psi\phi(x), \quad \text{if } x \in \phi^{-1}(Q),$$

is a continuous mapping of $\mathrm{Cl}H$ onto $\mathrm{Cl}H$ such that L is the only inverse set. Thus, we have the following theorem.

THEOREM 4. *Let $n > 3$ and let h be a homeomorphism of* $\mathrm{Cl}(D - B)$ *into* S^n. *If h is semi-linear on each finite polyhedron of* $\mathrm{Int}\,(A - B)$, *then* $h(\mathrm{Bd}A)$ *is tame in* S^n.

The semi-linear condition in Theorem 4 is used only to shrink L to a boundary point of $\mathrm{Cl}H$. It seems that one should be able to remove this condition and retain the conclusion, since the local embedding at each point t of L, different from p, is as "nice" as the local embedding of an interval at one of its points. In fact, for each $t \in L$, different from p one can find a homeomorphism h_t of S^n onto itself such that the subarc L_t of L from q to t is carried onto a linear segment.

DEFINITION 1. Let h be a homeomorphism of $\mathrm{Cl}(D - A)$ into S^n. If there exists a neighborhood N of $(0,0,\cdots,0,1)$ in E^n such that h is semi-linear on each finite polyhedron of $\mathrm{Int}\,(D - A) \cap N$, then we say that h is semi-linear on a deleted neighborhood of $(0,0,\cdots,0,1)$.

THEOREM 5. *Let $n > 3$ and h a homeomorphism of* $\mathrm{Cl}\,(D - A)$ *into S^n such that h is semi-linear on a deleted neighborhood of* $(0,0,\cdots,0,1)$. *If G is the component of $S^n - h(\mathrm{Bd}A)$ which intersects $h(\mathrm{Bd}D)$, then* $\mathrm{Cl}G$ *is a closed n-cell.*

Proof. The technique of proof used here is that used by Mazur in [8].

Let D_1 be a cell, obtained from D by a slight contraction on E^n toward $(0,0,\cdots,0,1)$, such that $(\mathrm{Bd}D_1) - (0,0,\cdots,0,1)$ is contained in $D - A$. Let G_1 and G_2, respectively, be the components of $S^n - h(\mathrm{Bd}D_1)$ and $S^n - h(\mathrm{Bd}D)$ which are contained in G. We now observe that $\mathrm{Cl}G_1$ is homeomorphic to $\mathrm{Cl}G$. For, if g is a homeomorphism of E^n onto itself which is fixed on $\mathrm{Bd}D$ and carries $\mathrm{Bd}D_1$ onto $\mathrm{Bd}A$, then the mapping ϕ defined by

$$\phi(x) = x, \qquad \text{if } x \in G_2,$$

$$\phi(x) = hgh^{-1}(x), \quad \text{if } x \in \mathrm{Cl}(G_1 - G_2),$$

carries $\mathrm{Cl}G_1$ homeomorphically onto $\mathrm{Cl}G$. This suggests the following observation: if one attaches a copy of $\mathrm{Cl}G_1$ to $\mathrm{Cl}(D_1 - A)$ along $\mathrm{Bd}D_1$ with h^{-1}, the set thus obtained is equivalent to $\mathrm{Cl}G_1$ (it is simply $\mathrm{Cl}G$). This will be used to show that $\mathrm{Cl}G_1$ is a closed n-cell, and hence that $\mathrm{Cl}G$ is a closed n-cell.

Let N be a neighborhood of $(0,0,\cdots,0,1)$ such that h is semi-linear on $\mathrm{Int}(D-A) \cap N$. Let S be an n-simplex in $\mathrm{Cl}(D_1 - A) \cap N$, such that $(0,0,\cdots,0,1)$ is a vertex of S and let $K = S^n - h(S)$. By Theorem 4, $\mathrm{Cl}K$ is a closed n-cell. Let $H = S^n - \mathrm{Cl}G$, then $\mathrm{Cl}K$ can be realized by taking $P = \mathrm{Cl}(D_1 - A) - \mathrm{Int}\,S$ and attaching $\mathrm{Cl}H$ to P along $\mathrm{Bd}A$ with h^{-1}, and attaching $\mathrm{Cl}G_1$ to P along $\mathrm{Bd}D_1$ with h^{-1}. The set P is a closed n-cell (the closure of the exterior of S) with the interiors of two n-cells, sharing a common boundary point with $\mathrm{Bd}S$, removed. The cell obtained from P by attaching $\mathrm{Cl}G_1$ and $\mathrm{Cl}H$ to the interior boundary spheres of P with h^{-1} will be denoted by $\bar{P}$.

Let F be the part of the solid unit ball in E^n centered at $(0,0,\cdots,0,1,0)$, determined by $x_n \geqq 0$. Let $\{q_i\}_{i=0}^{\infty}$ be a sequence of points in the intersection of the plane $x_1 = x_2 = \cdots = x_{n-2} = 0$ and $\mathrm{Bd}F$ such that, if $q_i = (0,0,\cdots, a_{(n-1)i}, a_{ni})$, then $a_{(n-1)0} = 2$, $a_{n0} = 0$, the $a_{(n-1)i}$ converge monotonically to zero, and $a_{ni} > 0$ for $i > 0$. We then section F into a countable number of n-cells by projecting the $(n-2)$-plane $x_n = x_{n-1} = 0$ onto each of the q_i. The section determined by q_{i-1} and q_i is denoted by C_i. We then delete from C_i the interior of a cell C_i', similar in shape to C_i and, except for the boundary point $(0,0,\cdots,0,0)$, contained in the interior of C_i. Any two adjacent sections then form a copy of P, and are labeled P_i, P_i', as in Figure 1. Notice that P_i and P_i' have $w_{2i} = \mathrm{Bd}C_{2i}'$ in common, and P_i' and P_{i+1} have $w_{2i+1} = \mathrm{Bd}C_{2i+1}'$ in common.

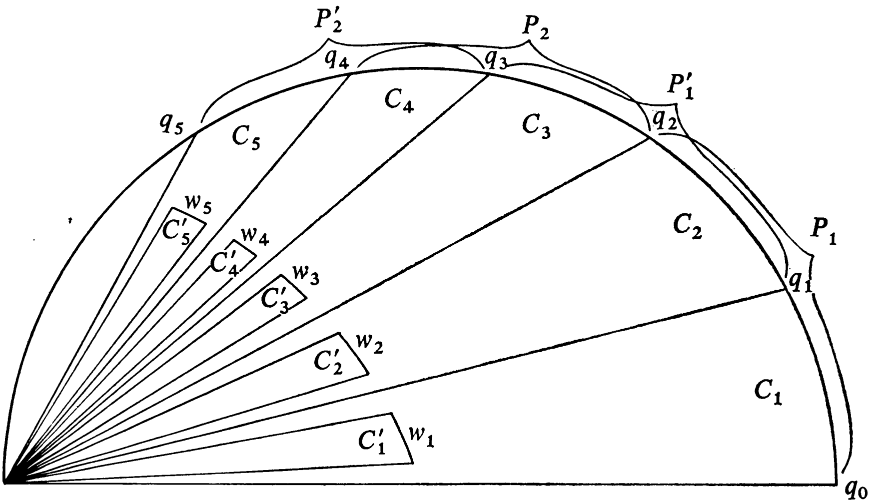

FIGURE 1

Let ϕ_i be a homeomorphism of P_i onto P_i' which leaves w_{2i} fixed and carries w_{2i-1} onto w_{2i+1}. Let ψ_i be a homeomorphism of P_i' onto P_{i+1} which leaves w_{2i+1} fixed and carries w_{2i} onto w_{2i+2}. We identify P_1 with P, with w_1 identified with $\mathrm{Bd}D_1$ and w_2 identified with $\mathrm{Bd}A$. The sets $\mathrm{Cl}G_1$ and $\mathrm{Cl}H$ are then sewn to P along w_1 and w_2, respectively, with h^{-1}. The resulting n-cell is denoted by $\bar{P}_1$. The sets $\mathrm{Cl}G_1$ and $\mathrm{Cl}H$ are then sewn into alternate holes bounded by w_{2i+1} and w_{2i+2} by the attaching homeomorphisms

$$\phi_i \cdots \phi_2\phi_1 h^{-1} : \mathrm{Bd}G_1 \to w_{2i+1},$$

$$\psi_i \cdots \psi_2\psi_1 h^{-1} : \mathrm{Bd}H \to w_{2i+2}.$$

The sets thus obtained from the P_i and P'_i are denoted by $\bar{P}_i$ and $\bar{P}'_i$ and we set $F_1 = \bigcup_{i=1}^{\infty} \bar{P}_i$.

Since ϕ_1 is the identity on w_2, we can extend ϕ_1 to a homeomorphism of $\bar{P}_1$ onto $\bar{P}'$, and conclude that $\bar{P}'_1$ is also a closed n-cell. In a similar manner we extend ψ_i to a homeomorphism of $\bar{P}'_i$ onto $\bar{P}_{i+1}$ and extend ϕ_i to a homeomorphism of $\bar{P}_i$ onto $\bar{P}'_i$. It then follows that each $\bar{P}_i$ and each $\bar{P}'_i$ is a closed n-cell.

We now observe that F_1 is a closed n-cell. We map the boundary of $C_{2i-1} \cup C_{2i}$ onto the boundary of $\bar{P}_i$ with the identity homeomorphism. Since $C_{2i-1} \cup C_{2i}$ and $\bar{P}_i$ are n-cells, this homeomorphism between their boundaries can be extended to a homeomorphism between the cells. These extensions for $i = 1, 2, \cdots,$ yield a homeomorphism of F onto F_1.

We next observe that F_1 is a copy of $\mathrm{Cl}(D_1 - A)$ with $\mathrm{Cl}G_1$ sewn along one of the boundary spheres. This can be established by showing that F_1, with G_1 removed from $\bar{P}_1$, is homeomorphic to F, with Int C'_1 removed. Let λ be the identity mapping on $C_1 - \mathrm{Int}\, C'_1$ and on $\mathrm{Bd}(C_{2i} \cup C_{2i+1})$, $i = 1, 2, \cdots$. Since $C_{2i} \cup C_{2i+1}$ and $\bar{P}'_i$ are closed n-cells and λ restricts to a homeomorphism between their boundaries, λ can be extended over their interiors. These extensions over each of the $C_{2i} \cup C_{2i+1}$ yield the desired homeomorphism.

We have seen that F_1 can first be viewed as a closed n-cell, and secondly as $\mathrm{Cl}G_1$ sewn into a boundary sphere of a copy of $\mathrm{Cl}(D_1 - A)$. We previously observed that a set of the second type is equivalent to $\mathrm{Cl}G_1$. Hence $\mathrm{Cl}G_1$, or equivalently $\mathrm{Cl}G$, is a closed n-cell, and Theorem 5 is proved.

If one were able to remove the semi-linear condition in Theorem 4, then the semi-linear condition in Theorem 5 could also be removed(2). In this general form Theorem 5 would imply that a wild $(n-1)$-sphere is S^n, $n > 3$, must be "knotted" at more than one point, and that such simple examples of wild spheres as the Fox-Artin examples [3] for $n = 3$ do not exist in the higher dimensional spaces.

3. Some 3-spheres in S^4.

Definition 2. In E^4 we take coordinates x_1, x_2, x_3, x_4 and let E^3 be described by $x_4 = 0$. Let $a = (0,0,0,1)$ and $b = (0,0,0,-1)$. For a set A in E^3 the suspension of A in E^4 is the join of A and $a \cup b$, and is denoted by Susp A.

The proof of Theorem 1 of [1] may be used directly to prove the following theorem.

Theorem 6. *Let S be a 2-sphere in E^3 and $K =$ Susp S. Let A_1 and A_2 be the bounded and unbounded components of $E^3 - S$ respectively, and B_1, B_2 the corresponding components of $E^4 - K$. Then the injection homomorphism $i_j : \pi_1(A_j) \to \pi_1(B_j)$, $j = 1, 2$, is an onto isomorphism.*

(2) *Added in proof.* After this paper was sent to press the author was able to remove the semi-linear conditions in Theorem s4 and 5. These results, together with certain generalizations, will appear in print at a later date.

Let $E^3_+ = \{(x_1, x_2, x_3, 0) \in E^4 \mid x_3 \geqq 0\}$ and let P be the plane $x_3 = x_4 = 0$. For $x = (x_1, x_2, x_3, 0)$ and $0 \leqq t < 2\pi$ we set $R_t(x) = (x_1, x_2, x_3 \cos t, x_3 \sin t)$, and for a subset M of E_3^+ we set $R(M) = \{R_t(x) \mid x \in M, 0 \leqq t < 2\pi\}$. For a subset N of E^4 we set $R^{-1}(N) = \{y \in E^3_+ \mid R_t(y) \in N$ for some $0 \leqq t < 2\pi\}$.

If M is a 2-cell in E^3_+ such that $M \cap P = \text{Bd}M = d$, and D is the bounded component of $P - d$, then the proof of Theorem 3 of [1] may be used to establish the following theorem.

THEOREM 7. *Let A_1 and A_2 be the bounded and unbounded components, respectively, of $E^3_+ - (M \cup D)$ and let B_1, B_2 be the corresponding components of $E^4 - R(M)$. Then $\pi_1(A_i) \approx \pi_1(B_i), i = 1, 2$.*

In [3] there are examples of 2-spheres in S^3 such that one complementary domain has a nontrivial fundamental group. Elementary modifications of these examples will give 2-spheres in S^3 such that the fundamental group of either complementary domain is nontrivial. These examples, together with Theorem 6 or Theorem 7, give the existence of 3-spheres in S^4 such that either one or both complementary domains have nontrivial fundamental groups. In passing, we observe one difference between the spheres Susp S and $R(M)$. Associated with each exceptional point $p \in S$ there will be an arc, Susp p, of exceptional points on Susp S, and for each exceptional point $p \in M$ there will be a simple closed curve, $R(p)$, of exceptional points on $R(M)$.

We now use the rotation of a disk about P to construct a 3-sphere in S^4, one complementary domain of which is simply connected but is not an open 4-cell. Let us first embed the 2-sphere S, discussed as Example 3.2 in [3], in

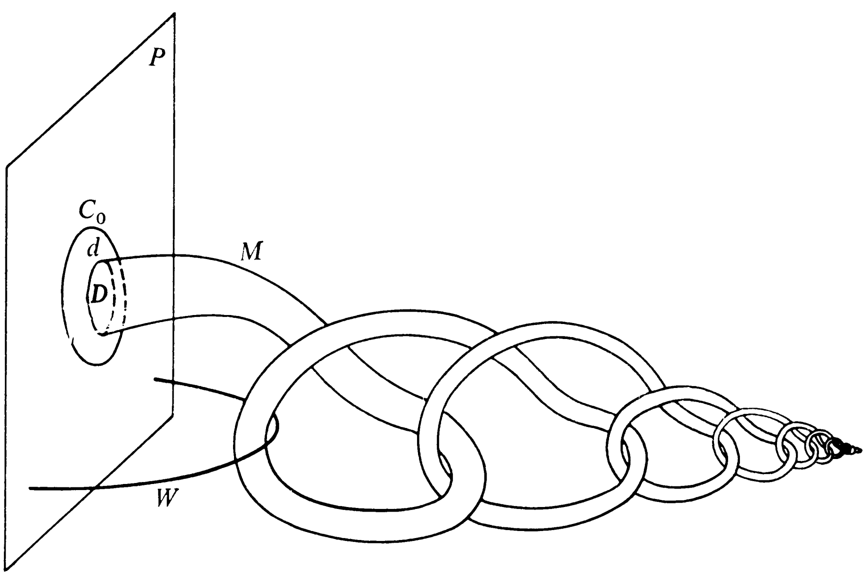

FIGURE 2

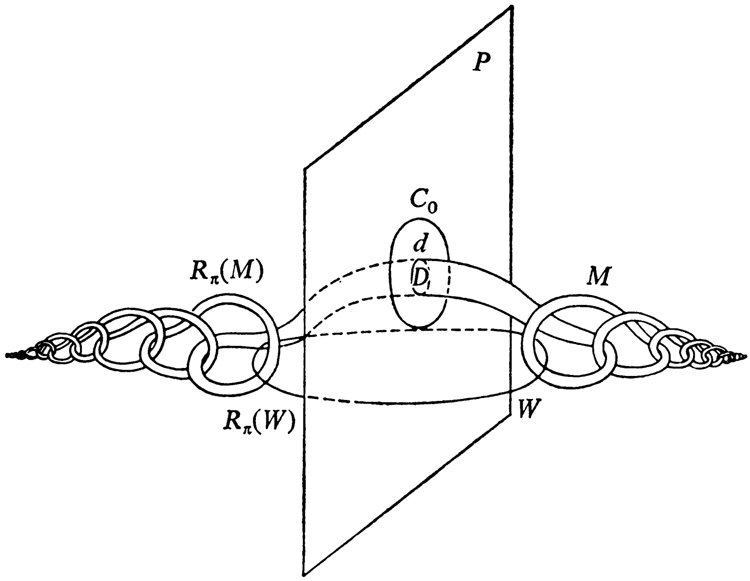

FIGURE 3

E^3_+ as indicated in Figure 2. The sphere S is to intersect P in a 2-cell D and $\mathrm{Cl}(S-D)$ is denoted by M. If L is the arc described as Example 1.3 in [3], the proof in [3] that E^3-L is simply connected may be used directly to show that A_2 (the exterior of S in E^3_+) is simply connected. Hence, by Theorem 7, B_2 (the exterior of $R(M)$ in $E^4(S^4)$) is simply connected.

The cross section $M \cup R_\pi(M)$ of $R(M)$ is shown in Figure 3.

Let A_2' denote the exterior of $M \cup R_\pi(M)$ in E^3. It is shown in [3, Example 1.3] that C_0 cannot be contracted to a point in $A_2'-[W \cup R_\pi(W)]$. This fact is now used to show that $R(W)$ is contained in no closed 4-cell subset of B_2 whose complement in B_2 is simply connected. Hence, B_2 is not an open 4-cell.

Suppose that such a 4-cell J did exist. Choose the base point for computing $\pi_1(B_2-J)$ in P and so close to d that there is a path c_0 in $(B_2-J)\cap P$ which cannot be contracted to a point in $A_2'-[W \cup R_\pi(W)]$. Let E be a unit disk in E^2 with boundary e, and let h be a continuous mapping of e onto c_0. Since $\pi_1(B_2-J)$ is trivial, there exists an extension H of h which carries E into B_2-J. We then follow H by R^{-1} and obtain a singular 2-cell, $R^{-1}H(E)$, in $A_2-R^{-1}(J)$ which is bounded by c_0. Since $A_2-R^{-1}(J)\subset A_2-W$, we see that c_0 can be contracted to a point in A_2-W and hence in the larger set $A_2'-[W \cup R_\pi(W)]$. This contradiction establishes the desired conclusion.

We now describe a third method for constructing $(n-1)$-spheres in S^n and refer to this method as capping a cylinder.

In E^n we again take coordinates $x_1, x_2, \cdots, x_n$ and let E^{n-1} be described by $x_n=0$.

LEMMA 1. *Let S be an $(n-2)$-sphere in E^{n-1} with the bounded and unbounded components of $E^{n-1}-S$ denoted by A_1 and A_2, respectively. If* $\mathrm{Cl}A_2$ (*compactified at infinity*) *is a closed $(n-1)$-cell, then* $\{S\times[0,1]\}\cup\{\mathrm{Cl}A_1\times[1]\}$ *is a closed $(n-1)$-cell.*

Proof. Let h be a homeomorphism of $\mathrm{Cl}A_2$ onto a standard unit ball B in E^{n-1}. Let $S_1=\mathrm{Bd}B$ and let S_2 be the sphere concentric with S_1 and with radius one-half. Then $h^{-1}(S_2)$ is an $(n-2)$-sphere in E^{n-1} and if C is the component of $E^{n-1}-h^{-1}(S_2)$ which contains A_1, then, by Theorem 1, $\mathrm{Cl}C$ is a closed $(n-1)$-cell. We now observe that $\mathrm{Cl}C$ consists of a closed annulus with $\mathrm{Cl}A_1$ sewn along one boundary component and is, therefore, a copy of $\{S\times[0,1]\}\cup\{\mathrm{Cl}A_1\times[1]\}$.

THEOREM 8. *Let S, A_1, and A_2 be as in Lemma* 1. *If* $\mathrm{Cl}A_2$ (*compactified at infinity*) *is a closed $(n-1)$-cell, then* $\{S\times[-1,1]\}\cup\{\mathrm{Cl}A_1\times[-1]\}\cup\{\mathrm{Cl}A_1\times[1]\}$ *is an $(n-1)$-sphere.*

Proof. By Lemma 1, each of $\{S\times[-1,0]\}\cup\{\mathrm{Cl}A_1\times[-1]\}$ and $\{S\times[0,1]\}\cup\{\mathrm{Cl}A_1\times[1]\}$ is a closed n-cell. These cells meet along their common boundary sphere S, and hence their union is an $(n-1)$-sphere.

We now consider a 2-sphere S, locally polyhedral except at a single point, in $E^3(S^3)$ such that the bounded complementary domain A_1 is an open 3-cell, $\mathrm{Cl}A_1$ is not a closed 3-cell, the unbounded complementary domain (compactified at infinity) is an open 3-cell, and $\mathrm{Cl}A_2$ is a closed 3-cell. The assertion is that the 3-sphere

$$T=\{S\times[-1,1]\}\cup\{\mathrm{Cl}A_1\times[1]\}\cup\{\mathrm{Cl}A_1\times[-1]\}$$

is embedded in S^4 such that, if B_1 and B_2, respectively, are the components of S^4-T which contain A_1 and A_2, then B_1 is an open 4-cell, $\mathrm{Cl}B_1$ is not a closed 4-cell, and $\mathrm{Cl}B_2$ is a closed 4-cell.

Since B_1 is the product of the open 3-cell A_1 and the open interval $(-1,1)$, B_1 is an open 4-cell. If $\mathrm{Cl}B_1=\mathrm{Cl}A_1\times[-1,1]$ were a closed 4-cell, a theorem due to Bing [4] would imply that $\mathrm{Cl}A_1$ is a closed 3-cell. Thus contradicting our assumption on the embedding of S in S^3.

We now show that $\mathrm{Cl}B_2$ is a closed 4-cell by constructing a homeomorphism $f: T\times[0,1/2]\to\mathrm{Cl}B_2$ such that $f_0(y)=f(y,0)=y$ for each $y\in T$ and then applying Theorem 1. Since $\mathrm{Cl}A_2$ is a closed 3-cell, there exists a homeomorphism $h: S\times[0,1/2]\to\mathrm{Cl}A_2$ such that $h_0(x)=h(x,0)=x$ for each $x\in S$. For $y\in T$, let x be the point of $\mathrm{Cl}A_1$ which lies under y ($y=(x,t)$ for some $t\in[-1,1]$). We define f by the following equations:

(1) $f_r(y)=(x,1+r)$, if $y=(x,1)$;

(2) $f_r(y)=(x,-1-r)$, if $y=(x,-1)$;

(3) $f_r(y)=(h_r(x),t)$, if $x\in S$ and $-1+r<t<1-r$;

(4) $f_r(y) = (h_{(1-t)}(x), 2t - (1 - r))$, if $x \in S$ and $1 - r \leqq t \leqq 1$;
(5) $f_r(y) = (h_{(1-t)}(x), 2t - (r - 1))$, if $x \in S$ and $-1 \leqq t \leqq -1 + r$.

The continuity of f follows rather quickly from the definition of f in terms of the continuous mapping h and a set of linear equations. The one-to-one property of f depends principally on the fact that each arc $f_r(x \times [0, 1])$ must lie over the arc $L_x = \{h_s(x) \mid s \in [0, 1/2]\}$ and that L_{x_1} and L_{x_2} intersect if and only if $x_1 = x_2$.

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