THE ROLE OF THE APPELL TRANSFORMATION IN THE THEORY OF HEAT CONDUCTION

BY

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1. Introduction. If \( v(x, t) \) is an arbitrary function of two variables its Appell transform is

\[
A[v] = A_{x,t}[v] = k(x, t) v \left( \frac{x}{t}, \frac{-1}{t} \right),
\]

where

\[
k(x, t) = \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}}.
\]

Here \( k(x, t) \) is the fundamental solution of the heat equation

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.
\]

As we shall see, this transformation serves in a remarkable way to establish a duality between types of solutions of (1.3). It was Appell [1] himself who showed that if \( v \) is a solution then \( A[v] \) is also. We shall be studying here various solutions which have integral representations and the effect of the Appell transformation thereon. A few of our results were outlined, mostly without proof, in [8].

Let us introduce notations for the various integral transforms to be considered, as follows:

**Poisson transform**

\[
P[\phi] = P_{x,t}[\phi] = \int_{-\infty}^{\infty} k(x - y, t) \phi(y) dy;
\]

**Fourier transform**

\[
F[\phi] = F_{x,t}[\phi] = \int_{-\infty}^{\infty} e^{ixy-ty^2} \phi(y) dy;
\]

**Laplace transform**

\[
L[\phi] = L_{x,t}[\phi] = \int_{-\infty}^{\infty} e^{xy+ty^2} \phi(y) dy.
\]
All of these have kernels which are solutions of (1.3) for each value of the parameter $y$ and consequently produce solutions for more or less arbitrary functions $\phi$. The transform (1.4) is also commonly referred to as the Weierstrass or Gauss transform. It is clear that $P_{ix} - [\phi]$ is also a solution of (1.3). We shall see that this one is paired with $F[\phi]$ while $L[\phi]$ is paired with $P[\phi]$ in the duality mentioned above.

We list a number of examples which will be useful in the sequel.

A. $P[x^n] = v_n(x, t) = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!} \frac{t^k}{k!}, \quad t > 0.$

This is called the heat polynomial of degree $n$. The Appell transform of this polynomial is the associated function studied in [6]:

$$A[v_n] = w_n(x, t) = k(x, t)v_n(x/t, -1/t)t^{-n}, \quad t > 0.$$ 

B. $F[e^{ax^2}] = 2\pi k(x, t-a), \quad t > a.$

C. $L[e^{-ax^2}] = 2\pi k(ix, a-t), \quad t < a.$

D. $A[k(x, t+a)] = (4\pi a)^{-1/2} k(x, t - a^{-1}), \quad t > a^{-1}.$

The transformed function is in every case a solution of (1.3) in the half-plane indicated.

We now state briefly a few of the results of the present paper. The Huygens property is defined in §2 below and the growth of an entire function, in §4. A fundamental result is the following.

A function $u(x, t)$ is equal to the integral $A[v]$, defined by (1.5) where $\phi$ is entire of growth $(2, 1/\sigma)$, if and only if it is the Appell transform $A[v]$ of a temperature function $v(x, t)$ which has the Huygens property for $|t| < \sigma$.

We also obtain several new characterizations of positive temperature functions. The first involves those which are positive for all negative time.

A solution $v(x, t)$ of (1.3) is $\geq 0$ for $t < 0$ if and only if

$$v(x, t) = \int_{-\infty}^{\infty} e^{-xy + x^2} dx(y), \quad t < 0,$$

where $x(y) \in \mathbb{T}.$

An example of such a solution is $e^{t \cosh x}$, for which $x$ is a step-function with jumps at $\pm 1$.

We also characterize a subclass of the above functions as follows.

A solution $v(x, t)$ of (1.3) is $\geq 0$ for $t < 0$ and satisfies the inequality

$$\int_{-\infty}^{\infty} v(x, t)e^{x^2/4t} dx < \infty, \quad t < 0$$

if and only if
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\[ v(x, t) = \int_{-\infty}^{\infty} k(y + ix, -t)\phi(y)dy, \quad t < 0, \]

where \( \phi \) is positive definite.

A case in point is \( v(x, t) \equiv 1 \) with \( \phi(y) \equiv 1 \). A less trivial example is \( v = k(ix, 1 - t) \), which evidently satisfies the conditions of the theorem. Here \( \phi \) is the positive definite function \( e^{-y^2/(4t)} \).

Finally we characterize those temperature functions which are positive for positive time and are absolutely integrable in the space variable.

A solution \( u(x, t) \) of (1.3) is \( \geq 0 \) for \( t < 0 \) and satisfies the inequality

\[ \int_{-\infty}^{\infty} u(x, t)dx < \infty, \quad t > 0 \]

if and only if

\[ u(x, t) = \int_{-\infty}^{\infty} e^{ixy - y^2}\phi(y)dy, \quad t > 0, \]

where \( \phi(y) \) is a positive definite function.

An example here is the fundamental solution itself with \( \phi \) equal to the constant \((2\pi)^{-1}\).

2. Relation between the Poisson and Fourier transforms. Let us reintroduce here some of the notation of [6].

**Definition 2.1.** A function \( u(x, t) \in H \), or is a temperature function, for \( a < t < b \) if and only if it is a class \( C^2 \) solution of the heat equation (1.3) there.

**Definition 2.2.** A function \( u(x, t) \in H^* \), or has the Huygens property, for \( a < t < b \) if and only if \( u \in H \) there and

\[ u(x, t) = \int_{-\infty}^{\infty} k(x - y, t - t')u(y, t')dy \]

for every \( t, t', a < t' < t < b \), the integral converging absolutely.

**Example A.** Any function

\[ u(x, t) = \int_{-\infty}^{\infty} k(x - y, t)\alpha(y)dy \]

for which the integral converges absolutely in \( a < t < b \) belongs to \( H^* \) there. Equation (2.1) then results immediately by use of Fubini's theorem. In particular, a positive function of \( H \) is also a function of \( H^* \). See [7].

**Example B.** The function \( k(x, t + i) \) belongs to \( H \) in the whole \( x, t \)-plane and belongs to \( H^* \) in any strip of the form \(-a < t < 1/a\). See [6, p. 242]. It is to be emphasized that a function may belong to \( H^* \) on each of two overlapping sets without doing so on the sum of the sets. Thus the present function has the Huygens property for \(-2 < t < 1/2\) and for \( 0 < t < \infty \) but certainly not for
\[ -2 < t < 2, \text{ for example. If we chose } t' = -1, t = 1, x = 0 \text{ in the integral (2.1) it becomes}\]
\[
\int_{-\infty}^{\infty} k(y, t) k(y, -1 + i) dy.
\]

It clearly does not converge absolutely.

We prove now that any Poisson transform of a function in class \( L \) is also a Fourier transform.

**Theorem 2.** If \( \phi \in L \) on \((-\infty, \infty)\), then \( P_x, [\phi] \in H^* \) for \( t > 0 \) and
\[
P_x, [\phi] = F_x, [\psi], \quad t > 0,
\]
where
\[
\psi(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{-ixy} \phi(y) dy.
\]

By definition
\[
(2.2) \quad P_x, [\phi] = \int_{-\infty}^{\infty} k(x - y, t) \phi(y) dy, \quad t > 0.
\]

The integral converges absolutely for \( t > 0 \) since \( k \leq (4\pi t)^{-1/2} \). It consequently belongs to \( H^* \) there by Example A of this section. By Example B of §1 the integral (2.2) becomes
\[
P_x, [\phi] = \int_{-\infty}^{\infty} e^{ixz - tz^2} \phi(y) dy = F_x, [\psi].
\]

The interchange in the order of integration is justified in an obvious way by Fubini’s theorem.

3. **Relation between the Poisson and Laplace transforms.** To prove our next result we need a lemma.

**Lemma 3.** If \( u(x, t) \in H^* \) for \( a < t < b \), then \( u(ix, -t) \in H^* \) for \( -b < t < -a \).

Since the integral (2.1), when it converges, defines an entire function of \( x \), the definition of \( u(ix, -t) \) is clear. We must prove that if \( a < t < t' < b \) then
\[
(3.1) \quad u(ix, t) = \int_{-\infty}^{\infty} k(x - y, t' - t) u(iy, t') dy,
\]
the integral converging absolutely. Choose \( t'' \) so that \( a < t'' < t \). Then by hypothesis
the integral converging absolutely. Substituting (3.2) in the integral (3.1) we have

$$\int_{-\infty}^{\infty} k(x-y, t'-t) dy \int_{-\infty}^{\infty} k(iy-z, t''-t) u(z, t'') dz$$

(3.3)

$$= \int_{-\infty}^{\infty} u(z, t'') dz \int_{-\infty}^{\infty} k(x-y, t'-t) k(iy-z, t''-t) dy.$$  

This last interchange is valid by virtue of the inequalities

$$\int_{-\infty}^{\infty} k(x-y, t'-t)e^{-\lambda(x-y)^2} dy < \infty,$$

(3.4)

$$\int_{-\infty}^{\infty} e^{\lambda(x-y)^2} u(z, t'') dz < \infty.$$  

The integral (3.4) clearly converges since $t'' < t$; (3.5) does also since (3.2) converges absolutely when $t = t'$. But the value of the inner integral in (3.3) is known to be $k(ix-z, t-t')$. See [5, p. 177]. That is, the integral (3.2) becomes

$$\int_{-\infty}^{\infty} k(ix-z, t-t') u(z, t'') dz,$$

and this by (3.2) is $u(ix, t)$. Thus (3.1) is proved. Finally, the absolute convergence of (3.1) follows as a consequence of Fubini's theorem.

**Theorem 3.** If $\phi \in L$ on $(-\infty, \infty)$, then $P_{-ix}^{-i\phi} \in H^*$ for $t < 0$ and

$$P_{-ix, -t}^{-i\phi} = L_{x, t}[\psi],$$

(3.6)

where

$$\psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixy} \phi(y) dy.$$  

By definition

$$P_{-ix, -t}^{-i\phi} = \int_{-\infty}^{\infty} k(ix+y, -t) \phi(y) dy.$$  

Since

$$|k(ix+y, -t)| \leq e^{-x^2/4t(-4\pi t)^{-1/2}},$$

it is clear that the integral (3.6) converges absolutely for $t < 0$. Since the integral $P_{x, t}[\phi]$ converges absolutely for $t > 0$ it defines a function of $H^*$ there. By Lemma 3 the function (3.6) belongs to $H^*$ for $t < 0$. By Example C of §1 ($a = 0$) the integral (3.6) becomes
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(y) dy \int_{-\infty}^{\infty} e^{z(x-iy)+iz^2} dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{z x + iz^2} dz \int_{-\infty}^{\infty} e^{-iyz} \phi(y) dy.
\]

The interchange of integrals is obviously permissible for \( t < 0 \), and the inner integral is \( \psi(z) \) as predicted.

Note that the initial temperature of \( F_{x,t}[[T]] \) is the Fourier transform of \( \psi \); the terminal temperature of \( L_{x,t}[[\psi]] \) is the Laplace transform of \( \psi \) when these transforms exist.

4. **A criterion for the Fourier representation.** Let us recall the following definition.

**Definition 4.** An entire function

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n
\]

belongs to the class \((2, \sigma)\), or has growth \((2, \sigma)\), if and only if

\[
(4.1) \quad \limsup_{n \to \infty} |a_n|^{2/n} \leq 2e\sigma.
\]

See, for example, [2, p. 8]. A function of this class is of order \( \leq 2 \), and if it is of order 2, then it is of type \( \leq \sigma \). For example, the functions \( xe^{-3x^2}, e^{x^2}, e^{5x} \) all belong to \((2, 3)\), whereas \( e^{-x^4} \) does not.

We first call attention to the following lemma.

**Lemma 4.1.** If \( s > 0, \ t > 0 \), then

\[
A_{x,t}[k(x-y, \ t+s)] = A_{y,s}[k(x-y, \ t+s)].
\]

This is a simple identity. Both sides are equal to

\[
(4\pi)^{-1}(st-1)^{-1/2}\exp[(2xy-ty^2-sx^2)(4st-4)^{-1}].
\]

A second preliminary result is contained in

**Lemma 4.2.** If \( v(x, \ t) \in H^* \) for \( |t| < \sigma \), then \( v(-2ix, \ 0) \in (2, 1/\sigma) \).

To prove this we appeal to Theorem 11.1 of [6] and thus admit that

\[
v(x, \ t) = \sum_{n=0}^{\infty} a_n v_n(x, \ t), \quad |t| < 0
\]

\[
v(x, \ 0) = \sum_{n=0}^{\infty} a_n x^n.
\]

By Theorem 5.5 of [6], \( v(x, 0) \in (2, 1/4\sigma) \). Thus (4.1) shows that \( v(-2ix, 0) \in (2, 1/\sigma) \), as stated.

**Theorem 4.** A necessary and sufficient condition that

\[
u(x, \ t) = F[\phi] = \int_{-\infty}^{\infty} e^{xy-ty^2} \phi(y) dy, \quad t > 1/\sigma,
\]

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where \( \phi(y) \in (2, \sigma^{-1}) \), is that there should exist a function \( v(x, t) \) of \( H^* \) in \( |t| < \sigma \) such that \( u = A[v] \).

We prove first the sufficiency. Since \( v \in H^* \) for \( |t| < \sigma \), then for any positive \( \sigma' < \sigma \)

\[
v(x, t) = \int_{-\infty}^{\infty} k(x - y, t + \sigma')v(y, -\sigma')dy,
\]

the integral converging absolutely when \( -\sigma' < t < \sigma \). By Lemma 4.1

\[
A[v] = \int_{-\infty}^{\infty} k(y, \sigma')k\left(x - \frac{y}{\sigma}, t - \frac{1}{\sigma'}\right)v(y, -\sigma')dy, \quad t > 1/\sigma',
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} k(y, \sigma')v(y, -\sigma')dy \int_{-\infty}^{\infty} \exp \left[ iz\left(x - \frac{y}{\sigma'}\right) - z^2 \left(t - \frac{1}{\sigma'}\right)\right] dz.
\]

Here we have used Example B of §1. If interchange of iterated integrals is permitted, we obtain

\[
A[v] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ixz - tz^2)dz \int_{-\infty}^{\infty} v(y, -\sigma')k(-2iz - y, \sigma')dy \quad t > 1/\sigma',
\]

where \( 2\pi \phi(z) = v(-2iz, 0) \). To justify the interchange it is sufficient to know, that

\[
\int_{-\infty}^{\infty} \exp(-tz^2 + \frac{z^2}{\sigma'})dz\int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{4\sigma'}\right) | v(y, -\sigma') | dy < \infty.
\]

The inner integral is finite since (4.2) converges absolutely when \( x = t = 0 \). The outer integral clearly converges when \( t > 1/\sigma' \). Since \( \phi(z) \) is independent of \( \sigma' \), it is clear that (4.3) continues to hold for all \( t > 1/\sigma \). Finally, that \( \phi \in (2, 1/\sigma) \) follows from Lemma 4.2.

To prove the necessity we now assume that \( u = F[\phi] \) with \( \phi \) belonging to \((2, 1/\sigma)\). We appeal to Theorem 12.1 of [6] to show that

\[
u(x, t) = \sum_{n=0}^{\infty} b_n w_n(x, t), \quad t > 1/\sigma,
\]

where

\[
b_n = \frac{2\pi \phi^{(n)}(\omega)}{n! (-2i)^n}.
\]

Since \( w_n = A[v_n] \) it is clear that \( u = A[v] \), where

\[
v(x, t) = \sum_{n=0}^{\infty} b_n v_n(x, t), \quad -\sigma < t < 0.
\]
By Theorem 11.1 of [6] the sum of any such series belongs to $H^*$ throughout its region of convergence, always an entire strip of the form $|t| < \sigma$. The convergence of (4.5) in $-\sigma < t < 0$ implies its convergence in $|t| < \sigma$, so that $v \in H^*$ there. The equation (4.4) yields

$$v(x, 0) = 2\pi \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{n!} \left(\frac{x}{-2i}\right)^n = 2\pi \phi \left(\frac{i}{2}\right)$$

$$v(-2iz, 0) = 2\pi \phi(z),$$

thus confirming the relation between $v$ and $\phi$ established in the sufficiency proof.

**Corollary 4.** If $v(x, t) \in H^*$ for $a < t < b$ and if $a < t_0 < b$, then

$$F_{x,t}[v(-2ix, t_0)] = 2\pi A_{x,t}[v(x, t + t_0)], \quad t > 1/\sigma,$$

where $\sigma = \min(t_0 - a, b - t_0)$.

This result was stated as Theorem II in [8]. It follows here by applying Theorem 4 to $v(x, t + t_0)$, which belongs to $H^*$ in $|t| < \sigma$.

Example D of §1 provides an illustration of Theorem 4. The function $v = k(x, t + a) \in H^*$ for $t > -a$ but the largest strip of the form $|t| < \sigma$ contained in that half-plane is $|t| < a$. Hence $A[v]$, according to Theorem 4, must equal $F[\phi]$ for $t > 1/a$, where

$$2\pi \phi(y) = v(-2iy, 0) = e^{y^2/\sigma} (4\pi \sigma)^{-1/2},$$

(4.6)

$$A[\phi] = \frac{(4\pi a)^{-1/2}}{2\pi} \int_{-\infty}^{\infty} e^{ixy - \frac{y^2}{2a}} e^{y^2/\sigma} dy,$$

$$A[\phi] = (4\pi a)^{-1/2} k\left(x, \frac{t - 1}{a}\right), \quad t > 1/a.$$  

Notice that $\phi \in (2, 1/a)$ as predicted and that (4.6) converges in no larger region than that predicted by the theorem.

5. **A sufficient condition for the Fourier representation.** Although the previous example shows that Theorem 4 cannot be improved to give a larger region of convergence for the integral (4.6), nevertheless for some examples the region is indeed larger. Consider the function

$$v(x, t) = k(ix, \delta - t) + k(x, t + \sigma).$$

It belongs to $H^*$ in $-\sigma < t < \delta$. If $\sigma > \delta$, Theorem 4 would predict the convergence of the corresponding integral $A[\phi]$ for $t > 1/\delta$. However

$$\phi(y) = \frac{e^{-y^2/\delta}}{4\pi \sqrt{\delta}} + \frac{e^{y^2/\sigma}}{4\pi \sqrt{\sigma}},$$

so that $F[\phi]$ actually converges in the larger region $t > 1/\sigma$. The following theorem would cover such an example.
Theorem 5. If \( v(x, t) \in H^* \) for \(-\sigma < t < \delta, \, \sigma > 0, \, \delta > 0\), then
\[
A[v] = F[\phi], \quad t > 1/\sigma,
\]
where \( 2\pi \phi(y) = v(-2iy, 0) \) is entire and
\[
\lim_{x \to \pm \infty} x^{-2} \log |\phi(x)| \leq 1/\sigma.
\]

Note that if \( \sigma > \delta \) then \( \phi \in (2, 1/\delta) \) and Theorem 4 would guarantee the convergence of the integral \( F[\phi] \) for \( t > 1/\delta \), in contrast with the present \( t > 1/\sigma \).

The proof of (5.1) proceeds exactly as in the sufficiency of the conditions of Theorem 4. We find as before that \( \phi(y) = v(-2iy, 0)/2\pi \). We show instead that (5.2) holds. From equation (4.2) we have
\[
\lim_{x \to \pm \infty} x^{-2} \log |\phi(x)| \leq 1/\sigma'.
\]

Observe that the condition \( \delta > 0 \) in Theorem 5 cannot be improved. The function \( v = k(ix, -t) \in H^* \) in \(-\infty < t < 0\). Its Appell transform is the constant \( 1/(4\pi) \). This certainly has no Fourier representation \( A[\phi] \), since any such function must vanish at infinity.

6. A necessary condition for the Fourier representation. It should be observed that \( F[\phi] \) may define a temperature function even when \( \phi \notin (2, \sigma) \) for any \( \sigma \). For example, if \( \phi(y) = e^{-y^4}, \phi \) is entire of order 4. But the integral
\[
F[\phi] = \int_{-\infty}^{\infty} e^{ixy - ty^2} e^{-y^4} dy
\]
converges over the whole \( x, t \)-plane. Although the example is not covered by Theorem 4, \( F[\phi] \) is nevertheless the Appell transform of some function, which of course cannot belong to \( H^* \) in any strip including the \( x \)-axis. The following result would include such an example.

Theorem 6. If \( u(x, t) = F[\phi] = \int_{-\infty}^{\infty} e^{ixy - ty^2} \phi(y)dy, t > 1/\sigma \geq 0, \) where
\[
\lim_{x \to \pm \infty} x^{-2} \log |\phi(x)| \leq 1/\sigma,
\]
then there exists \( v(x, t) \) belonging to \( H^* \) in \(-\sigma < t < 0\) such that \( A[v] = u \).

We define \( v \) explicitly:
\[
v(x, t) = 2\pi \int_{-\infty}^{\infty} k(y + ix, -t) \phi(-y/2) dy.
\]
By hypothesis

\[ \phi(y) = O(e^{y^2/\sigma'}) \quad \text{as} \quad y \to \pm \infty \]

for any positive \( \sigma' < \sigma \). Thus the integral (6.2) is dominated by

\[ (-4\pi t)^{-1/2} e^{-x^2/4t} \int_{-\infty}^{\infty} e^{y^2/4t} O(e^{y^2/4\sigma'}) dy. \]

As a consequence it converges for \(- \sigma' < t < 0\) and hence also for \(- \sigma < t < 0\). The same relation (6.3) shows that the integral

\[ w(x, t) = 2\pi \int_{-\infty}^{\infty} k(x - y, t) \phi(-y/2) dy \]

converges absolutely for \(0 < t < \sigma\), and defines a function of \(H^*\) there. Since \(v(x, t) = w(-ix, -t)\), we see by Lemma 3 that \(v(x, t) \in H^*\) for \(- \sigma < t < 0\). But from the definition of the Appell transformation (1.1) we have

\[ A[v] = \frac{1}{2} \int_{-\infty}^{\infty} e^{-ixy/2} e^{y^2/4} \phi(-y/2) dy \]

\[ = \int_{-\infty}^{\infty} e^{ixy} \phi(y) dy = F[\phi] = u. \]

This completes the proof.

In the example of this section the left-hand side of (6.1) is \(- \infty \) so that \(\sigma\) may be taken \(+ \infty \). The function (6.2) belongs to \(H^*\) for \(- \infty < t < 0\).

7. The Laplace transform of an Appell transform. If a function \(v(x, t) \in H^*\) in a strip including the \(x\)-axis, then the Laplace transform of \(A[v]\) has a remarkably simple form. It is in fact equal to \(v(2x, 0)e^{tx^2}\) for all \(t\) sufficiently large. The explicit result follows.

**Theorem 7.** If \(v(x, t) \in H^*\) for \(|t| < \sigma\), then for any \(t > 1/\sigma\)

\[ v(2x, 0)e^{tx^2} = \int_{-\infty}^{\infty} e^{xy} A_y[v] dy. \]

Make the change of variable \(y = tz\) in the integral (7.1) multiplied by \(e^{-tx^2}\). We obtain

\[ te^{-tx^2} \int_{-\infty}^{\infty} e^{txk(tz, t)v(z, -1/t)} dz = (t/4\pi)^{1/2} \int_{-\infty}^{\infty} e^{-(2x-z)^2/4t} v(z, -1/t) dz \]

\[ = v(2x, 0). \]

This last equation follows from the Huygens property provided only that \(- \sigma < -1/t < 0\), and this is guaranteed by hypothesis. This completes the proof.

This result as applied to a function \(v(x, t + t_0)\) was stated without proof as Theorem III in [7]. As an example take \(v = k(x, t + a)\). Then for \(t > 1/\alpha\)
But this is a familiar result, which can be verified by Example C of §1.

8. Positive temperatures in negative time. Temperature functions which are positive for all negative time, such as \( e^t \cosh x \), have a particularly simple representation in terms of the Laplace integral. The result follows.

**Theorem 8.1.** A necessary and sufficient condition that

\[
e^{tx^2} e^{-x^2/\alpha} = \int_{-\infty}^{\infty} e^{xy k(y, t - \frac{1}{\alpha})} dy.
\]

where \( \alpha(y) \in \Upsilon \) and the integral converges for \(-\infty < t < 0\) is that \( \alpha(x, t) \geq 0 \) and \( \alpha(x, t) \in H \) for \(-\infty < t < 0\).

The necessity of the condition is trivial. The kernel of (8.1) belongs to \( H \) for each \( y \), and differentiation under the integral sign is valid for Laplace integrals.

Conversely, apply the Appell transform to the non-negative temperature function \( \alpha \):

\[
A[\alpha] = k(x, t) \alpha \left( \frac{x}{t}, -\frac{1}{t} \right).
\]

Clearly \( A[\alpha] \in H \) and is \( \geq 0 \) for \( t > 0 \). Hence by an earlier result of the author [7] it has a Poisson representation

\[
\alpha \left( \frac{x}{t}, -\frac{1}{t} \right) = e^{x^2/4t} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} d\beta(y),
\]

where \( \beta(y) \in \Upsilon \). Now by obvious change of variable

\[
\alpha(x, t) = \int_{-\infty}^{\infty} e^{xy + ty^2} d\beta(y/2), \quad -\infty < t < 0.
\]

The proof is now concluded by defining \( \alpha(y) \) as \( \beta(y/2) \).

**Corollary 8.1.** A necessary and sufficient condition that

\[
\alpha(x, t) = \int_{-\infty}^{\infty} e^{xy + ty^2} d\alpha(y),
\]

where \( \alpha(y) \in \Upsilon \) and the integral converges for \(-\infty < t < c\), is that \( \alpha(x, t) \geq 0 \) and \( \alpha(x, t) \in H \) there.

This is proved by applying the theorem to \( \alpha(x, t + c) \).
As an application of this theorem we prove an earlier result of I. I. Hirschman [4].

**Theorem 8.2.** If \( v(x, t) \in H \) and is \( \geq 0 \) for \(-\infty < t \leq c \) and if \( \text{Max}_{|x| \leq r} v(x, t) = M(r) \), then \( \liminf_{r \to \infty} \log M(r) \leq 0 \) implies that \( u(x, t) \) is constant for \(-\infty < t \leq c \).

We may assume that \( c = 0 \) and show that the function \( a(y) \) of (8.1) has at most one point of increase, at \( y = 0 \). Suppose on the contrary that \( y = y_0 \neq 0 \) is such a point. Choose \( \delta \) so that \( y = 0 \) is not in the interval \( (y_0 - \delta, y_0 + \delta) \). Then

\[
v(x, 0) \geq \int_{y_0 - \delta}^{y_0 + \delta} e^{xy} dx(y) > p e^{(y_0 - \delta)x}, \quad x > 0 \quad [x < 0]
\]

where \( p \) is the positive number \( \alpha(y_0 + \delta) - \alpha(y_0 - \delta) \). This shows that

\[
M(r) \geq p e^{(y_0 - \delta)r}, \quad y_0 - \delta > 0
\]

\[
\geq p e^{-(y_0 + \delta)r}, \quad y_0 + \delta < 0.
\]

In either case

\[
\liminf_{r \to \infty} r^{-1} \log M(r) > 0,
\]

contradicting the hypothesis. This concludes the proof.

As a consequence of this theorem it is clear that any temperature function which is uniformly bounded for \(-\infty < t \leq c \) is a constant.

**9. A subclass of positive temperatures in negative time.** If an additional condition is imposed on the function \( v(x, t) \) of Theorem 8.1 it will have in addition to (8.1) a quite different integral representation.

**Theorem 9.** A necessary and sufficient condition that

\[
v(x, t) = \int_{-\infty}^\infty k(y + ix, -t) \phi(y) dy, \quad -\infty < t < 0,
\]

where \( \phi(y) \) is positive definite, is that for \(-\infty < t < 0 \) \( v(x, t) \in H, \geq 0 \) and that for some \( t_0 < 0 \)

\[
\int_{-\infty}^\infty v(x, t_0) e^{x^2/4t_0} dx < \infty.
\]

We prove first the necessity. Assume the representation (9.1). Since \( \phi \) is positive definite we have by Bochner's theorem [3, p. 76], that for some \( \alpha(z) \) that is nondecreasing and bounded

\[
\phi(y) = \int_{-\infty}^\infty e^{iyz} d\alpha(z).
\]

Hence
The interchange is valid for $t < 0$ since
$$\int_{-\infty}^{\infty} dx(z) \int_{-\infty}^{\infty} e^{yz^2} dy < \infty.$$ 
Evaluating the inner integral (9.4) we obtain
$$v(x, t) = \int_{-\infty}^{\infty} e^{xz^2 + rz^4} dx(z),$$
from which it is clear that $v \in H, \geq 0$. Moreover,
$$\int_{-\infty}^{\infty} v(x, t) e^{z^2/4t} dx = \int_{-\infty}^{\infty} e^{tz^2} dx(z) \int_{-\infty}^{\infty} e^{xz^2/4t} dx$$
$$= \sqrt{(4\pi(-t))} \int_{-\infty}^{\infty} dx(z) < \infty, \quad t < 0.$$ 
Hence (9.2) is also established.

Conversely, if $v(x, t) \in H, \geq 0$, for $-\infty < t < 0$, then by Theorem 8.1 equation (9.5) holds for some nondecreasing function $a(z)$. Since (9.2) is now assumed, it is the left side of (9.6) that is known to be finite for some $t_0$. From the right side we then conclude that $a(z)$ is also bounded. Hence we may define $\phi(y)$ by (9.3) as a positive definite function. Finally, the equality of the integrals (9.5) and (9.1), already established, concludes the proof.

To show that the functions here considered really form a proper subclass of those considered in §8 we have only to exhibit the function $v = k(ix, -t)$. It fails to satisfy (9.2) for any $t_0$. Yet it is positive for $t < 0$ and so has the representation (8.1). In fact $a(y) = y/(2\pi)$.

10. Positive temperatures in positive time. As we have noted above, temperature functions which are positive for $t > 0$ have a Poisson-Stieltjes integral representation. However, there is a subset of these functions which also have the Fourier integral representation.

**Theorem 10.** A necessary and sufficient condition that
$$u(x, t) = F[\phi] = \int_{-\infty}^{\infty} e^{ixy - t\phi(y)} dy, \quad 0 < t < \infty,$$
where $\phi(y)$ is positive definite, is that $u(x, t) \in H, \geq 0$ for $0 < t < \infty$ and $\in L(-\infty < x < \infty)$ for some $t = t_0 > 0$.

We prove first the sufficiency. Assuming $u \in H, \geq 0$, a new appeal to [7] gives
$$u = \int_{-\infty}^{\infty} k(x - y, t) dx(y), \quad 0 < t < \infty$$
for some $a(y) \in \dagger$. Since $u(x, t_0) \in L$ we have
Hence $\alpha(y)$ is bounded and we can form the positive definite function

$$
\phi(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iyz} d\alpha(z).
$$

Substitute this integral for $\phi(y)$ in (10.1) to obtain

$$
\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyz} e^{-ty^2} dy \int_{-\infty}^{\infty} e^{-iyz} d\alpha(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha(z) \int_{-\infty}^{\infty} e^{t(x-z)^2} e^{-ty^2} dy
$$

$$
= \int_{-\infty}^{\infty} k(x-z, t) d\alpha(z) = u(x, t), \quad 0 < t < \infty.
$$

The interchange of integrals is valid for $t > 0$ since $\alpha(z)$ is bounded. Conversely, if we assume the representation (10.1) then $\phi(y)$ is given by (10.4) according to Bochner’s theorem. And now $\alpha(z)$ is known to be bounded and equation (10.5) results as before. It shows that $u \in H$, $\geq 0$, for $t > 0$. Finally $u(x, t_0) \in L$ for every $t_0 > 0$ from (10.2) and (10.3). This completes the proof.

A simple example of a positive temperature function which does not belong to $L$ on any line $t = t_0$ is $u = x^2 + 2t$. It has the representation (10.5) with $\alpha(z) = z^3/3$, but of course none of the form (10.1).

An example of the theorem is $k(x, t + a)$, $a > 0$. Here $\phi(x) = e^{-ax^2/(2\pi)}$ and $k \in L$ for each $t > 0$.

**BIBLIOGRAPHY**


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