THE ROLE OF THE APPELL TRANSFORMATION IN THE THEORY OF HEAT CONDUCTION(1)

BY

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1. **Introduction.** If $v(x, t)$ is an arbitrary function of two variables its Appell transform is

$$A[v] = A_{x,t}[v] = k(x, t)\, v\left(\frac{x}{t}, \frac{-1}{t}\right), \tag{1.1}$$

$$k(x, t) = \frac{e^{-x^2/4t}}{(4\pi t)^{1/2}}. \tag{1.2}$$

Here $k(x, t)$ is the fundamental solution of the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}. \tag{1.3}$$

As we shall see, this transformation serves in a remarkable way to establish a duality between types of solutions of (1.3). It was Appell [1] himself who showed that if v is a solution then $A[v]$ is also. We shall be studying here various solutions which have integral representations and the effect of the Appell transformation thereon. A few of our results were outlined, mostly without proof, in [8].

Let us introduce notations for the various integral transforms to be considered, as follows:

Poisson transform

$$P[\phi] = P_{x,t}[\phi] = \int_{-\infty}^{\infty} k(x-y, t)\,\phi(y)dy; \tag{1.4}$$

Fourier transform

$$F[\phi] = F_{x,t}[\phi] = \int_{-\infty}^{\infty} e^{ixy - ty^2}\phi(y)dy; \tag{1.5}$$

Laplace transform

$$L[\phi] = L_{x,t}[\phi] = \int_{-\infty}^{\infty} e^{xy + ty^2}\phi(y)dy. \tag{1.6}$$

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All of these have kernels which are solutions of (1.3) for each value of the parameter y and consequently produce solutions for more or less arbitrary functions ϕ. The transform (1.4) is also commonly referred to as the Weierstrass or Gauss transform. It is clear that $P_{ix,-t}[\phi]$ is also a solution of (1.3). We shall see that this one is paired with $F[\phi]$ while $L[\phi]$ is paired with $P[\phi]$ in the duality mentioned above.

We list a number of examples which will be useful in the sequel.

A. $$P[x^n] = v_n(x, t) = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!} \frac{t^k}{k!}, \qquad t > 0.$$

This is called the heat polynomial of degree n. The Appell transform of this polynomial is the *associated function* studied in [6]:

$$A[v_n] = w_n(x, t) = k(x, t) v_n(x/t, -1/t) t^{-n}, \qquad t > 0.$$

B. $$F[e^{ax^2}] = 2\pi k(x, t-a), \qquad t > a.$$

C. $$L[e^{-ax^2}] = 2\pi k(ix, a-t), \qquad t < a.$$

D. $$A[k(x, t+a)] = (4\pi a)^{-1/2} k(x, t-a^{-1}), \qquad t > a^{-1}.$$

The transformed function is in every case a solution of (1.3) in the half-plane indicated.

We now state briefly a few of the results of the present paper. The Huygens property is defined in §2 below and the growth of an entire function, in §4. A fundamental result is the following.

A function $u(x, t)$ is equal to the integral $F[\phi]$ defined by (1.5) *where ϕ is entire of growth* $(2, 1/\sigma)$, *if and only if it is the Appell transform $A[v]$ of a temperature function $v(x,t)$ which has the Huygens property for $|t| < \sigma$.*

We also obtain several new characterizations of positive temperature functions. The first involves those which are positive for all negative time.

A solution $v(x, t)$ of (1.3) *is $\geqq 0$ for $t < 0$ if and only if*

$$v(x, t) = \int_{-\infty}^{\infty} e^{xy+ty^2} d\alpha(y), \qquad t < 0,$$

where $\alpha(y) \in \uparrow$.

An example of such a solution is $e^t \cosh x$, for which α is a step-function with jumps at ± 1.

We also characterize a subclass of the above functions as follows.

A solution $v(x, t)$ of (1.3) *is $\geqq 0$ for $t < 0$ and satisfies the inequality*

$$\int_{-\infty}^{\infty} v(x, t) e^{x^2/4t} dx < \infty, \qquad t < 0$$

if and only if

$$v(x,\ t)=\int_{-\infty}^{\infty} k(y+ix,\ -t)\phi(y)dy, \qquad t<0,$$

where ϕ is positive definite.

A case in point is $v(x,\ t)\equiv 1$ with $\phi(y)\equiv 1$. A less trivial example is $v=k(ix,\ 1-t)$, which evidently satisfies the conditions of the theorem. Here ϕ is the positive definite function $e^{-y^2/4}/\sqrt{(4\pi)}$.

Finally we characterize those temperature functions which are positive for positive time and are absolutely integrable in the space variable.

A solution $u(x,\ t)$ of (1.3) *is $\geqq 0$ for $t<0$ and satisfies the inequality*

$$\int_{-\infty}^{\infty} u(x,\ t)dx < \infty, \qquad t>0$$

if and only if

$$u(x,t) = \int_{-\infty}^{\infty} e^{ixy-ty^2}\phi(y)dy, \qquad t>0,$$

where $\phi(y)$ is a positive definite function.

An example here is the fundamental solution itself with ϕ equal to the constant $(2\pi)^{-1}$.

2. **Relation between the Poisson and Fourier transforms.** Let us reintroduce here some of the notation of [6].

DEFINITION 2.1. *A function $u(x,\ t)\in H$, or is a temperature function, for $a<t<b$ if and only if it is a class C^2 solution of the heat equation* (1.3) *there.*

DEFINITION 2.2. *A function $u(x,\ t)\in H^*$, or has the Huygens property, for $a<t<b$ if and only if $u\in H$ there and*

$$u(x,\ t) = \int_{-\infty}^{\infty} k(x-y,\ t-t')u(y,\ t')\,dy \tag{2.1}$$

for every t, t', $a<t'<t<b$, the integral converging absolutely.

EXAMPLE A. Any function

$$u(x,\ t)=\int_{-\infty}^{\infty} k(x-y,\ t)d\alpha(y)$$

for which the integral converges absolutely in $a<t<b$ belongs to H^* there. Equation (2.1) then results immediately by use of Fubini's theorem. In particular, a positive function of H is also a function of H^*. See [7].

EXAMPLE B. The function $k(x,\ t+i)$ belongs to H in the whole x, t-plane and belongs to H^* in any strip of the form $-a<t<1/a$. See [6 p. 242]. It is to be emphasized that a function may belong to H^* on each of two overlapping sets without doing so on the sum of the sets. Thus the present function has the Huygens property for $-2<t<1/2$ and for $0<t<\infty$ but certainly not for

$-2<t<2$, for example. If we chose $t'=-1$, $t=1$, $x=0$ in the integral (2.1 it becomes

$$\int_{-\infty}^{\infty} k(y,\ t)k(y,\ -1+i)dy.$$

It clearly does not converge absolutely.

We prove now that any Poisson transform of a function in class L is also a Fourier transform.

THEOREM 2. *If $\phi \in L$ on $(-\infty,\infty)$, then $P_{x,t}[\phi]\in H^*$ for $t>0$ and*

$$P_{x,t}[\phi] = F_{x,t}[\psi], \qquad t>0,$$

where

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixy}\phi(y)\,dy.$$

By definition

$$P_{x,t}[\phi] = \int_{-\infty}^{\infty} k(x-y,t)\phi(y)dy, \qquad t>0. \tag{2.2}$$

The integral converges absolutely for $t>0$ since $k \leqq (4\pi t)^{-1/2}$. It consequently belongs to H^* there by Example A of this section. By Example B of §1 the integral (2.2) becomes

$$\begin{aligned} P_{x,t}[\phi] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(y)dy\int_{-\infty}^{\infty} e^{i(x-y)z-tz^2}dz \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixz-tz^2}dz\int_{-\infty}^{\infty} e^{-iyz}\phi(y)dy \\ &= F_{x,t}[\psi]. \end{aligned}$$

The interchange in the order of integration is justified in an obvious way by Fubini's theorem.

3. **Relation between the Poisson and Laplace transforms.**

To prove our next result we need a lemma.

LEMMA 3. *If $u(x,\ t)\in H^*$ for $a<t<b$, then $u(ix,-t)\in H^*$ for $-b<t<-a$.*

Since the integral (2.1), when it converges, defines an entire function of x, the definition of $u(ix,-t)$ is clear. We must prove that if $a<t<t'<b$ then

$$u(ix,\ t) = \int_{-\infty}^{\infty} k(x-y,\ t'-t)u(iy,\ t')dy, \tag{3.1}$$

the integral converging absolutely. Choose t'' so that $a<t''<t$. Then by hypothesis

$$u(x,\ t)=\int_{-\infty}^{\infty}k(x-z,\ t-t'')u(z,\ t'')dz, \tag{3.2}$$

the integral converging absolutely. Substituting (3.2) in the integral (3.1) we have

$$\begin{aligned}\int_{-\infty}^{\infty}k(x-y,\ t'-t)dy\int_{-\infty}^{\infty}k(iy-z,\ t'-t'')u(z,\ t'')dz \\ =\int_{-\infty}^{\infty}u(z,\ t'')dz\int_{-\infty}^{\infty}k(x-y,\ t'-t)k(iy-z,\ t'-t'')dy.\end{aligned} \tag{3.3}$$

This last interchange is valid by virtue of the inequalities

$$\int_{-\infty}^{\infty}k(x-y,\ t'-t)e^{y^2/4(t'-t'')}dy<\infty, \tag{3.4}$$

$$\int_{-\infty}^{\infty}e^{z^2/4(t'-t'')}\ u(z,\ t'')\,|\,dz<\infty. \tag{3.5}$$

The integral (3.4) clearly converges since $t''<t$; (3.5) does also since (3.2) converges absolutely when $t=t'$. But the value of the inner integral in (3.3) is known to be $k(ix-z,\ t-t'')$. See [5, p. 177]. That is, the integral (3.2) becomes

$$\int_{-\infty}^{\infty}k(ix-z,\ t-t'')u(z,\ t'')dz,$$

and this by (3.2) is $u(ix,\ t)$. Thus (3.1) is proved. Finally, the absolute convergence of (3.1) follows as a consequence of Fubini's theorem.

THEOREM 3. *If $\phi\in L$ on $(-\infty,\ \infty)$, then $P_{-ix,-t}[\phi]\in H^*$ for $t<0$ and*

$$P_{-ix,-t}[\phi]=L_{x,t}[\psi],\qquad t<0,$$

where

$$\psi(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ixy}\phi(y)dy.$$

By definition

$$P_{-ix,-t}[\phi]=\int_{-\infty}^{\infty}k(ix+y,\ -t)\phi(y)\,dy. \tag{3.6}$$

Since

$$|\,k(ix+y,\ -t)\,|\leqq e^{-x^2/4t}(-4\pi t)^{-1/2},\qquad t<0,$$

it is clear that the integral (3.6) converges absolutely for $t<0$. Since the integral $P_{x,t}[\phi]$ converges absolutely for $t>0$ it defines a function of H^* there. By Lemma 3 the function (3.6) belongs to H^* for $t<0$. By Example C of §1 $(a=0)$ the integral (3.6) becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(y)dy\int_{-\infty}^{\infty}e^{z(x-iy)+tz^2}dz = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{zx+tz^2}dz\int_{-\infty}^{\infty}e^{-iyz}\phi(y)dy.$$

The interchange of integrals is obviously permissible for $t<0$, and the inner integral is $\psi(z)$ as predicted.

Note that the initial temperature of $F_{x,t}[\psi]$ is the Fourier transform of ψ; the terminal temperature of $L_{x,t}[\psi]$ is the Laplace transform of ψ when these transforms exist.

4. **A criterion for the Fourier representation.** Let us recall the following definition.

DEFINITION 4. *An entire function*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

belongs to the class $(2, \sigma)$, *or has growth* $(2, \sigma)$, *if and only if*

$$\limsup_{n\to\infty} n\,|a_n|^{2/n} \leqq 2e\sigma. \tag{4.1}$$

See, for example, [2, p. 8]. A function of this class is of order $\leqq 2$, and if it is of order 2, then it is of type $\leqq \sigma$. For example, the functions xe^{-3x^2}, e^{x^2}, e^{5x} all belong to $(2, 3)$, whereas e^{-x^4} does not.

We first call attention to the following lemma.

LEMMA 4.1. *If* $s>0$, $t>0$, *then*

$$A_{x,t}[k(x-y,\ t+s)] = A_{y,s}[k(x-y,\ t+s)].$$

This is a simple identity. Both sides are equal to

$$(4\pi)^{-1}(st-1)^{-1/2}\exp[(2xy-ty^2-sx^2)(4st-4)^{-1}].$$

A second preliminary result is contained in

LEMMA 4.2. *If* $v(x,t)\in H^*$ *for* $|t|<\sigma$, *then* $v(-2ix, 0)\in(2, 1/\sigma)$.

To prove this we appeal to Theorem 11.1 of [6] and thus admit that

$$v(x,t) = \sum_{n=0}^{\infty} a_n v_n(x,t), \qquad |t|<0$$

$$v(x,0) = \sum_{n=0}^{\infty} a_n x^n.$$

By Theorem 5.5 of [6], $v(x,0)\in(2, 1/4\sigma)$. Thus (4.1) shows that $v(-2ix, 0) \in (2, 1/\sigma)$, as stated.

THEOREM 4. *A necessary and sufficient condition that*

$$u(x,t) = F[\phi] = \int_{-\infty}^{\infty} e^{xy-ty^2}\phi(y)dy, \qquad t>1/\sigma,$$

where $\phi(y) \in (2, \sigma^{-1})$, *is that there should exist a function* $v(x,t)$ *of* H^* *in* $|t| < \sigma$ *such that* $u = A[v]$.

We prove first the sufficiency. Since $v \in H^*$ for $|t| < \sigma$, then for any positive $\sigma' < \sigma$

$$v(x, t) = \int_{-\infty}^{\infty} k(x - y, t + \sigma')v(y, -\sigma')dy, \tag{4.2}$$

the integral converging absolutely when $-\sigma' < t < \sigma$. By Lemma 4.1

$$A[v] = \int_{-\infty}^{\infty} k(y, \sigma')k\left(x - \frac{y}{\sigma'}, t - \frac{1}{\sigma'}\right)v(y, -\sigma')dy, \qquad t > 1/\sigma',$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} k(y, \sigma')v(y, -\sigma')dy \int_{-\infty}^{\infty} \exp\left[iz\left(x - \frac{y}{\sigma'}\right) - z^2\left(t - \frac{1}{\sigma'}\right)\right]dz.$$

Here we have used Example B of §1. If interchange of iterated integrals is permitted, we obtain

$$\begin{aligned} A[v] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ixz - tz^2)dz \int_{-\infty}^{\infty} v(y, -\sigma')k(-2iz - y, \sigma')dy \\ &= \int_{-\infty}^{\infty} e^{ixz - tz^2}\phi(z)dz, \qquad t > 1/\sigma', \end{aligned} \tag{4.3}$$

where $2\pi\phi(z) = v(-2iz, 0)$. To justify the interchange it is sufficient to know, that

$$\int_{-\infty}^{\infty} \exp\left(-tz^2 + \frac{z^2}{\sigma'}\right)dz \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{4\sigma'}\right)\left|v(y, -\sigma')\right| dy < \infty.$$

The inner integral is finite since (4.2) converges absolutely when $x = t = 0$. The outer integral clearly converges when $t > 1/\sigma'$. Since $\phi(z)$ is independent of σ', it is clear that (4.3) continues to hold for all $t > 1/\sigma$. Finally, that $\phi \in (2, 1/\sigma)$ follows from Lemma 4.2.

To prove the necessity we now assume that $u = F[\phi]$ with ϕ belonging to $(2, 1/\sigma)$. We appeal to Theorem 12.1 of [6] to show that

$$u(x, t) = \sum_{n=0}^{\infty} b_n w_n(x, t), \qquad t > 1/\sigma,$$

where

$$b_n = \frac{2\pi\phi^{(n)}(0)}{n!(-2i)^n}. \tag{4.4}$$

Since $w_n = A[v_n]$ it is clear that $u = A[v]$, where

$$v(x,t) = \sum_{n=0}^{\infty} b_n v_n(x, t), \qquad -\sigma < t < 0. \tag{4.5}$$

By Theorem 11.1 of [6] the sum of any such series belongs to H^* throughout its region of convergence, always an entire strip of the form $|t| < \sigma$. The convergence of (4.5) in $-\sigma < t < 0$ implies its convergence in $|t| < \sigma$, so that $v \in H^*$ there. The equation (4.4) yields

$$v(x,\ 0) = 2\pi \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{n!}\left(\frac{x}{-2i}\right)^n = 2\pi\phi\left(\frac{ix}{2}\right)$$

$$v(-2iz,\ 0) = 2\pi\phi(z),$$

thus confirming the relation between v and ϕ established in the sufficiency proof.

Corollary 4. *If $v(x,\ t) \in H^*$ for $a < t < b$ and if $a < t_0 < b$, then*

$$F_{x,t}[v(-2ix,\ t_0)] = 2\pi A_{x,t}[v(x,\ t+t_0), \qquad t > 1/\sigma,$$

where $\sigma = \min(t_0 - a\ \ b - t_0)$.

This result was stated as Theorem II in [8]. It follows here by applying Theorem 4 to $v(x,\ t+t_0)$, which belongs to H^* in $|t| < \sigma$.

Example D of §1 provides an illustration of Theorem 4. The function $v = k(x,\ t+a) \in H^*$ for $t > -a$ but the largest strip of the form $|t| < \sigma$ contained in that half-plane is $|t| < a$. Hence $A[v]$, according to Theorem 4, must equal $F[\phi]$ for $t > 1/a$, where

$$2\pi\phi(y) = v(-2iy,\ 0) = e^{y^2/a}(4\pi a)^{-1/2},$$

$$A[\phi] = \frac{(4\pi a)^{-1/2}}{2\pi}\int_{-\infty}^{\infty} e^{ixy - ty^2} e^{y^2/a}\, dy, \tag{4.6}$$

$$A[\phi] = (4\pi a)^{-1/2} k\left(x,\ t - \frac{1}{a}\right), \qquad t > 1/a.$$

Notice that $\phi \in (2,\ 1/a)$ as predicted and that (4.6) converges in no larger region than that predicted by the theorem.

5. **A sufficient condition for the Fourier representation.** Although the previous example shows that Theorem 4 cannot be improved to give a larger region of convergence for the integral (4.6), nevertheless for some examples the region is indeed larger. Consider the function

$$v(x,\ t) = k(ix,\ \delta - t) + k(x,\ t + \sigma).$$

It belongs to H^* in $-\sigma < t < \delta$. If $\sigma > \delta$, Theorem 4 would predict the convergence of the corresponding integral $A[\phi]$ for $t > 1/\delta$. However

$$\phi(y) = \frac{e^{-y^2/\delta}}{4\pi\sqrt{\delta}} + \frac{e^{y^2/\sigma}}{4\pi\sqrt{\sigma}},$$

so that $F[\phi]$ actually converges in the larger region $t > 1/\sigma$. The following theorem would cover such an example.

THEOREM 5. *If* $v(x, t) \in H^*$ *for* $-\sigma < t < \delta$, $\sigma > 0$, $\delta > 0$, *then*

$$A[v] = F[\phi], \qquad t > 1/\sigma, \tag{5.1}$$

where $2\pi\phi(y) = v(-2iy, 0)$ *is entire and*

$$\limsup_{x\to\pm\infty} x^{-2} \log|\phi(x)| \leqq 1/\sigma. \tag{5.2}$$

Note that if $\sigma > \delta$ then $\phi \in (2, 1/\delta)$ and Theorem 4 would guarantee the convergence of the integral $F[\phi]$ for $t > 1/\delta$, in contrast with the present $t > 1/\sigma$.

The proof of (5.1) proceeds exactly as in the sufficiency of the conditions of Theorem 4. We find as before that $\phi(y) = v(-2iy, 0)/2\pi$. But now we cannot show that $\phi \in (2, 1/\sigma)$. We show instead that (5.2) holds. From equation (4.2) we have

$$|v(-2ix, 0| \leqq \frac{e^{x^2/\sigma'}}{\sqrt{4\pi\sigma'}} \int_{-\infty}^{\infty} e^{-y^2/4\sigma'} |v(y, -\sigma')| dy,$$

from which it follows that

$$\limsup_{x\to\pm\infty} x^{-2} \log|\phi(x)| \leqq 1/\sigma'.$$

On account of the arbitrary nature of σ' this inequality implies (5.2), and the proof is complete.

Observe that the condition $\delta > 0$ in Theorem 5 cannot be improved. The function $v = k(ix, -t) \in H^*$ in $-\infty < t < 0$. Its Appell transform is the constant $1/(4\pi)$. This certainly has no Fourier representation $A[\phi]$, since any such function must vanish at infinity.

6. **A necessary condition for the Fourier representation.** It should be observed that $F[\phi]$ may define a temperature function even when $\phi \notin (2, \sigma)$ for any σ. For example, if $\phi(y) = e^{-y^4}$, ϕ is entire of order 4. But the integral

$$F[\phi] = \int_{-\infty}^{\infty} e^{ixy - ty^2} e^{-y^4} dy$$

converges over the whole x, t-plane. Although the example is not covered by Theorem 4, $F[\phi]$ is nevertheless the Appell transform of some function, which of course cannot belong to H^* in any strip including the x-axis. The following result would include such an example.

THEOREM 6. *If* $u(x, t) = F[\phi] = \int_{-\infty}^{\infty} e^{ixy - ty^2}\phi(y)dy, t > 1/\sigma \geqq 0$, *where*

$$\limsup_{x\to\pm\infty} x^{-2} \log|\phi(x)| \leqq 1/\sigma, \tag{6.1}$$

then there exists $v(x, t)$ *belonging to* H^* *in* $-\sigma < t < 0$ *such that* $A[v] = u$.

We define v explicitly:

$$v(x, t) = 2\pi \int_{-\infty}^{\infty} k(y + ix, -t)\ \phi(-y/2)dy. \tag{6.2}$$

By hypothesis

$$\phi(y) = O(e^{y^2/\sigma'}), \qquad y \to \pm\infty \tag{6.3}$$

for any positive $\sigma' < \sigma$. Thus the integral (6.2) is dominated by

$$(-4\pi t)^{-1/2} e^{-x^2/4t} \int_{-\infty}^{\infty} e^{y^2/4t} O(e^{y^2/4\sigma'}) dy.$$

As a consequence it converges for $-\sigma' < t < 0$ and hence also for $-\sigma < t < 0$. The same relation (6.3) shows that the integral

$$w(x, t) = 2\pi \int_{-\infty}^{\infty} k(x-y, t)\phi(-y/2) dy$$

converges absolutely for $0 < t < \sigma$, and defines a function of H^* there. Since $v(x,t) = w(-ix, -t)$, we see by Lemma 3 that $v(x, t) \in H^*$ for $-\sigma < t < 0$. But from the definition of the Appell transformation (1.1) we have

$$\begin{aligned} A[v] &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-ixy/2 - y^2 t/4} \phi(-y/2)\, dy \\ &= \int_{-\infty}^{\infty} e^{ixy - ty^2} \phi(y) dy = F[\phi] = u. \end{aligned}$$

This completes the proof.

In the example of this section the left-hand side of (6.1) is $-\infty$ so that σ may be taken $+\infty$. The function (6.2) belongs to H^* for $-\infty < t < 0$.

7. **The Laplace transform of an Appell transform.** If a function $v(x, t) \in H^*$ in a strip including the x-axis, then the Laplace transform of $A[v]$ has a remarkably simple form. It is in fact equal to $v(2x, 0)e^{tx^2}$ for all t sufficiently large. The explicit result follows.

THEOREM 7. *If $v(x, t) \in H^*$ for $|t| < \sigma$, then for any $t > 1/\sigma$*

$$v(2x, 0)e^{tx^2} = \int_{-\infty}^{\infty} e^{xy} A_{y,t}[v]\, dy. \tag{7.1}$$

Make the change of variable $y = tz$ in the integral (7.1) multiplied by e^{-tx^2}. We obtain

$$\begin{aligned} te^{-tx^2} \int_{-\infty}^{\infty} e^{xtz} k(tz, t) v(z, -1/t) dz &= (t/4\pi)^{1/2} \int_{-\infty}^{\infty} e^{-(2x-z)^2 t/4} v(z, -1/t) dz \\ &= v(2x, 0). \end{aligned}$$

This last equation follows from the Huygens property provided only that $-\sigma < -1/t < 0$, and this is guaranteed by hypothesis. This completes the proof.

This result as applied to a function $v(x, t + t_0)$ was stated without proof as Theorem III in [7]. As an example take $v = k(x, t + a)$. Then for $t > 1/a$

$$e^{tx^2}e^{-x^2/a} \quad = \int_{-\infty}^{\infty} e^{xy}k\left(y, t - \frac{1}{a}\right)dy.$$

But this is a familiar result, which can be verified by Example C of §1.

8. **Positive temperatures in negative time.** Temperature functions which are positive for all negative time, such as $e^t \cosh x$, have a particularly simple representation in terms of the Laplace integral. The result follows.

THEOREM 8.1. *A necessary and sufficient condition that*

$$v(x, t) = \int_{-\infty}^{\infty} e^{xy+ty^2} d\alpha(y), \tag{8.1}$$

where $\alpha(y) \in \uparrow$ *and the integral converges for* $-\infty < t < 0$ *is that* $v(x, t) \geqq 0$ *and* $v(x, t) \in H$ *for* $-\infty < t < 0$.

The necessity of the condition is trivial. The kernel of (8.1) belongs to H for each y, and differentiation under the integral sign is valid for Laplace integrals.

Conversely, apply the Appell transform to the non-negative temperature function v:

$$A[v] = k(x, t)v\left(\frac{x}{t}, -\frac{1}{t}\right).$$

Clearly $A[v] \in H$ and is $\geqq 0$ for $t > 0$. Hence by an earlier result of the author [7] it has a Poisson representation

$$A[v] = \int_{-\infty}^{\infty} k(x - y, t) d\beta(y), \qquad 0 < t < \infty$$

$$v\left(\frac{x}{t}, -\frac{1}{t}\right) = e^{x^2/4t} \int_{-\infty}^{\infty} e^{-(x-y)^2/4t} d\beta(y),$$

where $\beta(y) \in \uparrow$. Now by obvious change of variable

$$v(x, t) = \int_{-\infty}^{\infty} e^{xy+ty^2} d\beta(y/2), \qquad -\infty < t < 0.$$

The proof is now concluded by defining $\alpha(y)$ as $\beta(y/2)$.

COROLLARY 8.1. *A necessary and sufficient condition that*

$$v(x, t) = \int_{-\infty}^{\infty} e^{xy+ty^2} d\alpha(y),$$

where $\alpha(y) \in \uparrow$ *and the integral converges for* $-\infty < t < c$, *is that* $v(x, t) \geqq 0$ *and* $v(x, t) \in H$ *there.*

This is proved by applying the theorem to $v(x, t + c)$.

As an application of this theorem we prove an earlier result of I. I. Hirschman [4].

THEOREM 8.2. *If* $v(x, t) \in H$ *and is* $\geqq 0$ *for* $-\infty < t \leqq c$ *and if* $\operatorname{Max}_{|x| \leqq r} v(x, t) = M(r)$, *then* $\liminf_{r \to \infty} \log M(r) \leqq 0$ *implies that* $u(x, t)$ *is constant for* $-\infty < t \leqq c$.

We may assume that $c = 0$ and show that the function $\alpha(y)$ of (8.1) has at most one point of increase, at $y = 0$. Suppose on the contrary that $y = y_0 \neq 0$ is such a point. Choose δ so that $y=0$ is not in the interval $(y_0 - \delta, y_0 + \delta)$. Then

$$v(x, 0) \geqq \int_{y_0-\delta}^{y_0+\delta} e^{xy} d\alpha(y) > pe^{(y_0-\delta)x}[> pe^{(y_0+\delta)x}], \qquad x > 0 \ [x < 0]$$

where p is the positive number $\alpha(y_0 + \delta) - \alpha(y_0 - \delta)$. This shows that

$$\begin{aligned} M(r) &\geqq pe^{(y_0-\delta)r}, & y_0 - \delta > 0 \\ &\geqq pe^{-(y_0+\delta)r}, & y_0 + \delta < 0. \end{aligned}$$

In either case

$$\liminf_{r \to \infty} r^{-1} \log M(r) > 0,$$

contradicting the hypothesis. This concludes the proof.

As a consequence of this theorem it is clear that any temperature function which is uniformly bounded for $-\infty < t \leqq c$ is a constant.

9. A subclass of positive temperatures in negative time. If an additional condition is imposed on the function $v(x, t)$ of Theorem 8.1 it will have in addition to (8.1) a quite different integral representation.

THEOREM 9. *A necessary and sufficient condition that*

$$v(x, t) = \int_{-\infty}^{\infty} k(y + ix, -t)\phi(y)dy, \qquad -\infty < t < 0, \tag{9.1}$$

where $\phi(y)$ *is positive definite, is that for* $-\infty < t < 0$ $v(x, t) \in H$, $\geqq 0$ *and that for some* $t_0 < 0$

$$\int_{-\infty}^{\infty} v(x, t_0) e^{x^2/4t_0} dx < \infty. \tag{9.2}$$

We prove first the necessity. Assume the representation (9.1). Since ϕ is positive definite we have by Bochner's theorem [3, p. 76], that for some $\alpha(z)$ that is nondecreasing and bounded

$$\phi(y) = \int_{-\infty}^{\infty} e^{iyz} d\alpha(z). \tag{9.3}$$

Hence

$$(9.4)\quad v(x,t)=\int_{-\infty}^{\infty} k(y+ix,-t)dy\int_{-\infty}^{\infty} e^{iyz}d\alpha(z)=\int_{-\infty}^{\infty} d\alpha(z)\int_{-\infty}^{\infty} k(y+ix,\ -t)e^{iyz}dy.$$

The interchange is valid for $t<0$ since

$$\int_{-\infty}^{\infty} d\alpha(z)\int_{-\infty}^{\infty} e^{y^2/4t}dy < \infty.$$

Evaluating the inner integral (9.4) we obtain

$$(9.5)\qquad v(x,\ t)=\int_{-\infty}^{\infty} e^{xz+tz^2}\, d\alpha(z),$$

from which it is clear that $v\in H,\ \geqq 0$. Moreover,

$$(9.6)\qquad \begin{aligned}\int_{-\infty}^{\infty} v(x,\ t)e^{x^2/4t}dx &= \int_{-\infty}^{\infty} e^{tz^2}d\alpha(z)\int_{-\infty}^{\infty} e^{xz}e^{x^2/4t}\,dx \\ &= \sqrt{(4\pi(-t))}\int_{-\infty}^{\infty} d\alpha(z) < \infty, \qquad t<0.\end{aligned}$$

Hence (9.2) is also established.

Conversely, if $v(x,\ t)\in H,\ \geqq 0$, for $-\infty<t<0$, then by Theorem 8.1 equation (9.5) holds for some nondecreasing function $\alpha(z)$. Since (9.2) is now assumed, it is the left side of (9.6) that is known to be finite for some t_0. From the right side we then conclude that $\alpha(z)$ is also bounded. Hence we may define $\phi(y)$ by (9.3) as a positive definite function. Finally, the equality of the integrals (9.5) and (9.1), already established, concludes the proof.

To show that the functions here considered really form a proper subclass of those considered in §8 we have only to exhibit the function $v=k(ix,\ -t)$. It fails to satisfy (9.2) for any t_0. Yet it is positive for $t<0$ and so has the representation (8.1). In fact $\alpha(y)=y/(2\pi)$.

10. **Positive temperatures in positive time.** As we have noted above, temperature functions which are positive for $t>0$ have a Poisson-Stieltjes integral representation. However, there is a subset of these functions which also have the Fourier integral representation.

Theorem 10. *A necessary and sufficient condition that*

$$(10.1)\qquad u(x,\ t)=F[\phi]=\int_{-\infty}^{\infty} e^{ixy-ty^2}\phi(y)dy, \qquad 0<t<\infty,$$

where $\phi(y)$ is positive definite, is that $u(x,\ t)\in H,\ \geqq 0$ for $0<t<\infty$ and $\in L(-\infty<x<\infty)$ for some $t=t_0>0$.

We prove first the sufficiency. Assuming $u\in H,\ \geqq 0$, a new appeal to [7] gives

$$u=\int_{-\infty}^{\infty} k(x-y,\ t)d\alpha(y), \qquad 0<t<\infty$$

for some $\alpha(y)\in\uparrow$. Since $u(x,\ t_0)\in L$ we have

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k(x-y,\ t_0)\,d\alpha(y) = \int_{-\infty}^{\infty} d\alpha(y) \int_{-\infty}^{\infty} k(x-y,\ t_0)\,dx \tag{10.2}$$

$$= \int_{-\infty}^{\infty} d\alpha(y) < \infty. \tag{10.3}$$

Hence $\alpha(y)$ is bounded and we can form the positive definite function

$$\phi(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iyz}\,d\alpha(z). \tag{10.4}$$

Substitute this integral for $\phi(y)$ in (10.1) to obtain

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixy-ty^2}\,dy \int_{-\infty}^{\infty} e^{-iyz}\,d\alpha(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha(z) \int_{-\infty}^{\infty} e^{i(x-z)y-ty^2}\,dy$$

$$= \int_{-\infty}^{\infty} k(x-z,\ t)\,d\alpha(z) = u(x,\ t), \qquad 0 < t < \infty. \tag{10.5}$$

The interchange of integrals is valid for $t > 0$ since $\alpha(z)$ is bounded.

Conversely, if we assume the representation (10.1) then $\phi(y)$ is given by (10.4) according to Bochner's theorem. And now $\alpha(z)$ is known to be bounded and equation (10.5) results as before. It shows that $u \in H$, $\geqq 0$, for $t > 0$. Finally $u(x,\ t_0) \in L$ for every $t_0 > 0$ from (10.2) and (10.3). This completes the proof.

A simple example of a positive temperature function which does not belong to L on any line $t = t_0$ is $u = x^2 + 2t$. It has the representation (10.5) with $\alpha(z) = z^3/3$, but of course none of the form (10.1).

An example of the theorem is $k(x,\ t+a), a > 0$. Here $\phi(x) = e^{-ax^2}/(2\pi)$ and $k \in L$ for each $t > 0$.

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