TOPOLOGICAL LOOPS WITH INVARIANT UNIFORMITIES

BY

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1. **Introduction.** A topological loop $\mathscr{L}$ is said to have a right (left) invariant uniformity $\mathscr{U}$ if there exists a uniform structure $\mathscr{U}$ compatible with the topology of $\mathscr{L}$ such that for any $a \in \mathscr{L}$ and any entourage $A \in \mathscr{U}$, (x, y) belongs to A if and only if (xa, ya) (respectively, (ax, ay)) belongs to A. In the case that $\mathscr{L}$ is a group, this terminology is consistent with that of topological groups. K. H. Hofmann [6] has shown that a compact topological loop has a uniform structure $\mathscr{U}$ having the property that for any $a \in \mathscr{L}$ and any entourage $A \in \mathscr{U}$, there exists an entourage $B \in \mathscr{U}$ contained in A such that $(x, y) \in B$ implies that (ax, ay), (xa, ya), $((a^{-1}x), (a^{-1}y))$, and $((xa^{-1}), (ya^{-1}))$ all belong to A(2). There exist loops whose topologies are that of a Euclidean 1-sphere which do not have a left or a right invariant uniform structure (see §5). As in the case of topological groups, a loop with a right invariant uniformity which is metrizable has a right invariant metric.

The purpose of this paper is, firstly, to make use of a "natural tool," namely, a "group of translates," for the study of certain loops possessing various uniformities. It is shown in Theorem 1 that:

A compact loop $\mathscr{L}$ has an invariant uniformity if and only if the group of all translates of $\mathscr{L}$ is relatively compact.

Next, with the aid of the group of translates it is proved in Theorem 4 that:

A loop with an invariant uniformity on a Euclidean n-sphere is isomorphic to one of the following: the cyclic group of order two, or the complex, the quaternion, or the Cayley numbers of norm one, respectively.

Finally, using this result on n-spheres with invariant uniformity, it is shown that certain diassociative loops with zero on Euclidean n-spaces are isomorphic to the multiplicative structure of either the real, complex, quaternion, or Cayley numbers

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2. **Invariant uniformities.** In the study of a topological loop $\mathscr{L}$(3), the

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(2) In a loop, $(a^{-1}b)$ denotes the solution for x of the equation $ax=b$ for given a and b, and (ba^{-1}) the solution for y of the equation $ya=b$.

(3) See [6], for example, for definition.

right (left) translate by $a \in \mathscr{L}$ is the homeomorphism of $\mathscr{L}$ onto itself, denoted by R_a (respectively, L_a), that maps $x \in \mathscr{L}$ onto xa (respectively, ax). The set of all right translates generates a subgroup of the group of all homeomorphisms of $\mathscr{L}$ onto itself. Similarly the set of all left translates and the set of all translates (both right and left) generate subgroups. If $\mathscr{L}$ is locally compact, then the three above-mentioned groups are transitive, topological transformation groups of $\mathscr{L}$ when given the g-topology [1].

THEOREM 1. *Let $\mathscr{L}$ be a compact loop and $\mathfrak{G}$ denote the group generated by all right and left translates of $\mathscr{L}$. Then $\bar{\mathfrak{G}}$, the closure of $\mathfrak{G}$ in the set of all continuous functions from $\mathscr{L}$ to $\mathscr{L}$ with the g-topology (equivalently, in this case, the compact-open topology), is a compact, transitive transformation group of $\mathscr{L}$ if and only if $\mathscr{L}$ has an invariant uniformity.*

Proof. First suppose that $\mathscr{L}$ has an invariant uniformity $\mathscr{U}$. Ascoli's theorem will be applied to show $\bar{\mathfrak{G}}$ is compact. It is sufficient to show that $\mathfrak{G}$ is equicontinuous and, secondly, $x \in \mathscr{L}$ implies that $\mathfrak{G}(x)$, the orbit at x, is relatively compact in $\mathscr{L}$. The second condition obviously holds, as does the transitivity of $\bar{\mathfrak{G}}$. To show the first condition holds, if $a \in \mathscr{L}$, $f \in \mathfrak{G}$, and A is any entourage of $\mathscr{U}$, then

$$f = R_{a_1}L_{a_2}R_{a_3}^{-1}L_{a_4}^{-1} \cdots R_{a_{n-3}}L_{a_{n-2}}R_{a_{n-1}}^{-1}L_{a_n}^{-1}$$

for some $a_1, \ldots, a_n \in \mathscr{L}$. Using the invariance property of the uniformity of $\mathscr{L}$ and inducting on n, it is easily seen that $(a,x) \in A$ implies that $(f(a), f(x)) \in A$. Hence $f(A(a)) \subset A(f(a))$ and $\mathfrak{G}$ is equicontinuous. It is noted that to show $\mathfrak{G}$ equicontinuous it would be sufficient to assume the weaker property that the uniformity of $\mathscr{L}$ satisfy the property that for any entourage A there exists an entourage $B \subset A$ such that $(x, y) \in B$ implies $(f(x), f(y)) \in A$ for all $f \in \mathfrak{G}$.

Furthermore $\bar{\mathfrak{G}}$ is a group; for let $g \in \bar{\mathfrak{G}}$. There is a net $\{g_\alpha\}$ converging to g with each g_α belonging to $\mathfrak{G}$. Since $\bar{\mathfrak{G}}$ is compact, $\{g_\alpha^{-1}\}$ has a convergent subnet $\{g_{\alpha_\beta}^{-1}\}$. Hence g has a right and left inverse in $\bar{\mathfrak{G}}$, namely, $\lim_\beta g_{\alpha_\beta}^{-1}$. $\bar{\mathfrak{G}}$ is algebraically closed under multiplication. It is easily seen that $\bar{\mathfrak{G}}$ is a topological semigroup, and it follows that $\bar{\mathfrak{G}}$ is a topological group since $\bar{\mathfrak{G}}$ is compact [4].

Conversely, suppose that $\bar{\mathfrak{G}}$ is a compact transformation group of $\mathscr{L}$. Then $\bar{\mathfrak{G}}$ has an invariant uniformity. The right coset space $\bar{\mathfrak{G}} / \bar{\mathfrak{G}}_1$ (for 1, the identity of $\mathscr{L}$, $\bar{\mathfrak{G}}_1$ is the subgroup of all $f \in \bar{\mathfrak{G}}$ for which $f(1) = 1$) has a right invariant uniformity, the one induced from $\bar{\mathfrak{G}}$. Since $\bar{\mathfrak{G}}$ is compact, the function from $\bar{\mathfrak{G}} / \bar{\mathfrak{G}}_1$ onto $\mathscr{L}$ mapping $\bar{\mathfrak{G}}_1 g$ onto $g(1)$ is a homeomorphism; and it follows that $\mathscr{L}$ has a right invariant uniformity. Similarly the left coset space has a left invariant uniformity and $\mathscr{L}$ has also left invariant uniformity. Since $\mathscr{L}$ is compact, these two uniformities are the same.

The following corollary follows from the above remarks.

COROLLARY. *If $\mathscr{L}$ is a compact loop, then $\mathscr{L}$ has an invariant uniformity if* *and only if for any entourage A there exists an entourage $B \subset A$ such that $(x,y) \in B$ implies $(f(x),f(y)) \in A$ for all $f \in \mathfrak{G}$.*

The notation introduced in Theorem 1 will be observed throughout this note. In particular, $\bar{\mathfrak{G}}$, rather than $\mathfrak{G}$, will be called the group of all translates of $\mathscr{L}$.

THEOREM 2. *Let $\mathscr{L}$ be a locally compact, connected loop. Then $\mathfrak{G}$ and $\bar{\mathfrak{G}}$ are connected spaces.*

Proof. Define a function $F: \mathscr{L} \to \mathfrak{G}$ by $F(x) = R_x$. If $\{x_\alpha\}$ is a net in $\mathscr{L}$ converging to $x \in \mathscr{L}$, then $R_{x_\alpha}(y_\beta) = y_\beta x_\alpha$ converges to yx whenever $\{y_\beta\}$ converges to y, because of continuity of multiplication in $\mathscr{L}$. $R_{x_\alpha}^{-1}$ also converges to R_x^{-1}. Hence $F(x_\alpha)$ converges to $F(x) = R_x$ in the g-topology [1], and F is continuous. Similarly the function from $\mathscr{L}$ to $\mathfrak{G}$ mapping x onto L_x is continuous. For each natural number n, define a function T_{4n} from $\mathscr{L}^{4n}$ (the $4n$-fold cartesian product of $\mathscr{L}$ by itself) to $\mathfrak{G}$ by

$$T_{4n}(x_1, \cdots, x_{4n}) = R_{x_1} L_{x_2} R_{x_3}^{-1} L_{x_4}^{-1} \cdots R_{x_{4n-1}}^{-1} L_{x_{4n}}^{-1}.$$

From the previous remarks on the continuity of F and the fact that multiplication and inversion are continuous in $\mathfrak{G}$, it follows that T_{4n} is continuous. Hence $T_{4n}(\mathscr{L}^{4n})$ is connected for each n. Since $\mathfrak{G} = \bigcup \{T_{4n}(\mathscr{L}^{4n}) : n = 1, 2, \ldots\}$ and the identity of $\mathfrak{G}$ belongs to each $T_{4n}(\mathscr{L}^{4n})$, $\mathfrak{G}$ is connected; and $\bar{\mathfrak{G}}$ is also connected.

The proof of Theorem 2 implies that, if $\mathscr{L}$ is compact, then $\mathfrak{G}$ is the countable union of a tower of compact sets.

THEOREM 3. *If $\mathscr{L}$ is a locally compact, connected loop with an invariant uniformity, then $\bar{\mathfrak{G}}$ is a locally compact transformation group.*

The proof of this theorem follows from [1, Theorem 7].

The above statements may be appropriately modified for right (left) invariant uniformities and the group generated by the right (respectively, left) translates. Theorems 1 and 2 have interest in the case that $\mathscr{L}$ is additionally a nonabelian group.

3. **Loops on n-spheres.** In the section the results of the second section are applied to loops on n-spheres with an invariant uniformity.

THEOREM 4. *Let $(\mathscr{L}, \circ)$ be a loop with an invariant uniformity whose topology is that of a Euclidean n-sphere S^n. Then $(\mathscr{L}, \circ)$ is isomorphic to one of the following: the cyclic group of order two, the complex numbers of norm one, the quaternions of norm one, or the Cayley numbers of norm one.*

Proof. We assume $n > 0$, for a two element loop is a group. We may also assume that S^n is the unit sphere in R^{n+1}, Euclidean $(n+1)$-dimensional space. The purpose of the following constructions is to obtain a division algebra over the reals, where the multiplication in the division algebra is obtained from that of $(\mathscr{L}, \circ)$.

Let $\bar{\mathfrak{G}}$ be the group of translates of $\mathscr{L}$. Then $\bar{\mathfrak{G}}$ is a compact, connected, effective, transitive transformation group of homeomorphisms acting on the n-sphere. Hence $\bar{\mathfrak{G}}$ is a Lie group [9]. For any $g \in \bar{\mathfrak{G}}$ we extend the domain of the mapping g to R^{n+1} by defining $g(O) = O$ and $g(x) = ag(x')$, where $O \neq x \in R^{n+1}$, $x = ax'$, a is a positive real number, and x' is the unique vector on the n-sphere lying on the ray starting at the origin through x (where $ag(x')$ is the usual Euclidean scalar product of a by the vector $g(x')$). Hence the elements of $\bar{\mathfrak{G}}$ are transformations of R^{n+1} and it is easily verified that $\bar{\mathfrak{G}}$ is a topological transformation group of R^{n+1}.

$\bar{\mathfrak{G}}$ is a compact, connected Lie group acting effectively on R^{n+1} and has at least one n-dimensional orbit. By a theorem of J. Poncet [11] there exists a homeomorphism π of R^{n+1} into E^{n+1} (also Euclidean $(n+1)$-space) and an isomorphism H of $\bar{\mathfrak{G}}$ into $\mathrm{SO}(n+1)$ such that for any $g \in \bar{\mathfrak{G}}$ and any $x \in R^{n+1}$

$$\pi(g(x)) = [H(g)](\pi(x)), \tag{1}$$

where the element $H(g)$ of $\mathrm{SO}(n+1)$ acts on $\pi(x)$ in E^{n+1} in the usual manner of matrix multiplication.

It follows from equation (1) and from the fact that π is a homeomorphism, that the origin is mapped upon the origin under π and orbit spheres about the origin are mapped onto such. We may thus assume that the unit sphere in R^{n+1} is mapped on the unit sphere in E^{n+1}, because we may follow π with a linear transformation which preserves equivalence of transformation groups. Define a multiplication on the unit sphere $\pi(\mathscr{L})$ in E^{n+1} by

$$x \cdot y = \pi(\pi^{-1}(x) \circ \pi^{-1}(y)).$$

It is obvious that $\mathscr{L}$ and $\pi(\mathscr{L})$ are isomorphic as topological loops. We extend the multiplication of $\pi(\mathscr{L})$ to E^{n+1} in the following manner. If x, y are vectors in E^{n+1}, both nonzero, x and y may be represented uniquely as $x = ax'$, $y = by'$; where a and b are real numbers, x' and y' are on the unit n-sphere, x' is on the ray starting at the origin through x, and y' is on the ray starting at the origin through y. Define $x \cdot y = ab(x' \cdot y')$, where $x' \cdot y'$ is the vector on the unit n-sphere obtained by the loop product in $\pi(\mathscr{L})$ of x' and y'. If x or y is the zero vector O, define $x \cdot y = O$.

We will now show that E^{n+1}, with the operation of multiplication defined in the previous paragraph, the usual Euclidean vector addition, and the usual Euclidean scalar multiplication, is a division algebra over the real numbers. If the usual Euclidean norm is denoted by $|\ |$, then this norm will satisfy the equation $|x \cdot y| = |x|\,|y|$.

E^{n+1} is a vector space over the reals. We first show that

$$a(x \cdot y) = (ax) \cdot y = x \cdot (ay) \tag{2}$$

for any two vectors x, y and any real number a. Let cx' and dy' be the unique representations of x and y, respectively, described above. If a is positive, then $a(x\cdot y) = acd(x'\cdot y') = ax\cdot y = x\cdot ay$, since acx' and ady' are the unique representations of ax and ay, respectively. If a is zero, all three of the vectors in (2) are the zero vector. If a is negative, we have initially

$$ax\cdot y = |a|(-x)\cdot y = |a|cd(-x'\cdot y') = |a|cd\pi(\pi^{-1}(-x')\circ\pi^{-1}(y')).$$

The purpose of this paragraph is to show that there exists an element g in the center of $\bar{\mathfrak{G}}$ such that g has order two and $\pi^{-1}(-x) = g(\pi^{-1}(x))$ for all x in E^{n+1}. Since $\mathscr{L}$ is a Hopf space, $n = 1, 3,$ or 7. Montgomery and Samelson [8] and A. Borel [2] have shown that $\bar{\mathfrak{G}}$ must be isomorphic to one of the following groups: for $n = 1$, SO(2); for $n = 3$, SU(2), SO(4), and Sp(1); for $n = 7$, SU(4), SO(8), Sp(2), and the covering group of SO(7). It is claimed that each one of the above groups has an element g of order two in its center for which $g(x) \neq x$ for all $x \in S^n$. In [3, p. 414] it is seen that there is an element g of order two in the center of each of the above groups (and only one such g). For some $z \in S^n$, $g(z) \neq z$ because of effectiveness. Furthermore, if for some $y \in S^n$, $g(y) = y$, then there exists $h \in \bar{\mathfrak{G}}$ such that $h(y) = z$ by transitivity. Hence

$$z \neq g(z) = g(h(y)) = h(g(y)) = h(y) = z,$$

establishing the above claim. Furthermore,

$$H(g)(H(g)(x) + x) = [H(g)](H(g)(x)) + H(g)(x) = x + H(g)(x)$$

for $x \in E^{n+1}$. Every element of SO$(n+1)$ leaves the origin fixed. If $H(g)(x) + x$ is not the zero vector, then the line through the origin and $H(g)(x) + x$ is left fixed by $H(g)$. There is an element $p \in \pi(S^n)$ which is left fixed since $\pi(S^n)$ is a sphere about the origin. But $H(g)(p) = p$ contradicts the fact that $g(x) \neq x$ for all $x \in S^n$. Hence $H(g)(x) = -x$ and $\pi^{-1}(-x) = \pi^{-1}(H(g)(x)) = g(\pi^{-1}(x))$.

Letting $z = \pi^{-1}(y')$ the following equations hold:

$$\begin{aligned}
&|a|cd\pi(\pi^{-1}(-x')\circ\pi^{-1}(y')) \\
&\quad = |c|cd\pi(g(\pi^{-1}(x'))\circ z) \\
&\quad = |a|cd\pi(R_z(g(\pi^{-1}(x')))) = |a|cd\pi(g(R_z(\pi^{-1}(x')))) \\
&\quad = |a|cd(-\pi(\pi^{-1}(x')\circ z)) = |a|cd(-(x'\cdot y')) = acd\,(x'\cdot y') = a(x\cdot y).
\end{aligned}$$

Similarly it can be shown that $x\cdot ay$ also equals $a(x\cdot y)$ by using left translates in $\bar{\mathfrak{G}}$. Thus equation (2) holds.

Furthermore, multiplication is left distributive, since

$$x\cdot(y + z) = ax'\cdot(y + z) = ab\pi(\pi^{-1}(x')\circ\pi^{-1}(y/b + z/b)),$$

where $x = ax'$ and $y + z = bt$ are the unique representations described above for x and $y + z$, respectively. We may assume that x and $y + z$ are not the zero vector. Letting $u = \pi^{-1}(x')$, it follows further that

$$
\begin{aligned}
ab\pi\left(L_u\left(\pi^{-1}\left(\frac{1}{b}y + \frac{1}{b}z\right)\right)\right)& \\
&= ab\left([H(L_u)]\left(\pi\left(\pi^{-1}\left(\frac{1}{b}y + \frac{1}{b}z\right)\right)\right)\right) \\
&= ab\left([H(L_v)]\left(\frac{1}{b}y\right) + [H(L_u)]\left(\frac{1}{b}z\right)\right) = a[H(L_u)](y) + a[H(L_u)](z) \\
&= a\pi(L_u(\pi^{-1}(y))) + a\pi(L_u(\pi^{-1}(z))) = a(x'\cdot y) + a(x'\cdot z) \\
&= (ax')\cdot y + (ax')\cdot z = x\cdot y + x\cdot z.
\end{aligned}
$$

In a similar manner, it can be shown that multiplication distributes over vector addition on the right by using right translates.

The equations $u\cdot x = v$ and $y\cdot u = v$ in R^{n+1} may be solved for x and y, if u and v are not the zero vector, by letting $x = (a/b)(u'^{-1}v')$ and $y = (a/b)(v'u'^{-1})$, where $u = bu'$ and $v = av'$ are the representations described above.

Finally, the norm defined above satisfies the equation $|x||y| = |x\cdot y|$; for $|x\cdot y| = ab = |x|\,|y|$, where $x = ax'$ and $y = by'$ are the representations described above.

It is known that such a division algebra as R^{n+1} (with $n > 0$) is either the complex numbers, the quaternions, or the Cayley numbers [12]. Therefore $\mathscr{L}$, being isomorphic to the subloop of R^{n+1} of norm one, is determined.

4. **Certain loops on** R^n. A topological loop with zero is a Hausdorff space $\mathscr{L}$ with a continuous multiplication and an element $O \in \mathscr{L}$ which satisfies the following conditions:

(1) $\mathscr{L}\backslash O$ is a topological loop with respect to the multiplication;

(2) $Ox = xO = O$ for all $x \in \mathscr{L}$;

(3) $\mathscr{L}\backslash O$ is dense in $\mathscr{L}$.

The study of locally compact, connected loops with zero was initiated by K. H. Hofmann [7] and the study of locally compact, connected groups with zero (two-ended groups) by H. Freudenthal and L. Zippin [13].

THEOREM 5. *Let R^n be a loop with zero (assume that the origin O of R^n is the zero) and suppose that some topological $(n-1)$-sphere S^{n-1} about the origin containing the identity of R^n satisfies $\delta(gx,gy) = \delta(x,y) = \delta(xg,yg)$ for all $g \in S^{n-1}$ and all x, y in R^n, where δ is a metric on R^n compatible with its usual topology. Then S^{n-1} is a subloop of R^n and S^{n-1} is determined by Theorem 4.*

Proof. Let H denote the subloop of R^n generated by S^{n-1}. It will be shown that the closure of H, $\bar{H}$, is a compact subloop of R^n and is also connected if $n > 1$.

Since $\bar{H}$ is closed in R^n, it is sufficient to show that if $\{g_i\}$ is a sequence in H, then $\{g_i\}$ does not converge to ∞ or 0, where ∞ is the "ideal point" in the one-point compactification of R^n. Suppose $\lim_i g_i = O$. We have

$$g_i = R_{x_1}R_{x_2}^{-1}L_{x_3}L_{x_4}^{-1} \cdots L_{x_{s-1}}L_{x_s}^{-1}(1) = f_i(1) \text{ for some } x_j \in S^{n-1}, \ j \leqq s.$$

Then $\delta(1,0) = \delta(f_i(1), f_i(0)) = \delta(g_i, 0)$ for all i and consequently $\delta(1,0) = 0$; but this equality cannot hold. Also suppose that $\lim_i g_i = \infty$. By [7, Proposition 3.12] $\lim_i (g_i^{-1}1) = 0$, again leading to a contradiction. Furthermore a modification of the proof of Proposition 2.12 of [7] implies that H is arcwise connected if $n > 1$, Since H is generated by an arcwise connected set containing 1. Consequently $\bar{H}$ is a compact, connected (for $n > 1$) loop containing S^{n-1} and H.

If $n = 1$, let $1 \neq g \in S^0$. If I is the closed interval $[g, 1]$, then gI is an interval with endpoints g and g^2 and with O as an inner-point; hence $gI = [g, g^2]$ with $g^2 > O$. We use the usual order on the real numbers. The component R^+ of 1 in $R^1 \backslash O$ is a subloop containing $g^2 = h$. The following argument of Hofmann (unpublished) shows that if $h > 1$, then $1 < h < h^2 < (h^2)^2 < \cdots$. In $R^+ \times R^+$ let A be the graph of the relation $<$ and let Δ be the diagonal. For $(x,y) \in R^+ \times R^+$ define $(x,y)h = (xh, yh)$. Then h is a homeomorphism of $R^+ \times R^+$ and either $Ah \subset A$ or $Ah \subset R^+ \times R^+ \backslash (A \cup \Delta)$. If $h > 1$, then $(1h^{-1})$ must be less than 1. Then $((1h^{-1}), 1)h = (1, h)$. We have, therefore, that h maps one element of A into A. Hence $Ah \subset A$. The above inequalities then hold as claimed. H being compact implies that the sequence $\{1, h, h^2, (h^2)^2, \cdots\}$ converges to an idempotent which must be 1. Hence h cannot be greater than 1. Similarly $h \nless 1$. Then $h = g^2 = 1$ and S^0 is a subloop.

Now assume $n > 1$. Let $\bar{\mathfrak{G}}$ be the group of all translates of $\bar{H}$. Then $\bar{\mathfrak{G}}$ is a compact, connected group of homeomorphisms acting effectively and transitively on $\bar{H}$. It is easily seen that $\bar{H}$ is $(n-1)$-dimensional and topologically equal to $\bar{\mathfrak{G}} / \bar{\mathfrak{G}}_1$. It follows that $\bar{H}$ is locally the topological product of an $(n-1)$-cell and a totally disconnected set [9, p. 239]. Let $\mathscr{U}$ denote the relative topology in R^n of $\bar{H}$ and $(\bar{H}, \mathfrak{M}(\mathscr{U}))$ denote the space obtained from $(\bar{H}, \mathscr{U})$ with the associated locally arcwise connected topology [5]. Then $(\bar{H}, \mathfrak{M}(\mathscr{U}))$ is an locally Euclidean space. If C denotes the component of 1 in $(\bar{H}, \mathfrak{M}(\mathscr{U}))$, C is an $(n-1)$-dimensional manifold. Also C contains S^{n-1} and H [5, p. 634, §3.2]. It follows that $C = H = S^{n-1}$. Since S^{n-1} has an invariant metric, Theorem 4 may now be applied.

A loop is called diassociative if every pair of elements belongs to a subgroup.

THEOREM 6. *Let R^n be as in Theorem 5. Furthermore let $R^n \backslash O$ be diassociative. Then $R^n \backslash O$ is the direct product of a subgroup M isomorphic to the positive real numbers under multiplication and S^{n-1}, which is isomorphic to* *either the cyclic group of order two, or the complex, the quaternion, or the Cayley numbers of norm one, respectively.*

Proof. K. H. Hofmann [7] has shown the following:

(a) $R^n \backslash O$ has a closed subgroup isomorphic to the positive real numbers with the usual multiplication;

(b) if M denotes any subgroup of the type in (a), then $R^n \backslash O = MS$, where $S = \{x \in R^n \backslash O : \Gamma(x) \text{ is compact in } R^n \backslash O\}$, and $\Gamma(x)$ is the closed subgroup generated by x;

(c) $R^n \backslash O$ is homeomorphic to $M \times S$, and S is compact in $R^n \backslash O$.

We have that $R^n \backslash O$ is homeomorphic to both $M \times S^{n-1}$ and $M \times S$, implying that dim $S =$ dim S^{n-1}. Theorem 5 concludes that S^{n-1} is a subloop, and we also know that $S^{n-} \subset S$. Thus S^{n-1} is homogeneous and open in S. Connectedness of S implies $S^{n-1} = S$.

We first establish that for a subgroup M as described above

$$(m_1 m_2)s = m_1(m_2 s), \tag{3}$$

for m_1, $m_2 \in M$ and $s \in S$. Let p and q be integers. Then $m_1({m_1}^{p/q}s) = (m_1 {m_1}^{p/q})s$ since $m_1^{1/q}$, $m_1^{p/q}$, m_1, and s all belong to the subgroup generated by $m_1^{1/q}$ and s. Since $\{m_1^{p/q}: p \text{ and } q \text{ are integers}\}$ is dense in M, (3) follows.

Also for such a subgroup as M, we will show that each element of $R^n \backslash O$ can be expressed uniquely as the product of an element from M and an element from S. Suppose $g = ms = nt$ for m, $n \in M$ and s, $t \in S$. Then $m^{-1}(nt) = s = (m^{-1}n)t$ and $st^{-1} = m^{-1}n$. Since $M \cap S = \{1\}$, it follows that $m = n$, $s = t$, and g is represented uniquely.

The existence of a subgroup M as in (a) with the property that $ms = sm$ for $m \in M$ and $s \in S$ is next established. Let N be a subgroup as in (a). The following two notations are used: for $s \in S$, $O(s)$ is the order of s, and $\langle A, B \rangle$ is the closure in R^n of the subloop of $R^n \backslash O$ generated by the sets A and B. We use also the following properties of the Cayley numbers of norm 1: if C_1, C_2, and C_3 are three different circle subgroups, $\langle C_1, C_2 \rangle$ is the 3-sphere group; and if $C_3 \not\subset \langle C_1, C_2 \rangle$, then $\langle C_1, C_2 \cup C_3 \rangle$ is the Cayley numbers of norm 1. Let C be a circle group contained in S. Then $\langle C, N \rangle$ is a connected, locally compact loop with zero. Using methods similar to the verification of (3), $\langle C, N \rangle$ is also a group with zero. According to the theory of such groups, $\langle C, N \rangle \backslash O$ is the direct product of P and K, P isomorphic to the positive real numbers, and K a compact, connected group. If for each such circle group in S, $\langle C, N \rangle$ is two-dimensional, then it follows that N commutes with S, and M is defined to be N.

However, if there exists a circle group C such that $\langle C, N \rangle$ is not two-dimensional, then $\langle C, N \rangle \backslash O$ is the direct product of P and K as above and is four-dimensional since K must be three-dimensional. Also this compact direct factor K is isomorphic to the quaternions of norm one. Define $P = M$. It is now shown that M commutes elementwise not only with elements of K but with all elements of S. If C_1 is a circle group not contained in K, let $\langle C_1, M\rangle\backslash O$ be expressed as the direct product of L and T, L isomorphic to the positive real numbers and T compact and connected. If T is one-dimensional, M commutes with T element for element, and it follows that M commutes in the same manner with $\langle T, K\rangle = S$. We may thus suppose that T is three-dimensional and a three-sphere group. Every subgroup of the quaternions which is isomorphic to the positive real numbers commutes with some circle group, so there is a circle group C_2 which commutes element for element with M. If C_2 is not contained in K, then again M commutes with all of S. We may finally suppose that $C_2 \subset K \cap T$. This supposition will be contradicted. Let C' be a circle group in T different from C_1 and let C'' be a circle group in K different from C_2. Also let $j \in L, m \in M$, $c_1 \in C'$, $d_1 \in C''$ with the properties that $m \neq 1 \neq j$ and that $1 < o(c_1) = o(d_1) < \infty$. Let G_1 be the subgroup generated by jc_1 and md_1. For each integer $n > 1$, let G_n be the subgroup generated by $j^{\alpha_n}c_n$ and $m^{\alpha_n}d_n$, where the following are satisfied: $c_n \in C'$ and $c_n^2 = c_{n-1}$, $d_n \in C''$ and $d_n^2 = d_{n-1}$, $\{c_n\}$ and $\{d_n\}$ converge to 1, $\alpha_n = (2^{n-1}(1+2q)(1+4q)\cdots(1+2^{n-1}q))^{-1}$, and finally $q = o(c_1)$. Then it follows that $G_n \supset G_{n-1}$. Let H be the closure of $\bigcup\{G_n : n=1,2,\cdots\}$. For each n

$$\begin{aligned}(m^{\alpha_n}c_n)^{2^{n-1}q} &= m^{q/((1+2q)\dots(1+2^{n-1}q))}c_n^{\;2^{n-1}q}\\ &= m^{q/((1+2q)\dots(1+2^{n-1}q))}\end{aligned}$$

and this element belongs to H. Because elements of M arbitrarily close to 1 belong to H, $M \subset H$. Hence $C' \subset H$. Similarly, L and C'' are contained in H. But the product of M and L contains C_2, as the following argument shows. Recall that we have assumed that $M \cup L \cup C_2$ is contained in the product of L and T and that M commutes with C_2. Since $M \neq L$, $4 \geqq \dim\langle M,L\rangle \geqq 2$, and there is a circle subgroup C_3 of $\langle M,L\rangle$. If $\dim\langle M,L\rangle = 2$, elements of M commute with elements of C_3. Either $\langle C_2,C_3\rangle = T$ and $M = L$ (this is not so), or else $\langle C_2,C_3\rangle = C_3$ and $C_2 \subset \langle M,L\rangle$. The dimensiom of $\langle M,L\rangle$ cannot be 3. If $\dim\langle M,L\rangle = 4$, $C_2 \subset T = \langle M,L\rangle$.

Now we have that $C_2 \cup C' \cup C'' \subset H$. But $\langle C_2, C'\rangle = T$ and $\langle C_2, C''\rangle = K$. T and K, being different maximal subgroups of S, cannot generate a subgroup of S. The supposition that $C_2 \subset K \cap T$ is contradicted, and M has the desired properties.

It is now shown that for $m \in M$ and $s_1, s_2 \in S$

$$(ms_1)s_2 = m(s_1s_2). \tag{4}$$

There exist in S elements t_i for each natural number i, of finite order p_i, such that $\{t_i\}$ converges to s_2 and $\{p_i\}$ converges to ∞. Both $m^{1/p_i}t_i$ and s_1 belong to a group. Because of the commutative law, $(m^{1/p_i}t_i)^{p_i} = m$. Hence $m, m^{1/p_i}t_i$, and s_1 associate. By continuity of multiplication (4) follows.

The following equations can be established in a manner similar to the proof of (4):

$$(s_1 m)s_2 = s_1(ms_2), \qquad \text{for } m \in M \text{ and } s_1, s_2 \in S, \tag{5}$$

$$m(s_1(ns_2)) = (ms_1)(ns_2), \qquad \text{for } m, n \in M \text{ and } s_1, s_2 \in S. \tag{6}$$

By equations (3)–(6) and commutativity it follows that $(ms_1)(ns_2) = (mn)(s_1 s_2)$ for m, $n \in M$ and s_1, $s_2 \in S$, and the proof is concluded.

5. **Example.** For complex numbers $e^{i\alpha}$ and $e^{i\beta}$ on the unit circle S^1 in the complex plane, define a binary operation in the following manner:

$$e^{i\alpha} \circ e^{i\beta} = e^{i(\alpha+\beta+\sin\alpha+\sin\beta)}.$$

Then $(S^1, \circ)$ is a commutative loop in which the binary operations are real analytic and which does not have an invariant uniformity. This example may be generalized to S^3 and S^7. Hence the hypotheses concerned with invariant uniformities in several of the theorems of this paper are indispensable for the conclusions desired.

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