ON THE RESIDUAL NILPOTENCE OF SOME VARIETAL PRODUCTS

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1. **Introduction.** A nonempty class $\mathfrak{v}$ of groups is a *variety* if it is closed with respect to the formation of subgroups, factor groups and cartesian products. If G is any group, we define $\mathfrak{v}(G)$ to be the intersection of all normal subgroups N of G such that $G/N \in \mathfrak{v}$; it is not difficult to show that $G/\mathfrak{v}(G) \in \mathfrak{v}$.

Now suppose $\langle A_\lambda ; \lambda \in \Lambda \rangle$ is a given family of groups. Let P be the free product of the groups A_λ; then

$$F = P/\mathfrak{v}(P)$$

is called the *free* $\mathfrak{v}$*-product of the groups* G_λ (cf. S. Moran [1]). If the groups A_λ are all infinite cyclic, then F is termed a *free* $\mathfrak{v}$-group and the cardinality $|\Lambda|$ of Λ is the *rank* of F. The purpose of this paper is to show that, for certain varieties $\mathfrak{v}$, the residual nilpotence(2) of a free $\mathfrak{v}$-group of infinite rank implies the residual nilpotence of the free $\mathfrak{v}$-product of every family of torsion-free abelian groups.

We need the notion of the composition $\mathfrak{u}\mathfrak{w}$ of two varieties $\mathfrak{u}$ and $\mathfrak{w}$ introduced by Hanna Neumann [2]. By definition $\mathfrak{u}\mathfrak{w}$ consists of those groups G which possess a normal subgroup $N \in \mathfrak{u}$ such that $G/N \in \mathfrak{w}$; notice that $\mathfrak{u}\mathfrak{w}$ is itself a variety (Hanna Neumann [2]).

Now let $\mathfrak{a}$ be the variety of all abelian groups and let $\mathfrak{u}$ be any given variety. The purpose of this paper is the proof of the

THEOREM. *The free* $\mathfrak{u}\mathfrak{a}$*-product* F *of every family* $\langle A_\lambda ; \lambda \in \Lambda \rangle$ *of torsion-free abelian groups is residually torsion-free nilpotent if and only if some free* $\mathfrak{u}\mathfrak{a}$*-group* X *of countably infinite rank is residually torsion-free nilpotent*(3).

K. W. Gruenberg [3] has shown that every free $\mathfrak{a}^n$-group is residually torsion-free nilpotent ($n = 1, 2, \cdots$), where inductively

$$\mathfrak{a}^{r+1} = \mathfrak{a}\mathfrak{a}^r \qquad (r > 0).$$

Consequently, by the theorem, the free a^n-product of any family of torsion-free

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(2) If $\mathcal{P}$ is a property pertaining to groups then, according to P. Hall [9], a group G is residually $\mathcal{P}$ if every element $x \in G$ ($x \neq 1$) can be omitted from a normal subgroup N_x such that G/N_x is $\mathcal{P}$.

(3) Cf. Theorem 6.2 of K. W. Gruenberg [3].

abelian groups is residually torsion-free nilpotent ($n = 1, 2, \cdots$). When $n = 2$ this reduces to a theorem of Rimhak Ree [4]; the more general result answers the question raised by Ree in [4, p. 394].

It may be well to mention that we first prove the theorem when the A_λ are free abelian (Proposition 1) and then make use of an embedding theorem of A. I. Mal'cev [5] to prove the theorem in general (see Proposition 2).

2. **The proof of Proposition** 1. This is the first stage of the proof of the theorem of this paper. Incidentally, Proposition 1 seems interesting in itself.

PROPOSITION 1. *The free* $\mathfrak{U}\mathfrak{a}$*-product F of a family* $\langle A_\lambda; \lambda \in \Lambda \rangle$ *of free abelian groups is residually a free* $\mathfrak{U}\mathfrak{a}$*-group.*

The proof of Proposition 1 will be accomplished by introducing four lemmas. We begin with the first of these, which amounts to a generalization of a theorem of Gilbert Baumslag [6].

LEMMA 1. *Let F be the free* $\mathfrak{U}\mathfrak{a}$*-product of a family* $\langle A_\lambda; \lambda \in \Lambda \rangle$ *of free abelian groups. Furthermore, let U_λ be an infinite cyclic subgroup of A_λ for each $\lambda \in \Lambda$. Then E, the subgroup generated by the U_λ, is a free* $\mathfrak{U}\mathfrak{a}$*-group; indeed E is the free* $\mathfrak{U}\mathfrak{a}$*-product of its subgroups U_λ.*

Proof. Suppose

$$U_\lambda = \mathrm{gp}(u_\lambda) \qquad (\lambda \in \Lambda).$$

Let

$$W_\lambda = \mathrm{gp}(w_\lambda) \qquad (\lambda \in \Lambda)$$

be infinite cyclic groups and let W the free $\mathfrak{U}\mathfrak{a}$-product of the $W_\lambda(\lambda \in \Lambda)$.

Now for every group $C \in \mathfrak{U}\mathfrak{a}$ and every system ϕ_λ of homomorphisms of A_λ into C ($\lambda \in \Lambda$) there is a homomorphism ϕ of F into C which coincides with ϕ_λ on A_λ (cf., e.g., S. Moran [1]). Since the A_λ are free abelian it is easy to concoct a homomorphism ϕ_λ of A_λ to W_λ such that $u_\lambda \phi_\lambda \neq 1$, say

$$u_\lambda \phi_\lambda = w_\lambda^{r_\lambda} \qquad (r_\lambda \neq 0).$$

Let ϕ be the homomorphism of F into W continuing the ϕ_λ. We claim, that the restriction of ϕ to E is a monomorphism.

To see this, notice that

$$E\phi = \mathrm{gp}(w_\lambda^{r_\lambda}; \lambda \in \Lambda).$$

By Theorem 3 of Gilbert Baumslag [6], $E\phi$ is the free $\mathfrak{U}\mathfrak{a}$-product of the groups $\mathrm{gp}(w_\lambda^{r_\lambda})$ ($\lambda \in \Lambda$). Hence the mappings

$$\theta_\lambda : w_\lambda^{r_\lambda} \to u_\lambda$$

can be extended to an epimorphism θ of $E\phi$ to E. Since $\phi\theta$ is the trivial automorphism when restricted to E, the restriction of ϕ to E is a monomorphism; this completes the proof.

It is worthwhile, at this point, to explain the motivation of the introduction of Lemma 1. We shall prove Proposition 1 by showing that if $f \in F (f \neq 1)$, then there exists an endomorphism $\hat{\eta}$ of F such that $F\hat{\eta}$ is a free $\mathfrak{U}\mathfrak{a}$-group and $f\hat{\eta} \neq 1$; it is Lemma 1 that affords us with subgroups of F which are free $\mathfrak{U}\mathfrak{a}$-groups. The remaining lemmas that we shall need before proceeding to the actual proof of Proposition 1 are aimed at establishing the existence of such an endomorphism $\hat{\eta}$.

LEMMA 2 (K. W. GRUENBERG [3]). *Let Λ be a totally ordered set and let P be the free product of a family $\langle A_\lambda; \lambda \in \Lambda \rangle$ of torsion-free abelian groups. Then P', the commutator subgroup of P, is freely generated by*

$$S = \{[b_\lambda, b_\mu, \cdots, b_\rho] \mid b_\lambda \in A_\lambda, b_\mu \in A_\mu, \cdots, b_\rho \in A_\rho,$$
$$b_\lambda \neq 1, b_\mu \neq 1, \cdots, b_\rho \neq 1,\ \lambda > \mu, \mu < \cdots < \rho\}.$$

[We have used in Lemma 2 the usual commutator notation. Consequently,

$$[x, y] = x^{-1} y^{-1} x y$$

and, inductively,

$$[x_1, x_2, \cdots, x_n] = [[x_1, x_2, \cdots, x_{n-1}], x_n] \qquad (n > 2).]$$

We need next a simple combinatorial fact concerning certain sequences of integers.

LEMMA 3. *Let*

$$(e_{i1}, e_{i2}, \cdots, e_{ir}) \qquad (i = 1, 2, \cdots, k)$$

be distinct r-termed sequences of integers. Then there exist integers

$$c_1, c_2, \cdots, c_r$$

such that the sums

$$\sum_{j=1}^{r} e_{ij} c_j \qquad (i \in \{1, 2, \cdots, k\})$$

are also distinct.

Proof. The proof of Lemma 3 is a straightforward argument by induction on r. To begin with, if $r = 1$, then we need only put

$$c_1 = c_2 = \cdots = c_r = 1$$

to obtain the desired result.

Thus let us suppose $r > 1$ and inductively choose

$$c_1, c_2, \cdots, c_{r-1}$$

so that

$$\sum_{j=1}^{r-1} e_{hj}c_j = \sum_{j=1}^{r-1} e_{ij}c_j$$

if and only if

$$(e_{h1}, e_{h2}, \cdots, e_{hr-1}) = (e_{i1}, e_{i2}, \cdots, e_{ir-1}),$$

where $h, i \in \{1, 2, \cdots, k\}$. We are left now with the choice of c_r. To this end let

$$h, i \in \{1, 2, \cdots, k\} \qquad (h \neq i).$$

Consider the equation

$$\sum_{j=1}^{r-1} e_{hj}c_j + xe_{hr} = \sum_{j=1}^{r-1} e_{ij}c_j + xe_{ir}. \tag{1}$$

This equation (1) has at most one solution. To see this notice that the sequences

$$(e_{h1}, e_{h2}, \cdots, e_{hr}), \qquad (e_{i1}, e_{i2}, \cdots, e_{ir})$$

are distinct. So if $e_{hr} = e_{ir}$,

$$\sum_{j=1}^{r-1} e_{hj}c_j \neq \sum_{j=1}^{r-1} e_{ij}c_j$$

and (1) does not have a solution. On the other hand, if $e_{hr} \neq e_{ir}$, then (1) clearly has a single solution. It follows that there are at most finitely many integers which satisfy at least one of the finitely many equations

$$\sum_{j=1}^{r-1} e_{hj}c_j + xe_{hr} = \sum_{j=1}^{r-1} e_{ij}c_j + xe_{ir} \qquad (h, i \in \{1, 2, \cdots, k\}, h \neq i).$$

We can therefore choose c_r in accordance with the requirements of the lemma and this then completes the proof.

Suppose now that we assume the notation of Lemma 2. We shall say that

$$s\ (= [b_\lambda, b_\mu, \cdots, b_\rho]) \in S$$

involves A_α $(\alpha \in \Lambda)$ if

$$\alpha \in \{\lambda, \mu, \cdots, \rho\},$$

and we say that the A_α *contribution* to s is b_α. Then the following lemma holds.

LEMMA 4. *Let α be a fixed element of Λ and let b be part of a basis for A_α. Further let*

$$s_1, s_2, \cdots, s_k \quad (s_i \neq s_j \textit{ if } i \neq j)$$

be elements of S all of which involve A_α. Then there exists an endomoromorphism η_α of P, which is the identity on A_β ($\beta \neq \alpha$), such that

(i) $s_1\eta_\alpha, s_2\eta_\alpha, \cdots, s_k\eta_\alpha$ *are distinct elements of S,*

(ii) $s_1\eta_\alpha, s_2\eta_\alpha, \cdots, s_k\eta_\alpha$ *involve A_α, and*

(iii) $A_\alpha\eta_\alpha \leqq \mathrm{gp}(b)$.

Proof. Let a_i be the A_α-contribution of s_i ($i = 1,2,\cdots,k$). Then we choose a subset

(2) $$b_1 = b, b_2, \cdots, b_r$$

of a basis B of A_α such that

$$a_i \in \mathrm{gp}(b_1, b_2, \cdots, b_r) \qquad (i = 1,2,\cdots,k).$$

Then

$$a_i = b_1^{e_{i1}} b_2^{e_{i2}} \cdots b_r^{e_{ir}} \qquad (i = 1,2,\cdots,k).$$

Consider the k sequences

(3) $$(e_{i1}, e_{i2}, \cdots, e_{ir}) \qquad (i = 1,2,\cdots,k).$$

Since $a_i \neq 1$ ($i = 1,2,\cdots,k$), none of the sequences $(e_{i1}, e_{i2}, \cdots, e_{ir})$ consists entirely of zeros. If we add to the sequences (3) the sequence

$$(0,0,\cdots,0),$$

then it follows from Lemma 3 that we can find integers $c_1, c_2, \cdots, c_r$ such that, firstly,

(4) $$\sum_{j \infty 1}^{r} e_{ij}c_j \neq 0$$

and secondly, if h, $i \in \{1,2,\cdots,k\}$,

(5) $$\sum_{j \infty 1}^{r} e_{hj}c_j = \sum_{j \infty 1}^{r} e_{ij}c_j \text{ if and only if } (e_{h1}, e_{h2}, \cdots, e_{hr}) = (e_{i1}, e_{i2}, \cdots, e_{ir}).$$

We are now in a position to define η_α. To begin with we define the effect of η_α on A_β ($\beta \neq \alpha$) to be the identity mapping. Next we define the action of η_α on A_α by specifying its action on a basis of A_α. We consider the basis B *involved* in (2); thus we put

$$x\eta_\alpha = 1 \text{ if } x \notin \{b_1, b_2, \cdots, b_v\} \qquad (x \in B),$$

and, finally, define

$$b_i\eta_\alpha = b^{c_i} \qquad (i = 1,2,\cdots,r).$$

By (4), (5) and Lemma 2 it follows that if h, $i \in \{1,2,\cdots,r\}$, then

$$a_h\eta_\alpha = a_i\eta_\alpha \text{ if and only if } a_h = a_i.$$

This completes the proof of Lemma 4 (cf. Lemma 2).

It is not difficult now to deduce Proposition 1. Thus let us suppose that F is the free $\mathfrak{u}\mathfrak{a}$-product of the free abelian groups A_λ ($\lambda \in \Lambda$). Then

$$F = P/\mathfrak{u}\mathfrak{a}(P),$$

where P is the free product of the groups A_λ ($\lambda \in \Lambda$). It is therefore sufficient, for the proof of Proposition 1, to show that if

$$w \in P, w \notin \mathfrak{u}\mathfrak{a}(P)$$

then there exists a homomorphism η of P into a free $\mathfrak{u}\mathfrak{a}$-group such that

$$w\eta \neq 1,$$

since the kernel K of η will necessarily contain $\mathfrak{u}\mathfrak{a}(P)$.

If $w \notin P'$, then it is easy, on noting that P/P' is free abelian, to find a homomorphism η of P into an infinite cyclic group (i.e., a free $\mathfrak{u}\mathfrak{a}$-group) so that $w\eta \neq 1$.

Thus we may suppose $w \in P'$. Now, by Lemma 2,

$$w = s_1^{\varepsilon_1}s_2^{\varepsilon_2}\cdots s_k^{\varepsilon_k} \qquad (\varepsilon_i = \pm 1,\ s_i \in S).$$

It follows easily, by a repeated application of Lemma 4, that there exists an endomorphism η^* of P such that

(i) $A_\lambda\eta^*$ is an infinite cyclic subgroup of A_λ ($\lambda \in \Lambda$),

(ii) $s_i\eta^* \in S$ ($i = 1,2,\cdots,k$),

(iii) $s_i\eta^* = s_j\eta^*$ if and only if $s_i = s_j$ ($i,j \in \{1,2,\cdots,k\}$).

Now by Lemma 2, S is a free set of generators of the free group

$$P' = \mathfrak{a}(P).$$

Consequently, as η^* is one-to-one on

$$\{s_1, s_2,\cdots, s_k\},$$

by (iii), there is certainly an automorphism μ of $\mathfrak{a}(P)$ such that

$$s_i\mu = s_i\eta^* \qquad (i = 1,2,\cdots,k).$$

Therefore

(6) $$w\eta^*(= w\mu) \notin \mathfrak{u}(\mathfrak{a}(P))(= \mathfrak{u}\mathfrak{a}(P))$$

since

$$w \notin \mathfrak{u}\mathfrak{a}(P).$$

Since $\mathfrak{u}\mathfrak{a}(P)$ is fully invariant the mapping

$$\eta : x \to x\eta^*\mathfrak{u}\mathfrak{a}(P)$$

is a homomorphism of P into $P\eta^* \mathfrak{u}\mathfrak{a}(P)/\mathfrak{u}\mathfrak{a}(P)$. But by (i) and Lemma 1, $P\eta$ is a free $\mathfrak{u}\mathfrak{a}$-group. Furthermore, by (6),

$$w\eta \neq 1.$$

This completes the proof of Proposition 1.

We would like to place on record the obvious conjecture that arises in connection with Proposition 1.

CONJECTURE. Let G be the free $\mathfrak{u}$-product of a family of free $(\mathfrak{u} \cap \mathfrak{a})$-groups. Then G is residually a free $\mathfrak{u}$-group.

3. **Proposition** 2. A group K is termed *radical* if extraction of roots is always possible in K. A. I. Mal'cev [5] has shown that a torsion-free nilpotent group can always be embedded in a torsion-free nilpotent radical group of the same class. Suppose now that K is a torsion-free nilpotent group and that K^* is a torsion-free nilpotent radical group containing K; K^* is called a *completion* of K if every radical subgroup of K^* containing K coincides with K^*. It is easy to see then that every torsion-free nilpotent group K has a completion K^*; moreover two completions of K are isomorphic (A. I. Mal'cev [5]). We need some additional information about K^*.

PROPOSITION 2. *Let K be a torsion-free nilpotent group. If K belongs to a variety $\mathfrak{v}$ then so does every completion K^* of K.*

Proof. It is easy to see that if every finitely generated subgroup of K^* lies in $\mathfrak{v}$ then so does K^* (cf. e.g. Hanna Neumann [2]).

Thus let H be a finitely generated subgroup of K^*:

$$H = \text{gp}(a_1, a_2, \cdots, a_n).$$

Put

$$L = H \cap K.$$

By a theorem of A. I. Mal'cev (cf. e.g. A. G. Kurosh [7, p. 248, Volume 2]), there is an integer r such that $a_i^r \in K$ $(i = 1, 2, \cdots, n)$, i.e.,

$$a_i^r \in L. \tag{7}$$

Let p be a prime chosen so that

$$(p, r) = 1. \tag{8}$$

Now, by a theorem of K. W. Gruenberg [3] the normal subgroups of p-power index in a finitely generated torsion-free nilpotent group intersect in the identity. Thus if $x \in H$ $(x \neq 1)$ we can find N, normal in H, such that $x \notin N$ and H/N is of order a power of p. But by (7) and (8) it follows that

$$a_i \in LN \qquad (i = 1, 2, \cdots, n).$$

Hence

$$LN/N = H/N.$$

But

$$LN/N \cong L/L \cap N.$$

Therefore $\mathfrak{v}(H) \leqq N$ since L (and so also $L/L \cap N) \in \mathfrak{v}$. Consequently

$$x \notin \mathfrak{v}(H),$$

and so $\mathfrak{v}(H) = 1$, i.e., $H \in \mathfrak{v}$. This completes the proof of Proposition 2.

4. **The proof of the main result.** We recall that the object of this paper is the proof of the

THEOREM. *Let $\mathfrak{u}$ be any variety and let $\mathfrak{a}$ be the variety of abelian groups. Then the free $\mathfrak{u}\mathfrak{a}$-product of every family of torsion-free abelian groups is residually torsion-free nilpotent if and only if some free $\mathfrak{u}\mathfrak{a}$-group of countably infinite rank is residually torsion-free nilpotent.*

Proof. The one part of the theorem is trivial.

For the other let us suppose that some free $\mathfrak{u}\mathfrak{a}$-group of infinite rank is residually torsion-free nilpotent. Then, clearly, every free $\mathfrak{u}\mathfrak{a}$-group is residually torsion-free nilpotent.

Now let $\langle A_\lambda; \lambda \in \Lambda\rangle$ be a family of torsion-free abelian groups and let F be their free $\mathfrak{u}\mathfrak{a}$-product. Furthermore, let

$$f \in F \quad (f \neq 1).$$

We can find (cf., e.g., L. Fuchs [8]) free abelian subgroups B_λ of A_λ such that

(i) A_λ/B_λ is periodic $(\lambda \in \Lambda)$,

(ii) $f \in \mathrm{gp}(B_\lambda; \lambda \in \Lambda)$.

We choose, for each $\lambda \in \Lambda$, an isomorphic copy $\bar{B}_\lambda$ of B_λ and consider their free $\mathfrak{u}\mathfrak{a}$-product $\bar{B}$. Then, by Proposition 1, $\bar{F}$ is residually torsion-free and nilpotent. Consequently, $\bar{B}$ is a subgroup of a cartesian product C of torsion-free nilpotent groups T_i $(i \in I)$ (cf., e.g., K. W. Gruenberg [3]):

$$C = \prod_{i=1}^{\infty} T_i.$$

Let T_i^* be the completion of T_i and let C^* be the cartesian product of the T_i^*:

$$C^* = \prod_{i=1}^{\infty} T_i^*.$$

Now it is easy to see, on noting that C^* is radical and torsion-free, that there is a completion C_λ of $\bar{B}_\lambda$ in C^*. So by the choice of the subgroup B_λ of A_λ (see (i) above) it follows that there is a monomorphism σ_λ of A_λ into C_λ mapping B_λ isomorphically onto $\bar{B}_\lambda$; let $\tilde{\sigma}_\lambda$ be the restriction of σ_λ to B_λ.

The groups $T_i \in \mathfrak{u}\mathfrak{a}$; hence by Proposition 2, $T_i^* \in \mathfrak{u}\mathfrak{a}$. But then $C^* \in \mathfrak{u}\mathfrak{a}$. This means that the system of mappings σ_λ $(\lambda \in \Lambda)$ can be continued to a homomorphism σ of F into C^*.

Let B be the subgroup generated by the B_λ $(\lambda \in \Lambda)$. Then σ induces an epimorphism $\tilde{\sigma}$ from B to $\bar{B}$. Indeed we claim that $\tilde{\sigma}$ is an isomorphism. To see this we recall that $\bar{B}$ is the free $\mathfrak{u}\mathfrak{a}$-product of the groups $\bar{B}_\lambda$. Hence the homomorphisms

$$\tilde{\sigma}_\lambda^{-1} : \bar{B}_\lambda \to B_\lambda$$

can be extended to a homomorphism $\bar{\sigma}$ from $\bar{B}$ to B. It follows immediately that $\tilde{\sigma}$ and $\bar{\sigma}$ are mutually inverse; hence $\tilde{\sigma}$ is one-to-one (and so B is the free $\mathfrak{u}\mathfrak{a}$-product of its subgroups B_λ $(\lambda \in \Lambda)$; this represents a partial generalization of a theorem of Gilbert Baumslag [6]).

But now the one-to-oneness of $\tilde{\sigma}$ implies

$$f\sigma(=f\tilde{\sigma}) \neq 1.$$

So there is a normal subgroup N of F, which does not contain f, such that F/N is torsion-free nilpotent. This completes the proof of the theorem.

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