TRANSFORMATIONS PRESERVING
THE GRASSMANNIAN

BY
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1. Introduction. For \( m \) a positive integer, let \( E_m \) be the arithmetic \( m \)-space over a commutative field \( F \). Let \( \mathcal{A}_m \) be the full linear group of \( E_m \), and let \( S_{m-1} \) be the projective space of homogeneous coordinates in \( E_m \). For the rest of the paper, we fix two positive integers \( n \) and \( k \), such that \( k < n \). Let \( N = \binom{n}{k} \), and let \( \Omega(k,n) \) be the \( k,n \) Grassmannian variety:

\[
\Omega(k,n) \subset S_{n-1}.
\]

Let \( \psi(k,n) \) be the set of those nonzero elements \( x \) of \( E_N \) such that there is some \( y \) satisfying

\[
x \in y \in \Omega(k,n).
\]

Let \( G \) be the set of nonsingular linear transformations of \( E_N \) which keep \( \psi(k,n) \) fixed as a set. If \( C_N \) is the center of the full linear group of \( E_N \), then \( G/C_N \) is the set of projective transformations of \( S_{n-1} \) which keep \( \Omega(k,n) \) fixed as a set.

Let \( A(n,k) \) be the group of all \( k \)-compounds \([1, \text{Vol. 1, p. 291}]\) of elements of \( \mathcal{A}_n \). Then \( A(n,k)/(C_N \cap A(n,k)) \) may be thought of as the group of projective transformations of \( S_{n-1} \) "induced" by the group of projective transformations of \( S_{n-1} \). Since \( A(n,k)/(C_N \cap A(n,k)) \) is isomorphic to \( (A(n,k) \cdot C_N)/C_N \), and since \( A(n,k) \cdot C_N \) is a subgroup of \( G \), \( (A(n,k) \cdot C_N)/C_N \) is a subgroup of \( G/C_N \).

The principal results to be proved here are:

1. If \( n \neq 2k \), then

\[
A(n,k) \cdot C_N = G,
\]

and thus

\[
(A(n,k) \cdot C_N)/C_N = G/C_N.
\]

2. If \( n = 2k \), let \( J \) denote the "star dual" mapping of \( \psi(k,n) \) onto itself (see 2). Since

\[
J^2 = (-1)^{(k^2)} I,
\]

where \( I \) is the identity element of \( \mathcal{A}_N \), \( J \) generates a cyclic subgroup of order 2 if \( k \) is even, and of order 4 if \( k \) is odd. Let \( \mathcal{J} \) denote this group. Let \( \mathcal{K} \) be the

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subgroup of $G/C_N$ made up of cosets of elements of $\mathcal{J}$. Thus $\mathcal{K}$ is of order 2. Then, in this case,

$$\mathcal{J} \cdot A(n,k) \cdot C_N = G,$$

and thus

$$\mathcal{K} \cdot ((A(n,k) \cdot C_N)/C_N) = G/C_N.$$

2. **Notation.** (For definitions of terms used here and proofs of results given here, see [2].) We shall denote the exterior product of vectors by "\(\wedge\)." Thus $x$ is an element of $\psi(k,n)$ if and only if there is a linearly independent set of $k$ elements of $E_n,x_1,x_2,x_3,\ldots,x_k$; and

$$x = x_1 \wedge x_2 \wedge x_3 \wedge \cdots \wedge x_k.$$  

For $A \in \mathcal{A}_n$, let $A^k$ be the $k$-compound of $A$. Thus if

$$x = x_1 \wedge x_2 \wedge x_3 \wedge \cdots \wedge x_k,$$

then

$$A^k x = A x_1 \wedge A x_2 \wedge A x_3 \wedge \cdots \wedge A x_k.$$  

For $E \subseteq E_m$, let $L(E)$ be the subspace of $E_m$ spanned by $E$. If $x \in \psi(k,n)$, such that

$$x = x_1 \wedge x_2 \wedge x_3 \wedge \cdots \wedge x_k,$$

let

$$\pi(x) = L(\{x_1,x_2,x_3,\ldots,x_k\}).$$

For any positive integer $m$, let

$$\mathcal{N}(m) = \{1,2,3,\ldots,m\}.$$  

For $t$ a positive integer, $t \leq m$, let

$$P(m,t) = \{p : p = \{p_1,p_2,p_3,\ldots,p_t\}, \ p_i \in \mathcal{N}(m) \text{ for } i \in \mathcal{N}(t), \ \text{and} \ p_1 < p_2 < p_3 < \cdots < p_t\}.$$  

For $p \in P(m,t)$, let $c(p)$ be that element of $P(m,m-t)$ such that

$$p \cup c(p) = \mathcal{N}(m).$$

For $x$ an element of $\psi(k,n)$, $\star x$ is that element of $\psi(n-k,n)$ defined by

$$(\star x)_q = \varepsilon(q)x_{c(q)},$$

where $q$ is any element of $P(n,n-k)$, and $\varepsilon(q)$ is $-1$ to the power of the parity of the permutation $(q_1,q_2,q_3,\ldots,q_{n-k},(cq)_1,(cq)_2,(cq)_3,\ldots,(cq)_k)$. Let $J$ be that mapping of $\psi(k,n)$ onto $\psi(n-k,n)$ defined by

$$J(x) = \star x.$$
Then \( J \) can be extended to a nonsingular linear mapping of \( E_N \) onto itself.

Since \( k < n \), we may consider \( E_{k+1} \) as a subspace of \( E_n \), and \( \psi(k, k + 1) \) as a subset of \( \psi(k, n) \). On occasion, we shall find it necessary to use the \( * \)-dual of a vector in \( \psi(k, k + 1) \) "relative to \( E_{k+1} \)." That is, for \( x \) an element of \( \psi(k, k + 1) \subset \psi(k, n) \),

\[
(*_{k+1}x)_i = (-1)^{i-1}x_{c(i)}, \quad \text{where } c(i) = \mathcal{N}(k + 1) - \{i\}, \text{if } 1 \leq i \leq k + 1;
\]

and

\[
(*_{k+1}x)_i = 0, \quad \text{if } i > k + 1.
\]

Then \( *_{k+1}x \in E_{k+1} \subset E_n \), and

\[
L(*_{k+1}x) = (\pi(x))^\perp_{k+1},
\]

where \( \perp_{k+1} \) denotes the orthogonal complement relative to \( E_{k+1} \).

For \( i \in \mathcal{N}(m) \), let \( e_i \) be that element of \( E_m \) whose \( j \)-th component is \( \delta_{ij} \). For \( p \in P(n, k) \), let

\[
e_p = e_{p_1} \wedge e_{p_2} \wedge \cdots \wedge e_{p_k}.
\]

Then the set \( \{e_p : p \in P(n, k)\} \) is a basis for \( E_N \).

For \( A \in G \), and \( p \in P(n, k) \), let \( A_p = Ae_p \). Then \( A_p \in E_N \), and it is the \( p \)-th column vector of the matrix of \( A \). For any \( q \in P(n, k - 1) \),

\[
dim(\bigcap \pi(e_p)) = k - 1,
\]

the intersection being taken over all \( p \in P(n, k) \) such that \( q \subset p \); and

\[
dim(L(\{e_p : q \subset p \in P(n, k)\})) = n - k + 1.
\]

So if \( A \in G \), and \( q \in P(n, k - 1) \), and if

\[
M = A(L(\{e_p : q \subset p \in P(n, k)\}))
\]

then \( \dim M = n - k + 1 \), and \( M \) is spanned by the set \( \{A_p : q \subset p \in P(n, k)\} \). Furthermore, for \( p \in P(n, k) \), \( A_p \in M \) if and only if \( q \subset p \).

Since we have excluded the zero vector from \( \psi(k, n) \), no linear subspace of \( E_N \) is contained in \( \psi(k, n) \). However, if \( M \) is a linear subspace of \( E_N \), we shall say \( M \subset \psi(k, n) \) if and only if for \( x \in M \), if \( x \neq 0 \), then \( x \in \psi(k, n) \).

3. **Principal results.** The principal results may now be stated in the following two theorems.

3.1. **Theorem.** If \( n \neq 2k \), and \( A \in G \), then there exists \( C \in C_N \) and \( B \in \mathcal{A}_m \) such that

\[
A = CB^k.
\]

3.2. **Theorem.** If \( n = 2k \), and if \( A \in G \), then there exists \( C \in C_N \) and \( B \in \mathcal{A}_n \) such that either
\[ A = CB^k \]

or

\[ A = CJB^k. \]

The proofs of these theorems depend on the following three lemmas, which will be proved in §§4 and 5.

3.3. Lemma. For \( m \) an integer, \( 2 \leq m \leq N \), let \( M \) be a subspace of \( E_N \), with \( \dim M = m \), such that there exists a set \( \{x_1, x_2, x_3, \ldots, x_m\} \subset \psi(k, n) \) and \( \{x_1, x_2, x_3, \ldots, x_m\} \) spans \( M \). Then,

1. if

\[
\dim \bigcap_{i=1}^{m} \pi(x_i) = k - 1,
\]

then \( M \subset \psi(k, n) \),

\[
\dim \bigcap_{x \in M} \pi(x) = k - 1,
\]

and

\[
\dim L(\{\pi(x) : x \in M\}) = k + m - 1;
\]

2. if

\[
\dim L(\{\pi(x_i) : 1 \leq i \leq m\}) = k + 1,
\]

then \( M \subset \psi(k, n) \), \( m \leq k + 1 \),

\[
\dim \bigcap_{x \in M} \pi(x) = k - m + 1,
\]

and

\[
\dim L(\{\pi(x) : x \in M\}) = k + 1.
\]

In either case, \( M \) is the set of all \( k \)-vectors of \( k \) dimensional subspaces of \( E_n \) which contain \( \bigcap_{x \in M} \pi(x) \) and are contained in \( L(\{\pi(x) : x \in M\}) \).

3.4. Lemma. For \( m \) an integer, \( 2 \leq m \leq N \), let \( M \) be a subspace of \( E_N \), with \( \dim M = m \), and assume that \( M \subset \psi(k, n) \). Let \( \{x_1, x_2, x_3, \ldots, x_m\} \) be any spanning set of \( M \). Then either

\[
\dim \bigcap_{i=1}^{m} \pi(x_i) = k - 1,
\]

or

\[
\dim L(\{\pi(x_i) : 1 \leq i \leq m\}) = k + 1.
\]

3.5. Lemma. If \( A \in G \), and if, for each \( q \in P(n, k-1) \),

\[
\dim \bigcap \pi(A_p) = k - 1,
\]

the intersection being taken over all \( p \) such that
then there exists \( C \in C_N \) and \( B \in \mathcal{A}_n \) such that

\[ A = CB^k. \]

**Proof of Theorem 3.1 assuming Lemmas 3.3, 3.4, and 3.5.** First assume that \( n > 2k \). For \( q \in P(n, k-1) \), let \( M(q) \) be the subspace of \( E_N \) spanned by the set \( \{ A_p : q \subseteq p \in P(n, k) \} \). Then \( M(q) \subseteq \psi(k, n) \), and \( \dim M(q) = n - k + 1 \). But \( n - k + 1 > k + 1 \). So by 3.3 and 3.4,

\[ \dim \bigcap \pi(A_p) = k - 1, \]

the intersection being taken over all \( p \) such that \( q \subseteq p \in P(n, k) \).

The result follows from 3.5. Now assume that \( n < 2k \). Then for \( x \in \psi(n-k, n) \), \( JAJ^{-1}(x) \in \psi(n-k, n) \). Hence there exists \( C \in C_N \) and \( B \in \mathcal{A}_n \) such that

\[ JAJ^{-1} = CB^{n-k}. \]

So

\[ A = CJ^{-1}B^{n-k}J. \]

By the Laplace expansion of a determinant,

\[ J^{-1}B^{n-k}J = (\det B)I(B^{-T})^k, \]

where \(-T\) denotes inverse transpose. Hence

\[ A = C(\det B)I(B^{-T})^k. \]

This completes the proof.

**Proof of Theorem 3.2 assuming Lemmas 3.3, 3.4, and 3.5.** We first show that if

\[ \dim L(\{ \pi(A_p) : q' \subseteq p \in P(n, k) \}) = k + 1, \]

for some \( q' \in P(n, k-1) \), then

\[ \dim L(\{ \pi(A_p) : q \subseteq p \in P(n, k) \}) = k + 1, \]

for every \( q \in P(n, k-1) \). It suffices to consider \( q' = \{1, 2, 3, \ldots, k-1\} \) and to assume that

\[ \dim L(\{ \pi(A_p) : q' \subseteq p \in P(n, k) \}) = k + 1. \]

Select \( q \in P(n, k-1) \), so ordered that if \( q_i \subseteq q' \), then \( q_i = i \). Let \( q^* = \{2, 3, 4, \ldots, k-1, q_1\} \). We will show that

\[ \dim L(\{ \pi(A_p) : q^* \subseteq p \in P(n, k) \}) = k + 1. \]
If \( q_1 = 1 \), there is nothing to prove. So assume that \( q_1 \neq 1 \). Let 
\( p^* = \{1, 2, 3, \ldots, k-1, q_1\} \), and let 
\[
M' = L(\{A_p: q^* \subset p \in P(n, k)\}),
\]
and 
\[
M'' = L(\{A_p: q^* \subset p \in P(n, k)\}).
\]
Then 
\[
M' \cap M'' = L(A_{p^*}),
\]
so 
\[
\dim(M' \cap M'') = 1.
\]
Now let \( Q' = L(\{\pi(A_p): q' \subset p \in P(n, k)\}) \), and \( Q^* = \bigcap \pi(A_p) \), the intersection being taken over all \( p \in P(n, k) \) such that \( q^* \subset p \), and assume that \( \dim Q^* = k - 1 \). Then 
\[
Q^* \subset \pi(A_{p^*}) \subset Q'.
\]
So the set of all \( y \in \psi(k, n) \) such that \( Q^* \subset \pi(y) \subset Q' \) is a subspace of \( M' \cap M'' \), but by [1, Vol. 2, Chapter XIV, Theorem I], the dimension of this subspace is 2. So \( \dim(M' \cap M'') \geq 2 \). This is a contradiction. So by Lemma 3.4, 
\[
\dim L(\{\pi(A_p): q^* \subset p \in P(n, k)\}) = k + 1.
\]
Continuing in this manner, working with one element of \( q \) at a time, we conclude that 
\[
\dim L(\{\pi(A_p): q \subset p \in P(n, k)\}) = k + 1.
\]
Hence either \( A \) or \( JA \) satisfies the conditions of Lemma 3.5, so the result follows from the fact that \( J^2 = (-1)^{(k^2)}I \).

4. Linear subspaces contained in \( \psi(k, n) \). Lemmas 3.3 and 3.4 describe the linear subspaces of \( E_N \) which are contained in \( \psi(k, n) \) in the sense of 2. In this section we give proofs of these two lemmas.

**Proof of Lemma 3.3.** Select a set \( \{x_1, x_2, x_3, \ldots, x_m\} \subset \psi(k, n) \), such that \( \{x_1, x_2, x_3, \ldots, x_m\} \) spans \( M \), and assume that 
\[
\dim \bigcap_{i=1}^{m} \pi(x_i) = k - 1.
\]
Then without loss of generality, we may assume that 
\[
x_i = e_1 \wedge e_2 \wedge e_3 \wedge \ldots \wedge e_{k-1} \wedge e_{k+i-1}, \quad \text{for } i = 1, 2, 3, \ldots, m.
\]
Now let \( x \in M \). Then there exist \( a_1, a_2, a_3, \ldots, a_m \), elements of \( F \), such that \( x = \sum_{i=1}^{m} a_i x_i \). So 
\[
x = e_1 \wedge e_2 \wedge e_3 \wedge \ldots \wedge e_{k-1} \wedge \left( \sum_{i=1}^{m} a_i e_{k+i-1} \right).
\]
Hence \( M \subseteq \psi(k, n) \), and consists of those \( k \)-vectors of \( k \)-spaces containing \( L(\{e_1, e_2, e_3, \ldots, e_{k-1}\}) \), and contained in \( L(\{e_1, e_2, e_3, \ldots, e_{k+m-1}\}) \). Now assume that

\[
\dim L(\{\pi(x_i): 1 \leq i \leq m\}) = k + 1.
\]

Then without loss of generality, we may assume that

\[
\pi(x_i) \subseteq L(\{e_1, e_2, e_3, \ldots, e_{k+1}\})
\]

for \( i = 1, 2, 3, \ldots, m \). Hence the \( x_i \) may be thought of as \( k \)-vectors in \( E_{k+1} \). So if \( x \in M \), \( x = \sum_{i=1}^{m} a_i x_i \), for suitable elements \( a_i \) of \( F \), then \( x \) is a \( k \)-vector in \( E_{k+1} \). Hence \( M \subseteq \psi(k, n) \), and

\[
\dim L(\{\pi(x): x \in M\}) = k + 1.
\]

Also, the set \( \{x_{k+1} x_i: 1 \leq i \leq m\} \) spans an \( m \)-space of \( E_{k+1} \), so \( m \leq k + 1 \), and since \( L(x_{k+1} x_i) = (\pi(x_i))^{-1} x_i \),

\[
\dim \bigcap_{i=1}^{m} \pi(x_i) = k - m + 1.
\]

But for \( x \in M \), \( L(x_{k+1} x) \subseteq L(\{x_{k+1} x_i: 1 \leq i \leq m\}) \), and so

\[
\bigcap_{i=1}^{m} \pi(x_i) \subseteq \pi(x).
\]

Hence

\[
\dim \bigcap \pi(x) = k - m + 1,
\]

the intersection being taken over all \( x \in M \). This completes the proof.

**Proof of Lemma 3.4.** Since \( M \subseteq \psi(k, n) \), the plane spanned by \( x_i \) and \( x_j \) lies in \( \psi(k, n) \), for \( i \neq j, i,j = 1, 2, 3, \ldots, m \). By [1, Vol. 2, Chapter XIV, Theorem 1],

\[
\dim (\pi(x_i) \cap \pi(x_j)) = k - 1.
\]

So, without loss of generality, we may assume that

\[
\pi(x_1) = L(\{e_1, e_2, e_3, \ldots, e_k\}),
\]

and

\[
\pi(x_2) = L(\{e_2, e_3, e_4, \ldots, e_{k+1}\}).
\]

Now assume that there is some \( x_j \), say \( x_3 \), such that

\[
\pi(x_1) \cap \pi(x_2) \subseteq \pi(x_3).
\]

Then we may assume that \( \pi(x_3) = L(\{e_2, e_3, e_4, \ldots, e_k, e_{k+2}\}) \). Now assume that there is some \( x_i \), such that \( \pi(x_i) \) does not contain \( \pi(x_1) \cap \pi(x_2) \). Since

\[
\dim (\pi(x_i) \cap \pi(x_1)) = \dim (\pi(x_i) \cap \pi(x_2)) = k - 1,
\]
we can choose a spanning set \( \{u_1, u_2, u_3, \ldots, u_k\} \) for \( \pi(x_i) \) such that \( u_i \in \pi(x_i) \), for \( i = 1, 2, 3, \ldots, k-1 \), and \( u_k \in \pi(x_2) \). Hence
\[
\pi(x_i) \subset L(\{e_1, e_2, e_3, \ldots, e_k+1\}),
\]
and so
\[
\dim(\pi(x_i) \cap \pi(x_3)) < k-1.
\]
But this contradicts the fact that \( \dim(\pi(x_i) \cap \pi(x_3)) = k-1 \). So
\[
L(\{e_2, e_3, e_4, \ldots, e_k\}) \subset \pi(x_i),
\]
and hence
\[
\dim \bigcap_{i=1}^{m} \pi(x_i) = k-1.
\]
Thus far, we have shown that if any three of the spaces \( \pi(x_1), \pi(x_2), \pi(x_3), \ldots, \pi(x_m) \) intersect in a \( k-1 \) space, then they all intersect in a \( k-1 \) space. Now assume that no three of these spaces intersect in a \( k-1 \) space. Hence, for \( i \neq 1, 2, \pi(x_i) \) does not contain \( \pi(x_1) \cap \pi(x_2) \). So, as before, \( \pi(x_i) \subset L(\{e_1, e_2, e_3, \ldots, e_{k+1}\}) \), and so
\[
\dim L(\{\pi(x_i) : 1 \leq i \leq m\}) = k + 1.
\]

5. Proof of Lemma 3.5. The proof is in two parts.

PART 1. We first prove that, given the assumptions of the lemma, there is a set \( \{x_1, x_2, x_3, \ldots, x_n\} \subset E_n \), such that
\[
(1) \quad \pi(A_p) = L(\{x_{p_1}, x_{p_2}, x_{p_3}, \ldots, x_{p_k}\})
\]
for any \( p \in P(n, k) \). The proof is by induction on the number of vectors which can be found satisfying (1). First note that the assumption that for any \( q \in P(n, k-1) \), the dimension of the intersection of the spaces \( \pi(A_p) \) for \( q \subset p \in P(n, k) \) is \( k-1 \), implies that to each \( q \in P(n, k-1) \) there is assigned a one-to-one manner, a \( k-1 \) space \( S(q) \) of \( E_n \), such that
\[
S(q) = \pi(A_p) \cap \pi(A_r),
\]
for any \( p \in P(n, k) \), and \( r \in P(n, k) \), such that \( p \neq r \), and \( q \subset p \cap r \). Obviously, there is a set \( \{x_1, x_2, x_3, \ldots, x_k\} \subset E_n \) such that if \( p = \{1, 2, 3, \ldots, k\} \), then (1) is true. So, assume that there exists a set \( \{x_1, x_2, x_3, \ldots, x_t\} \subset E_n \), for some integer \( t, k \leq t \leq n-1 \), such that (1) holds for any \( p \in P(t, k) \). Let \( p = \{1, 2, 3, \ldots, k-1, t+1\} \). Then there exists an \( x_{t+1} \in E_n \) such that (1) holds for this \( p \). Let \( q \) be an element of \( P(t, k-1) \), so ordered that if \( q_s \in p \), then \( q_s = s \). Let \( \bar{p} = q \cup \{t+1\} \). We wish to show that (1) holds for \( \bar{p} \). We now define a family of elements of \( P(n, k) \) as follows:
for \( j = 1, 2, 3, \ldots, k-1 \). We will show by induction on \( j \), that (1) holds for each \( p(j) \). This will complete the induction on \( t \), since \( \bar{p} = p(k-1) \). Obviously, (1) is true if \( j = 0 \). Assume that, for some \( j \), \( 0 \leq j < k-1 \), (1) holds for \( p(j) \). If \( q_{j+1} = j+1 \), then (1) holds for \( p(j+1) \). So assume that \( q_{j+1} \neq p \). We also assume that \( q_{j+1} \neq k \). Let

\[
p' = (p(j+1) - \{t+1\}) \cup \{k\},
\]

\[
r = (p(j+1) - \{t+1\}) \cup \{j+1\},
\]

\[
Z(i) = L(\{x_1, x_2, x_3, \ldots, x_i\}),
\]

for \( i \) equal \( t \) or \( t + 1 \). Then

\[
\pi(A_{p(j+1)}) \cap Z(t) = \pi(A_p) \cap \pi(A_{p(j+1)}) = S(p' \cap p(j+1))
\]

\[
= \pi(A_p) \cap \pi(A_j) = L(\{x_{p(j+1)}, x_{p(j+1)+1}, \ldots, x_{p(j+1) + k-1}\}),
\]

and

\[
\pi(A_{p(j)}) \cap \pi(A_{p(j+1)}) = S(p(j) \cap p(j+1)).
\]

Since \( p' \cap p(j+1) \neq p(j) \cap p(j+1) \),

\[
dim(\pi(A_{p(j+1)}) \cap \pi(A_{p(j)}) \cap Z(t)) < k-1.
\]

Also, since \( Z(t+1) \) is spanned by \( \pi(A_{p(j)}) \cup Z(t) \),

\[
dim(\pi(A_{p(j+1)}) \cap Z(t+1)) = k,
\]

and hence

\[
\pi(A_{p(j+1)}) \subset Z(t+1).
\]

Therefore, (1) holds for \( p(j+1) \). If \( q_{j+1} = k \), interchange \( k \) and \( j+1 \) in the argument above. This completes the proof of Part 1.

**Part 2.** As a consequence of Part 1, there is an \( H \in \mathscr{A}_n \) such that \( AH^k \) is diagonal. Hence we can assume that \( A \) is diagonal.

\[
A = \text{diag}(a_p), \quad \text{for } p \in P(n,k).
\]

Now select any two integers \( g \) and \( h \), such that \( 1 \leq g, h \leq n \), and \( g \neq h \). Let \( q \) and \( r \) be two elements of \( P(n,k-1) \), neither of which contains \( g \) or \( h \). Let

\[
p = q \cup \{g\},
\]

\[
p' = q \cup \{h\},
\]

\[
\bar{p} = r \cup \{g\},
\]

and
\[ p' = r \cup \{h\}. \]

We want to show that
\[ a_p a_{\tilde{r}} = a_{p'} a_{\tilde{r}}. \]

As in Part 1, we construct two families of elements of \( P(n, k) \).
\[ p(0) = p, \]
\[ p(j) = (p(j - 1) - \{q_j\}) \cup \{r_j\}, \]
for \( j = 1, 2, 3, \ldots, k - 1 \) and
\[ p'(0) = p', \]
\[ p'(j) = (p'(j - 1) - \{p'_j\}) \cup \{r_j\}, \]
for \( j = 1, 2, 3, \ldots, k - 1 \). Here we regard the \( p(j) \) and \( p'(j) \) as so ordered that \( g \) or \( h \) is always the last element. It suffices to prove that
\[ (2) \ a_{p(j - 1)} a_{p'(j)} = a_{p(j)} a_{p'(j - 1)} \]
for \( j = 1, 2, 3, \ldots, k - 1 \). Let \( y = e_{p(j - 1)} + e_{p(j)} + e_{p'(j)} + e_{p'(j - 1)} \). Then \( y \in \psi(k, n) \). Therefore \( Ay \in \psi(k, n) \). Thus \( Ay \) satisfies the Plucker identities, one of which may be written as (2), since only these four components of \( Ay \) are not zero.

Now let
\[ b(g, h) = a_p / a_{p'}. \]

Then \( b(g, h) \) is independent of \( q \), and for any three integers \( g, h, \) and \( s, \)
\[ 1 \leq g, h, s \leq n, \]
\[ b(g, s) = b(g, h) b(h, s). \]

Therefore, for \( r \in P(n, k) \), and \( r' = \{1, 2, 3, \ldots, k\} \),
\[ a_r = \prod_{i=1}^{k} b(p_i, i) a_{r'}, \]
where \( \prod \) here indicates product. So if \( B \in \mathcal{A}_n \)
\[ B = \text{diag} \( b(1, 1), b(2, 1), b(3, 1), \ldots, b(n, 1) \), \]
and
\[ \lambda = \left( \prod_{i=1}^{k} b(1, i) \right) a_{r'}, \]
then
\[ A = \lambda IB^k. \]

This completes the proof of Lemma 3.5.
6. The orthogonal group. In this section we let $F$ be the field of real numbers. For $m$ a positive integer, let $\cdot$ denote the usual inner product of $E_m$, and $|v|$ the usual norm. For $A \in \mathcal{A}_m$, let $A^{(i)}$ denote the $i$th row vector of the matrix of $A$.

6.1. Lemma. For $m$ and $A$ as above, if there exists a set $T \subseteq E_m$ such that
1. $e_i \in T$ for all integers $i$, $1 \leq i \leq m$,
2. $A^{(i)} \in T$ for all integers $i$, $1 \leq i \leq m$,
3. for all $v \in T$, $Av \in T$, and $A^{-1}v \in T$,
4. for all $v \in T$, $|Av| = |v|$,
then $A$ is orthonormal.

Proof. Since, for $v \in T$, $A^{-1}v \in T$, we have that
\[ |v| = |AA^{-1}(v)| = |A^{-1}(v)|. \]

Now let $x_i = A^{-1}e_i$ for any integer $i$, $1 \leq i \leq m$. Then $|x_i| = 1$, and $Ax_i = e_i$. Hence $A^{(i)}x_i = 1$, and thus $|A^{(i)}| \geq 1$. But
\[ |AA^{(i)}|^2 = \sum_{j=1}^{m} (A^{(j)} \cdot A^{(i)})^2 = A^{(i)} \cdot A^{(i)}. \]

So
\[ \sum_{j=1, j \neq i}^{m} \left( A^{(j)} \cdot A^{(i)} \right)^2 = A^{(i)} \cdot A^{(i)}(1 - A^{(i)} \cdot A^{(i)}). \]

Hence $|A^{(i)}| \leq 1$. Thus, for any integers $i$ and $j$, $1 \leq i, j \leq m$, $i \neq j$, $|A^{(i)}| = 1$, and $A^{(i)} \cdot A^{(j)} = 0$. Hence $A$ is orthonormal.

6.2. Theorem. Let $A \in G$ such that for all $v \in \psi(k, n)$, $|Av| = |v|$. Then $A$ is orthonormal, and there exist $B \in \mathcal{A}_n$, $B$ orthonormal, and $C \in C_n$, $C^2 = I$, such that either $A = CB^k$, or $A = CJB^k$.

Proof. This follows immediately from the previous lemma.

References