EXTENDING A DISK TO A SPHERE(1)

BY

JOSEPH M. MARTIN(2)

This paper deals with the problem of extending a disk in $E^3$ to a 2-sphere in $E^3$. In [5] R. H. Bing points out that there is a disk in $E^3$ which is not a subset of any 2-sphere in $E^3$ and gives an example of such a disk. This example is reproduced as Example 1 of §III, and Theorem 2 of §II is used to give a proof that it does not lie on a 2-sphere in $E^3$. In the same paper Bing proves that "small" subdisks of a disk in $E^3$ lie on 2-spheres in $E^3$.

**Theorem 2.1 of [5].** Suppose that $D$ is a disk in $E^3$ and $p \in \text{Int} D$. Then there is a 2-sphere $S$ in $E^3$ and a disk $D'$ in $D$ such that $p \in \text{Int} D'$ and $D' \subset S$.

In [9], it is shown that if $D$ is a disk in $E^3$ which is locally polyhedral at each point of $\text{Int} D$, then $D$ lies on a 2-sphere in $E^3$.

In this paper, Theorem 1 of §II gives a necessary and sufficient condition on the embedding of a disk in $E^3$ in order that it should lie on a 2-sphere in $E^3$. Theorem 2 of §II shows that in some cases the fact that a certain subdisk of a disk lies on a 2-sphere implies that the disk itself lies on a 2-sphere. This theorem is an extension of a result of R. J. Bean [2].

§III of this paper is devoted to examples. Example 2 of §III shows that a disk need not lie on a 2-sphere even though both (1) every closed proper subset lies on a 2-sphere, and (2) every arc on the disk is tame. Example 3 of §III shows that a disk $D$ need not lie on a 2-sphere even though $\text{Bd} D$ can be shrunk to a point in $E^3 \setminus \text{Int} D$.

I. **Notation and terminology.** If $n$ is a positive integer, $E^n$ will denote Euclidean $n$-space with the usual metric topology.

A *disk* is a space which is topologically equivalent to \{(x_1, x_2); (x_1, x_2) \in E^2 and $x_1^2 + x_2^2 \leq 1$\}. An *annulus* is a space which is topologically equivalent to the product of a circle and a closed interval. If $D$ is a disk, the interior of $D$, denoted by $\text{Int} D$, is the set of all points of $D$ which have neighborhoods homeomorphic with $E^2$, and the *boundary* of $D$, denoted by $\text{Bd} D$, is $D \setminus \text{Int} D$. Similar terminology is used for the interior and boundary of an annulus. The boundary of an annulus $A$ is the union of two disjoint simple

---

Presented to the Society, November 18, 1961 under the title *Disks on spheres*; received by the editors December 6, 1962.

(1) This paper is a major part of the author's doctoral dissertation at the State University of Iowa written under the direction of Professor S. Armentrout.

(2) This paper was written while the author was a Fellow of the National Science Foundation.

385
closed curves $J_1$ and $J_2$, and $J$ is a boundary component of $A$ if and only if either $J = J_1$ or $J = J_2$.

The word "interior" is used in another sense. If $S$ is a sphere with handles in $E^3$, then the interior of $S$, denoted by $\text{Int} S$, is the bounded complementary domain of $S$. This double usage should not cause confusion.

A 2-sphere is a space which is homeomorphic with

$$\{(x_1, x_2, x_3); (x_1, x_2, x_3) \in E^3 \text{ and } x_1^2 + x_2^2 + x_3^2 = 1\}.$$ 

Suppose that $n$ and $k$ are positive integers and $n \leq k$. A set $X$ in $E^k$ is an $n$-manifold if and only if each point of $X$ has a neighborhood in $X$ which is homeomorphic with $E^n$, and a set $X$ in $E^k$ is an $n$-manifold-with-boundary if and only if each point in $X$ has a neighborhood whose closure in $X$ is homeomorphic with the unit cube in $E^n$.

A set $X$ in $E^3$ is a polyhedron or is polyhedral if and only if for some $n$, $n = 0, 1, 2,$ or $3$, $X$ is the union of the elements of a rectilinear $n$-complex. A set $X$ in $E^3$ is locally polyhedral at a point $p$ of $X$ if and only if there exists an open set $N$ containing $p$ such that $N \cap X$ is a polyhedron.

Suppose that $X$ is a closed set in $E^3$ and $X$ is homeomorphic with a polyhedron. $X$ is tame if and only if there is a homeomorphism $h$ of $E^3$ onto itself such that $h(X)$ is a polyhedron, and $X$ is wild if and only if $X$ is not tame. Examples of wild sets may be found in [1], [3], [7], and [8].

A simple closed curve $J$ in a space $X$ can be shrunk to a point in $X$ if and only if there is a continuous map $f$ of a standard 2-simplex $E$ into $X$ such that $f$ carries $\text{Bd} E$ homeomorphically onto $J$.

A space $X$ is simply connected if and only if each simple closed curve in $X$ can be shrunk to a point in $X$.

A triangulated space $D$ is a Dehn disk if and only if there exists a semi-linear map $f$ of a standard 2-simplex $E$ onto $D$ such that there exists an annulus $A$ on $E$ such that (i) $\text{Bd} E \subset A$, (ii) $f$ is a homeomorphism on $A$, and (iii) $f(E - A) \cap f(A) = \emptyset$.

A map $f$ satisfying the above conditions is a defining map for $D$.

If $D$ is a Dehn disk and $f$ is a defining map for $D$, the boundary of $D$ is $f(\text{Bd} E)$.

If $D$ is a Dehn disk and $f$ is a defining map for $D$, then the set of $f$-singularities of $D$ is $\{z; z \in D$ and $f^{-1}(z)$ is nondegenerate$\}$.

If $n$ is a positive integer and $x$ and $y$ are points of $E^n$, the Euclidean distance from $x$ to $y$ is denoted by $d(x, y)$.

For terms not defined in this paper the reader may see [4] or [6].

II. Conditions under which a disk can be extended to a sphere.

**Theorem 1.** A disk $D$ in $E^3$ lies on a 2-sphere in $E^3$ if and only if there exists an annulus $A$ in $E^3$ such that $\text{Bd} D \subset \text{Bd} A$, $D \cap A = \text{Bd} D$, and $\text{Bd} A - \text{Bd} D$ can be shrunk to a point in $E^3 - D$. 
The proof of Theorem 1 given here relies heavily on an approximation theorem proved by Bing and Dehn’s Lemma as proved by Papakyriakopoulos. For completeness these are stated below.

**BING’S APPROXIMATION THEOREM.** If, in a triangulated 3-manifold $S$, $M$ is a 2-manifold-with-boundary and $f$ is a non-negative continuous real-valued function defined on $M$, there is in $S$ a 2-manifold-with-boundary $M'$ and a homeomorphism $h$ of $M$ onto $M'$ such that (1) $d(x,h(x)) \leq f(x)$ and (2) if $f(x) > 0$, $M'$ is locally polyhedral at $h(x)$ [4].

**DEHN’S LEMMA (AS PROVED BY PAPAKYRIAKOPOULOS).** Suppose that $D$ is a Dehn disk in $E^3$, $f$ is a defining map for $D$, and $U$ is a neighborhood of the $f$-singularities of $D$. Then there exists a polyhedral disk $D'$ in $E^3$ such that $\text{Bd}D' = f$-boundary of $D$ and $D' \subset D \cup U$ [10].

**Proof of Theorem 1.** The necessity of the condition is obvious.

Let $D$ be a disk in $E^3$ and $A$ an annulus in $E^3$ such that $D$ and $A$ satisfy the hypothesis of Theorem 1. Let $J_1$ denote the boundary component of $A$ different from $\text{Bd}D$. Since $J_1$ can be shrunk to a point in $E^3 - D$, there exists a continuous map $f_1$ of a standard 2-simplex $E$ into $E^3 - D$ taking the boundary of $E$ homeomorphically onto $J_1$. Let $K$ denote $f_1(E)$. There is no loss in generality in assuming that $K \cap A = J_1$. This is true since $K \cap D = \emptyset$, and if necessary the annulus $A$ can be trimmed back to a smaller annulus having the desired property.

A continuous non-negative real-valued function $g$ is defined on $A$ as follows: if $x \in A$, $g(x) = \min \{\text{dist}(x, D), \text{dist}(x, K)\}$. $g$ is zero on $J_1 \cup \text{Bd}D$ and positive elsewhere. By Bing’s Approximation Theorem there exists an annulus $A'$ such that $\text{Bd}A' = J_1 \cup \text{Bd}D$, $A' \cap D = \text{Bd}D$, $A' \cap K = J_1$, and if $x \in \text{Int}A'$, then $A'$ is locally polyhedral at $x$.

Let $J'$ be a polygonal simple closed curve in $\text{Int}A'$ which is homotopic in $A'$ to $J_1$. Let $B$ denote the annulus on $A'$ whose boundary components are $\text{Bd}D$ and $J'$, $C'$ denote $[(A' \cup K) - B]$, $v_1, v_2, \ldots$, and $v_n$ denote the vertices of $J'$, and $e$ denote $\text{dist}(C', D)$. There exists a continuous map $f_2$ of a standard 2-simplex $E$ onto $C'$ carrying $\text{Bd}E$ onto $J'$ and which is one-to-one on $\text{Bd}E$. Let $r$ be a positive number such that if $d(x, y) < r$, then $d(f_2(x), f_2(y)) < e$.

Let $T$ be a triangulation of $E$ such that $(\text{mesh} T) < r$ and $f_2^{-1}(v_1), f_2^{-1}(v_2), \ldots$, and $f_2^{-1}(v_n)$ are vertices of $T$. Now there exists a piecewise linear map $f_3$ of $E$ into $E^3$ defined as follows: Suppose that $\langle u_1, u_2, u_3 \rangle$ is a 2-simplex in $T$. For $i = 1, 2, \text{or} 3$, let $f_3(u_i)$ be $f_2(u_i)$, and let $f_3$ be extended linearly, taking $u_1, u_2, u_3$ onto the simplex in $E^3$ determined by $f_3(u_1), f_3(u_2), \text{and} f_3(u_3)$. Because of the choice of $e$ and $r$, $f_3(E) \cap D = \emptyset$.

Let $J''$ be a polygonal simple closed curve in $\text{Int}A'$ such that $J''$ is homotopic to $\text{Bd}D$ in $A' - J'$ and if $A_1$ denotes the annulus on $A'$ bounded by $J''$ and $\text{Bd}D$, then $A_1 \cap f_3(E) = \emptyset$. Let $K'$ denote $[(A' - A_1) \cup f_3(E)]$. $K'$ is a Dehn disk and
there exists a defining map \( f_4 \) from a standard 2-simplex \( E_1 \) onto \( K' \) such that \( J'' \) is the \( f_4 \)-boundary of \( K' \). Here it is convenient to think of \( E \) as being contained in the interior of \( E_1 \) and to think of \( f_4 \) as an extension of \( f_3 \).

Let \( U \) be a neighborhood of the \( f_4 \)-singularities of \( K' \) such that \( U \cup (D \cup A_1) = \emptyset \). Since \( K' \) and \( U \) satisfy the hypothesis of Dehn's Lemma, as proved by Papakyriakopoulos, there is a polyhedral disk \( K^* \) such that \( \text{Bd} \; K^* = J'' \) and \( K^* \subseteq K' \cup U \). The last condition implies that \( K^* \cap (D \cup A_1) = J'' \).

Let \( D_1 \) denote \( K^* \cup A_1 \). \( D_1 \) is a disk such that the common part of \( D \) and \( D_1 \) is the boundary of each. This shows that \( D \) lies on a 2-sphere in \( E^3 \).

**Corollary 1.1.** Suppose that \( D \) is a disk in \( E^3 \) such that (1) \( E^3 - D \) is simply connected, and (2) there is an open subset \( U \) of \( D \) such that \( \overline{U} \cap \text{Bd} \; D = \emptyset \) and \( D - U \) lies on a 2-sphere in \( E^3 \). Then \( D \) lies on a 2-sphere in \( E^3 \).

**Proof.** Suppose that \( D \) and \( U \) satisfy the hypothesis of the corollary. Let \( S \) be a 2-sphere in \( E^3 \) such that \( D - U \subseteq S \). Now there exists an annulus \( A \) on \( S - \text{Int} \; D \) such that \( \text{Bd} \; D \subseteq A \) and \( A \cap \overline{U} = \emptyset \). Since \( E^3 - D \) is simply connected, \( \text{Bd} \; A - \text{Bd} \; D \) can be shrunk to a point in \( E^3 - D \). It follows from Theorem 1 that \( D \) lies on a 2-sphere in \( E^3 \).

**Corollary 1.2.** If \( D \) is a disk in \( E^3 \) such that \( \text{Bd} \; D \) can be shrunk to a point in \( E^3 - \text{Int} \; D \) and \( D' \) is a subdisk of \( D \) such that \( D' \subseteq \text{Int} \; D \), then \( D' \) lies on a 2-sphere in \( E^3 \).

**Theorem 2.** Suppose that \( D \) is a disk in \( E^3 \) which lies on a 2-sphere in \( E^3 \). Suppose further that \( A \) is a polyhedral annulus in \( E^3 \) such that \( A \cap D = \text{Bd} \; D \) and \( \text{Bd} \; D \subseteq \text{Bd} \; A \). Then \( D \cup A \) lies on a 2-sphere in \( E^3 \).

Before proceeding with the proof of Theorem 2 some additional definitions and terminology will be introduced.

If \( J \) is a simple closed curve in \( E^3 \), the statement that \( J \) is **unknotted** means that there is a homeomorphism of \( E^3 \) onto itself carrying \( J \) into a plane.

A **torus** is a space which is topologically equivalent to the product of two circles. A **solid torus** is a space which is topologically equivalent to the product of a circle and a disk. A torus \( T \) is **unknotted** if and only if the closure of each complementary domain of \( T \) in \( S^3 \) is a solid torus. If \( T \) is an unknotted torus in \( E^3 \) the solid torus \( T \cup \text{Int} \; T \) will be denoted by \( T^* \).

Let \( T \) be an unknotted polyhedral torus in \( E^3 \). A polygonal simple closed curve \( J \) on \( T \) is **trivial** on \( T \) if and only if \( J \) is null homologous on \( T \). A polygonal simple closed curve \( J \) on \( T \) is **meridional** on \( T \) if and only if \( J \) is nontrivial on \( T \) and \( J \) bounds a disk in \( T^* \). A polygonal simple closed curve \( J \) on \( T \) is **longitudinal** on \( T \) if and only if \( J \) is nontrivial on \( T \) and \( J \) bounds a disk in \( E^3 - \text{Int} \; T \). In [11] it is shown that any two meridional curves are homologous on \( T \) and that
any two longitudinal curves are homologous on $T$. It is also shown that if $J_1$ is meridional (longitudinal) and $J_2$ is homologous to $J_1$ on $T$, then $J_2$ is meridional (longitudinal). A polyhedral disk $D$ such that $\text{Int } D \subset \text{Int } T$ and such that $\text{Bd } D$ is a meridional curve on $T$ is a meridional disk of $T^*$.

Let $T^*$ be separated by two disjoint meridional disks $D_1$ and $D_2$ into two 3-cells $K_1$ and $K_2$. Let $x$ be a point of $\text{Int } D_1$ and $y$ be a point of $\text{Int } D_2$. Let $a_1$ and $a_2$ be unknotted chords of $K_1$ and $K_2$, respectively, joining the points $x$ and $y$. The simple closed curve $a_1 \cup a_2$ is a centerline of $T^*$. In [11] it is shown that if $L_1$ and $L_2$ are centerlines of $T^*$, then there exists a semi-linear mapping of $E^3$ onto itself which is the identity of $E^3 - T^*$ and takes $L_1$ onto $L_2$.

The preceding terminology is that of Schubert in [11], and the first of the following lemmas is an immediate consequence of results in the same paper.

**Lemma 1.** If $T$ is an unknotted polyhedral torus in $E^3$, $D$ is a disk in $E^3$ bounded by a centerline of $T^*$, and $J$ is a nontrivial polygonal simple closed curve on $T$ such that $J \cap D = \emptyset$, then $J$ is longitudinal on $T$.

**Lemma 2.** If $T$ is an unknotted polyhedral torus in $E^3$ and $J_1$ and $J_2$ are polygonal simple closed curves on $T$ such that any nontrivial simple closed curve in $J_1 \cup J_2$ is longitudinal on $T$, then there exists an open annulus $O$ on $T$ such that $J_1$ and $J_2$ are homotopic in $T - O$.

**Proof.** Let $T$, $J_1$, and $J_2$ satisfy the hypothesis. Without loss of generality it is assumed that $J_1 \neq J_2$, $J_1 \cap J_2 \neq \emptyset$, and $J_1 \cap J_2$ contains no arc. Let $A$ be the annulus ring obtained by cutting $T$ along $J_1$ and let $P$ be the identification mapping of $A$ onto $T$.

Let the boundary components of $A$ be denoted by $L_1$ and $L_2$. Let $C$ be a component of $J_2 - J_1$. $P^{-1}(C)$ is an arc in $A$ with both endpoints on $\text{Bd } A$. Suppose that $P^{-1}(C)$ has an endpoint $q_1$ on $L_1$ and an endpoint $q_2$ on $L_2$. Let $q_i$ denote the point on $L_2$ having the same image under $P$ as $q_i$. Let $a$ be an arc in $L_2$ from $q_2$ to $q_1$. Then $P^{-1}(C) \cup a$ is an arc in $A$ from $q_1$ to $q_1'$, and hence $P(P^{-1}(C) \cup a)$ is a simple closed curve on $T$. But $P(P^{-1}(C) \cup a)$ is contained in $J_1 \cup J_2$ and is not longitudinal on $T$. This is contradictory to the hypothesis and it follows that either both endpoints of $P^{-1}(C)$ are on $L_1$ or both endpoints of $P^{-1}(C)$ are on $L_2$.

Let $J_1 \cap J_2$ be $\{p_1, p_2, \ldots, p_n\}$, the components of $J_2 - J_1$ be $C_1, C_2, \ldots$, and $C_n$ and assume that the notation is chosen so that $C_1 \cap C_n = p_1$, and if $i + 1 \leq n$, $C_i \cap C_{i+1} = p_{i+1}$. For each $i$, $i \leq n$, let $D_i$ denote the open disk on $\text{Int } A$ bounded by $P^{-1}(C_i)$. Then $D_i$ is a disk in $A$ which has an arc in common with $\text{Bd } A$, and $P(D_i)$ is a disk in $T$ bounded by the union of an arc in $J_1$ from $p_i$ to $p_{i+1}$ and an arc in $J_2$ from $p_i$ to $p_{i+1}$. Now $\text{Int } A - \bigcup_{i=1}^n D_i$ is an open annulus on $A$. Let $O$ denote $P(\text{Int } A - \bigcup_{i=1}^n D_i)$. $O$ is an open annulus on $T$ such that $O - O \subset J_1 \cup J_2$. For each $i$, $i \leq n$, let $h_i$ be a homotopy in $D_i$ pulling $P^{-1}(C_i)$ into $\text{Bd } A$ and leaving
$P^{-1}(p_i)$ and $P^{-1}(p_{i+1})$ fixed. Then $Ph_i$ is a homotopy in $P(D_i)$ pulling an arc in $J_2$ bounded by $p_i$ and $p_{i+1}$ onto an arc in $J_1$ bounded by $p_i$ and $p_{i+1}$, leaving $p_i$ and $p_{i+1}$ fixed. It follows that there is a homotopy $H$ in $T - O$ pulling $J_1$ onto $J_2$.

**Lemma 3.** Suppose that $T$ is a polyhedral unknotted torus in $E^3$, $D$ is a disk such that $Bd D$ is a centerline of $T^*$, and $J_1$ and $J_2$ are polygonal longitudinal simple closed curves on $T$ such that for $i = 1$ or $2$, $J_i \cap D = \emptyset$. Then there is an open annulus $O$ on $T$ such that $J_1$ and $J_2$ are homotopic in $T - O$ and if $M$ is a meridional polygonal simple closed curve on $T$, then $M \cap O \cap D \neq \emptyset$.

**Proof.** Let $T$, $J_1$, $J_2$, and $D$ satisfy the hypothesis of the lemma. Let $L$ be a polygonal simple closed curve in $D \cap T$ which is a longitude of $T$. Now if $J_1 \cap J_2 = \emptyset$, let $O$ be the component of $T - (J_1 \cup J_2)$ which contains $L$. If $J_1 \cap J_2 \neq \emptyset$, let $O$ be the open annulus promised by Lemma 2. In this case it follows from the proof of Lemma 2 that $L \subset O$. This is because the components of $T - (O \cup J_1 \cup J_2)$ are open disks. Now if $M$ is a meridian of $T$, then $M \cap L \neq \emptyset$, since it follows from Lemma 1 that every meridian of $T$ intersects every longitude of $T$. Hence $M \cap O \cap D \neq \emptyset$.

**Lemma 4.** Suppose that $A$ is a planar polyhedral annulus in $E^3$, $J_1$ and $J_2$ are the boundary components of $A$, and $e > 0$. Then there exists a polyhedral torus $T_e$ such that (1) $J_1$ is a centerline of $T^*_e$, (2) $J_2 \subset E^3 - T^*_e$, (3) $A \cap T_e$ is a longitudinal polygonal simple closed curve on $T$, and (4) if $x \in T_e$ there exists a meridional polygonal simple closed curve $J_x$ on $T_e$ such that $x \in J_x$ and $(\text{diam } J_x) < e$.

**Proof of Theorem 2.** Let $D$ and $A$ satisfy the hypothesis of Theorem 2. Let $S$ be a 2-sphere in $E^3$ such that $D \subset S$. Let $D'$ denote $S - \text{Int} D$. $D'$ is a disk such that $D \cap D' = Bd D$ and $Bd D' = Bd D$. By the Bing Approximation Theorem there is no loss in generality in assuming that $D'$ is locally polyhedral except at points on $Bd D'$. Let the simple closed curve $Bd A - D$ be denoted by $J$. It follows from Lemma 4 and the fact that $A$ is polyhedral that there exists a sequence $T_1, T_2, \ldots$ of polyhedral tori such that for each $i$, (1) $Bd D$ is a centerline of $T_i$, (2) $T_{i+1} \subset \text{Int} T_i$, (3) $J \subset E^3 - T_i^*$, (4) $A \cap T_i$ is a longitudinal simple closed curve on $T$, (5) $T_i$ is unknotted, and (6) if $x \in T_i$ there exists a polygonal meridional simple closed curve $M_x$ on $T_i$ such that $x \in M_x$ and $(\text{diam } M_x) < 1/i$.

Without loss of generality it may be assumed that the vertices of $D'$ are in relative general position with the vertices of each of $A, T_1, T_2, \ldots$. It follows that for each $i$, $D' \cap T$ is a finite collection of mutually disjoint simple closed curves. It follows from Lemma 1 and the fact that $Bd D'$ is a centerline of $T_i$
that each of these simple closed curves is either trivial on $T_i$ or longitudinal on $T_i$. Now $D'$ may be adjusted so that, for each $i$, $D' \cap T_i$ is a single longitudinal simple closed curve on $T_i$, and it is further assumed that $D'$ is so adjusted.

Let $e$ be $\text{dist}(T_2, E^3 - T_i^*)$. Then if $n > 2$ and $x \in T_n$, $\text{dist}(x, E^3 - T_i^*) > e$. Now there exists a positive number $r$ such that if $x$ and $y$ are points of $\text{Int} D$ and $d(x, y) < r$, then there exists an arc $a$ on $\text{Int} D$ from $x$ to $y$ whose diameter is less than $e$.

Let $n$ be a positive integer such that $n > 2$ and $1/n < r$. Let $J_1$ denote $A \cap T_n$ and $J_2$ denote $D' \cap T_n$. Now for each $i$, $i = 1$ or $2$, $J_i \cap D = \emptyset$ and it follows from Lemma 3 that there exists an open annulus $O$ on $T_n$ such that $J_1$ and $J_2$ are homotopic in $T_n - O$ and if $M$ is a polygonal meridional simple closed curve on $T_n$, then $M \cap D \cap O = \emptyset$.

Suppose that $D \cap (T_n - O) \neq \emptyset$. Let $x$ be a point of $D \cap (T_n - O)$ and let $M_x$ be a polygonal meridional simple closed curve on $T_n$ such that $x \in M_x$ and $(diam M_x) < r$. Let $y$ be a point of $M_x \cap D \cap O$. Now there exists an arc $a$ on $\text{Int} D$ from $x$ to $y$ such that $(diam a) < e$, and it follows from the choice of $e$ that $a \subset T_i^*$. Since $a \cap Bd D = \emptyset$, there exists an integer $k$, $k > n$, such that $a \cap T_k^* = \emptyset$. Hence $x$ and $y$ are in the same component of $[\text{Int} T_i^* - (T_k^* \cup A \cup D')]$.

Since $A$ and $D'$ are in relative general position, $J \subset E^3 - T_i^*$, and $D' \cap T_n$ is a single longitudinal simple closed curve on $T_n$, it follows that the intersection of $T_n$ and the component of $[\text{Int} T_i^* - (T_k^* \cup A \cup D')]$ containing $x$ is the component of $T_n - (J_1 \cup J_2)$ containing $x$. Since $x$ and $y$ are in different components of $T_n - (J_1 \cup J_2)$, it follows that $x$ and $y$ are in different components of $[\text{Int} T_i^* - (T_k^* \cup A \cup D')]$. This is a contradiction and hence $D \cap (T_n - O) = \emptyset$.

Since $J_1$ and $J_2$ are homotopic in $T_n - O$, and since $J_2$ is contained in the interior of the disk $D'$, it follows that $J_1$ can be shrunk to a point missing $D$. It follows from Theorem 1 that there exists a disk $D_1$ in $D \cup A$ such that $D \subset \text{Int} D_1$ and $D_1$ lies on a 2-sphere in $E^3$. It follows from [2] that $D \cup A$ lies on a 2-sphere in $E^3$. This establishes Theorem 2.

**Corollary 2.1.** If $D$ is a disk in $E^3$ and $A$ is a tame annulus such that $A \subset D$, $Bd D \subset Bd A$, and $Bd D$ can be shrunk to a point in $E^3 - \text{Int} D$, then $D$ lies on a 2-sphere in $E^3$.

**Proof.** Let $D$ and $A$ satisfy the hypothesis of the corollary. Since $A$ is tame there is a homeomorphism $h$ of $E^3$ onto itself carrying $A$ onto a polyhedral annulus $A'$ and $D$ onto a disk $D'$. Since $Bd D$ can be shrunk to a point in $E^3 - \text{Int} D$, $Bd D'$ can be shrunk to a point in $E^3 - \text{Int} D'$, and it follows from Theorem 1 that there exists a disk $D_1$ such that $D_1 \subset \text{Int} D'$, $Bd D_1$ is polygonal, $Bd D_1 \subset A'$, and $D_1$ lies on a 2-sphere $S$ in $E^3$. It follows from Theorem 2 that $D'$ lies on a 2-sphere $S'$ in $E^3$. Then $h^{-1}(S')$ is a 2-sphere in $E^3$ such that $D \subset h^{-1}(S')$. This establishes Corollary 2.1.
III. Examples.

Example 1. A disk $D$ in $E^3$ which does not lie on a 2-sphere in $E^3$.

Description. The disk $D$ is obtained by taking a horizontal disk $D'$ in $E^3$; removing two circular holes from $\text{Int} D'$; adding tubes from the holes, one going down and the other up and around; and finally adding hooked disks as in the construction of the Alexander Horned Sphere [1]. See Figure 1.

![Figure 1](image_url)

Proposition 1.1. The disk $D$ of Example 1 does not lie on a 2-sphere in $E^3$.

Proof. It follows from repeated application of Theorem 9 of [3] that the simple closed curve $J$ of Figure 1 cannot be shrunk to a point in $E^3 - D$. Let $A$ be a polyhedral annulus such that $D \cap A = \text{Bd} D$ and $\text{Bd} A = J \cup \text{Bd} D$. It follows from Theorem 2 that $D$ lies on a 2-sphere in $E^3$ if and only if $D \cup A$ lies on a 2-sphere in $E^3$. $D \cup A$ does not lie on a 2-sphere since $J$ cannot be shrunk to a point in $E^3 - D$. This establishes Proposition 1.1.

Example 2. A disk $D$ in $E^3$ such that (1) $D$ does not lie on a 2-sphere in $E^3$, (2) if $C$ is a proper closed subset of $D$ then there exists a 2-sphere $S_c$ such that $C \subset S_c$, and (3) every arc in $D$ is tame.
getDescription. Let $D_0$ be a horizontal disk in $E^3$ with a rectangular boundary. $D_0$ is now subdivided into two disks $E_1$ and $E_2$ whose common part is an arc. Each $E_i$ is thickened into a topological cube. The thickened $E_i$ may be regarded as $E_i \times [0,1]$ with $E_i$ identified with $E_i \times \{1/2\}$. From the center of $E_1 \times \{0\}$ and from the center of $E_2 \times \{1\}$ solid feelers with solid loops $H_1$ and $H_2$ are erected in such a way that the loop of $H_1$ goes around the stem of $H_2$ and the loop of $H_2$ goes around the stem of $H_1$. See Figure 2. The thickened $E_i$ plus $H_i$ is topologically a solid torus $T_i$. $T_i$ will be called a cube-with-eye-bolt. The disk $D$ will lie in $T_1 \cup T_2$. The construction which follows is motivated by Example 2 of [3].

For each $i$, $i = 1$ or 2, a slice is removed from the loop of $T_i$ resulting in a topological cube $K_i$. $\text{Bd}E_i$ separates $\text{Bd}K_i$ into two disks one of which has a "hook" in it. The interior of this hooked disk is pushed slightly into $\text{Int}T_i$ to form a disk $A_i$.

Let $W_1$ denote $T_1 \cup T_2$ and let $D_1$ denote $A_1 \cup A_2$. The 3-manifold-with-boundary $W_1$ is a first approximation to $D$ and the disk $D_1$ is also a first approximation to $D$.

Now each $A_i$ is subdivided into fifteen subdisks $E_{i1}, E_{i2}, \ldots, E_{i15}$ such that if $j \leq 14$, $E_{ij}$ and $E_{ij+1}$ share an edge, and $E_{i1}$ and $E_{i15}$ share an edge. The disks $E_{ij}$ are thickened slightly and solid feelers with solid loops $H_{ij}$ are added in such a way that the loop of $H_{ij}$ circles the stem of $H_{ij+1}$, $j \leq 14$, and the loop of $H_{i15}$ circles the stem of $H_{i1}$. The stems of $H_{i7}$ and $H_{i11}$ also intertwine as shown in
Figure 3. Let $T_{ij}$ denote $E_{ij} \cup H_{ij}$. The $T_{ij}$'s are chosen so that $\bigcup_{j=1}^{15} T_{ij} \subset T_i$. In each $T_{ij}$ a disk $A_{ij}$ is placed exactly as $A_i$ was placed in $T_i$ in the previous step. Let $W_2$ denote $\bigcup_{i=1}^{2} (\bigcup_{j=1}^{15} T_{ij})$ and $D_2$ denote $\bigcup_{i=1}^{2} (\bigcup_{j=1}^{15} A_{ij})$. $W_2$ and $D_2$ are second approximations to the disk $D$.

Now for each positive integer $k$, $W_{k+1}$ and $D_{k+1}$ are obtained from $D_k$ in a manner analogous to the way in which $W_2$ and $D_2$ are obtained from $D_1$.

Let $D$ be $\bigcap_{i=1}^{\infty} W_i$; $D$ is also $\lim D_i$. The same argument which is used to show that Example 2 of [3] is a 2-sphere can be used to show that $D$ is a disk.
Proposition 2.1. The simple closed curve $J$ of Figure 2 cannot be shrunk to a point in $E^3 - D$.

Proof. Suppose that $J$ can be shrunk to a point in $E^3 - D$. Then, since $D = \bigcap_{i=1}^{\infty} W_i$, there exists a positive integer $n$ such that $J$ can be shrunk to a point in $E^3 - W_n$. Let $k$ be $\min\{n; J$ can be shrunk to a point in $E^3 - W_n\}$.

For each $i$, let $W'_i$ be the set obtained from $W_i$ by filling each hole in each loop in $W_i$. Let $W''_i$ be the set obtained by removing slices from the loops of $W_i$ and then adding back the intertwining stems of the 7th and 11th feelers at the next stage.

Since $W'_i$ is a cube with handles and $J$ circles one of these handles exactly once, $J$ cannot be shrunk to a point in $E - W'_i$. It follows from Theorem 11 of [3] that $J$ cannot be shrunk to a point in $E^3 - W'_i$ and hence $k > 1$.

Since $J$ cannot be shrunk to a point in $E^3 - W'_k$, it follows from Theorem 9 of [3] that $J$ cannot be shrunk to a point in $E^3 - W''_k$. By considering a slight isotopy of $E^3$ together with the adding of handles it follows that $J$ cannot be shrunk to a point in $E^3 - W''_k$. It follows from Theorem 11 of [3] that $J$ cannot be shrunk to a point in $E^3 - W_k$. This is a contradiction and hence establishes Proposition 2.1.

Proposition 2.2. The disk $D$ of Example 2 does not lie on any 2-sphere in $E^3$.

Proof. Let $A$ be a polyhedral annulus in $E^3$ such that $D \cap A = \text{Bd} D$ and $\text{Bd} A = \text{Bd} D \cup J$. Now, by Theorem 2, $D$ lies on a 2-sphere in $E^3$ if and only if $D \cup A$ lies on a 2-sphere in $E^3$. $D \cup A$ does not lie on a 2-sphere in $E^3$ since, by Proposition 2.1, $J$ cannot be shrunk to a point in $E^3 - D$. This establishes Proposition 2.2.

It will next be shown that if $C$ is a proper closed subset of the disk $D$ of Example 2, then $C$ lies on a 2-sphere in $E^3$. Before proceeding with the proof of this assertion, two definitions and a theorem from [3] will be stated.

Associated annulus. If each of the thickened $E_i$'s is regarded as $E_i \times [0,1]$ with $E_i$ identified with $E_i \times \{1/2\}$, then the annulus associated with the topological torus $T_i$ is $\text{Bd} E_i \times [0,1]$.

Associated cube. Suppose that $C_i$ is a tame cube in the cube-with-eye-bolt $T_i$ such that $\text{Bd} C_i \cap \text{Bd} T_i$ is the annulus associated with $T_i$. Then $C_i$ is a cube associated with $T_i$.

Theorem 4 of [3]. Suppose that $n$ and $m$ are positive integers, $n > m$. Suppose that $T$ is a cube-with-eye-bolt at the $m$th stage in the description of $D$ and $T'$ is a cube-with-eye-bolt at the $n$th stage in the description of $D$ such that $T' \subset T$. Then if $C'$ is a cube associated with $T'$ there exists a cube $C$ associated with $T$ such that $C' \subset C$, and if $T''$ is a cube-with-eye-bolt at the $n$th stage in the description of $D$ and $T'' \neq T'$, then $T'' \subset C$. 


Let $C$ be a proper closed subset of $D$. Consider the following modification of the construction of $D$ which will result in a disk $D(C)$ such that $C \subseteq D(C)$. If $C \cap T_i \neq \emptyset$, let $T(C)_i$ be $T_i$. If $C \cap T_i = \emptyset$, let $T(C)_i$ be a cube associated with $T_i$ which contains $E_i$. Let $W(C)_1$ denote $T(C)_1 \cup T(C)_2$. The 3-manifold-with-boundary $W(C)_1$ is a first approximation to $D(C)$.

Disks $A(C)_1$ and $A(C)_2$ are now selected in $T(C)_1$ and $T(C)_2$ in the following way:

1. If $T(C)_i = T_i$, let $A(C)_i$ be $A_i$.
2. If $T(C)_i$ is a cube associated with $T_i$, let $A(C)_i$ be $E_i$.

Let $D(C)_1$ denote $A(C)_1 \cup A(C)_2$. The disk $D(C)_1$ is also a first approximation to $D(C)$.

Suppose that $i$ is an integer such that $T(C)_i = T_i$. Now if $j$ is an integer such that $C \cap T_{ij} = \emptyset$, let $T(C)_{ij}$ be a cube associated with $T_{ij}$ which contains $E_{ij}$. If $k$ is an integer such that $C \cap T_{ik} \neq \emptyset$, let $T(C)_{ik}$ be $T_{ik}$.

Disks $A_{ij}$ are now selected in $T(C)_{ij}$ in the following way:

1. If $T(C)_{ij} = T_{ij}$, let $A(C)_{ij}$ be $A_{ij}$.
2. If $T(C)_{ij}$ is a cube associated with $T_{ij}$, let $A(C)_{ij}$ be $E_{ij}$.

Suppose that $i$ is an integer such that $T(C)_i$ is a cube associated with $T_i$. In this case $A(C)_i$ is $E_i$, and the disk $A(C)_i$ will be left alone during all successive approximations and will lie in the disk $D(C)$. In order to preserve consistent notation, $A(C)_i$ may be subdivided into fifteen subdisks $E(C)_{i1}$, $E(C)_{i2}$, ..., and $E(C)_{i15}$; these are thickened slightly to form 3-cells $T(C)_{i1}$, $T(C)_{i2}$, ..., and $T(C)_{i15}$. In this case $A(C)_{ij}$ is $E(C)_{ij}$.

Let $W(C)_2$ be $\bigcup_{i=1}^{2} \{ \bigcup_{j=1}^{15} T(C)_{ij} \}$ and let $D(C)_2$ be $\bigcup_{i=1}^{2} \{ \bigcup_{j=1}^{15} A(C)_{ij} \}$. The 3-manifold-with-boundary $W(C)_2$ and the disk $D(C)_2$ are second approximations to the disk $D(C)$.

This process is continued, resulting in a sequence $W(C)_1$, $W(C)_2$, ... of 3-manifolds-with-boundary and a sequence $D(C)_1$, $D(C)_2$, ... of disks. The disk $D(C)$ is $\bigcap_{i=1}^{\infty} W(C)_i$ or $\lim_{i \to \infty} D(C)_i$. The same argument which is used to show that Example 2 of [3] is a 2-sphere can be used to show that $D(C)$ is a disk. Since, for each $i$, $C \subseteq W(C)_i$, it follows that $C \subseteq D(C)$.

**PROPOSITION 2.3.** *For each proper closed subset* $C$ *of* $D$, *the simple closed curve* $J$ *of Figure 2 can be shrunk to a point in* $E^3 - D(C)$.

**Proof.** There is an integer $n$ and a cube-with-eye-bolt $T_n$ at the $n$th stage in the construction of $D$ such that $T_n \cap C = \emptyset$. Let $k = \min \{ n ; \text{there is a cube-with-eye-bolt} \ T_n \text{ at the} \ n \text{th stage such that} \ T_n \cap C = \emptyset \}$. Let $T_k$ be a cube-with-eye-bolt at the $k$ th stage such that $T_k \cap C = \emptyset$. Then $T(C)_k$ is a cube associated with $T_k$. Without loss of generality it may be assumed that $T_k \subseteq T_1$.

It follows from Theorem 4 of [3] that there is a cube $K$ associated with $T_1$ such that $T(C)_k \subseteq K$ and $K$ contains every cube-with-eye-bolt at the $k$th stage.
in the construction of $D$ with the exception of $T_k$. Hence $D(C) \subseteq K \cup T_2$. It follows from the proof of Theorem 3 of [3] that there is a tame cube $W$ in $E^3 - J$ such that $K \cup T_2 \subseteq W$. Since $E^3 - W$ is simply connected, $J$ can be shrunk to a point in $E^3 - W$, and therefore $J$ can be shrunk to a point in $E^3 - D(C)$.

**Proposition 2.4.** If $C$ is a proper closed subset of $D$ then $C$ lies on a 2-sphere in $E^3$.

**Proof.** Let $A$ be a polyhedral annulus such that $\text{Bd } A = \text{Bd } D(C) \cup J$ and $A \cap D = \text{Bd } D(C)$. Such an annulus may be constructed in $(E^3 - W_1) \cup \text{Bd } D(C)$. Since, by Proposition 2.3, $J$ can be shrunk to a point in $E^3 - D(C)$, it follows from Theorem 1 that there exists a 2-sphere $S$ in $E^3$ such that $D(C) \cap S$. Since $C \subseteq D(C)$, $C \subseteq S$, and this establishes Proposition 2.4.

**Proposition 2.5.** Every arc on $D$ is tame.

Proposition 2.5 can be established using either the method in [3] or in [8].

**Example 3.** A disk $D$ such that $\text{Bd } D$ can be shrunk to a point in $E^3 - \text{Int } D$ but $D$ does not lie on a 2-sphere in $E^3$.

**Description.** Let $D_0$ be a horizontal disk in $E^3$. $D$ is obtained from $D_0$ as follows: remove a sequence of circular disks in $\text{Int } D_0$ which converge to a point $p$ of $\text{Bd } D_0$; add back a sequence $H_1, H_2, \ldots$ of disks with one handle in such a way that (i) $H_1$ goes down and around the boundary of $D_0$, (ii) for each $i$, the handle of $H_i$ loops the stem of $H_{i+1}$, and (iii) $H_1, H_2, \ldots$ converges to $p$; finally, for each $i$, a cylinder is removed from the handle of $H_i$ and is replaced by a pair of Alexander hooked disks [1]. See Figures 4 and 5.

![Figure 4](image-url)
Proposition 3.1. The simple closed curve $J$ of Figure 4 cannot be shrunk to a point in $E^3 - D$.

**Proof.** Proposition 3.1 follows from a repeated application of Theorems 9 and 11 of [3].

Proposition 3.2. The disk $D$ of Example 3 does not lie on any 2-sphere in $E^3$.

**Proof.** Let $A$ be a polyhedral annulus such that $A \cap D = \text{Bd} D$ and $\text{Bd} A = \text{Bd} D \cup J$. Since $J$ cannot be shrunk to a point in $E^3 - D$, $D \cup A$ does not lie on a 2-sphere and it follows from Theorem 2 that $D$ does not lie on a 2-sphere.

Proposition 3.3. The simple closed curve $J$ of Figure 4 can be shrunk to a point in $E^3 - \text{Int} D$.

**Proof.** Let $J$ bound a disk $D_1$ which extends over the right end of $H_1$ and which intersects the stem of $H_2$ in two simple closed curves $J_1$ and $J_2$, one above the loop of $H_1$ and one below the loop of $H_1$. Now $D_1$ can be adjusted to obtain a disk $D_2$ which extends over the right end of $H_2$ and which intersects the stem of $H_3$ in four simple closed curves. $D_1$ is adjusted by replacing the two subdisks on $D_1$ bounded by $J_1$ and $J_2$ by disks which extend over the right end of $H_2$. In a similar way $D_2$ may be adjusted to obtain a disk $D_3$ which extends over the right end of $H_3$.

Continuing this process a sequence $D_1, D_2, \ldots$ of disks is obtained. $D_{i+1}$ is obtained from $D_i$ by replacing $2^i$ small subdisks on $D_i$.

Now there is a continuous map $f$ of a standard 2-simplex $s$ onto $\lim D_i$ such that $f$ carries $\text{Bd} s$ homeomorphically onto $J$. Since $(\lim D_i) \cap \text{Int} D = \emptyset$, it follows that $J$ can be shrunk to a point in $E^3 - \text{Int} D$.

Proposition 3.4. $\text{Bd} D$ can be shrunk to a point in $E^3 - \text{Int} D$. 

**Figure 5**
Proof. This follows from Proposition 3.3 and the fact that \( \text{Bd} D \) and \( J \) are homotopic in \( E^3 - \text{Int} D \).

References


5. ———, *Each disk in each 3-manifold is pierced by a tame arc*, Abstract 559–114, Notices Amer. Math. Soc. 6 (1959), 510.


State University of Iowa, Iowa City, Iowa

The Institute for Advanced Study, Princeton, New Jersey