SUBSTITUTION MINIMAL SETS

BY

W. H. GOTTSCHALK

Introduction. We show that every substitution on finitely many symbols, which replaces each symbol by a block of length 2 or more, effectively determines at least one almost periodic point under the shift transformation of symbolic dynamics. The orbit-closures of these almost periodic points, called substitution minimal sets, are analyzed topologically to some extent and in particular it is proved that under certain conditions their structure groups are n-adic groups. Several of the known symbolic minimal sets, such as the Morse minimal set, are definable by the present method of construction. Minimal sets exhibiting new properties also appear. The first two sections contain general theorems on the trace relation of transformation groups and on \( \psi \)-adic transformation groups, some of these results being used to study substitution minimal sets in the third section. A few particular examples of substitution minimal sets are described in the last section. As general references for notions, notation, and terminology occurring here, consult [3; 1].

1. The trace relation.

1.01. Notation. Groups are taken to be multiplicative unless otherwise stated in a particular instance. All topological spaces and topological groups considered are assumed to be Hausdorff. If \( X \) is a set, then \( \Delta(X) \) or \( \Delta_X \) or \( \Delta \) denotes the diagonal of \( X \times X \), that is, the subset \( \{(x, x) \mid x \in X\} \) of \( X \times X \). If \( X \) is a set and if \( R \) is a relation in \( X \), then \( xR \) for \( x \in X \) denotes \( \{y \mid y \in X \& (x, y) \in R\} \), and \( X/R \) denotes \( \{xR \mid x \in X\} \). If \( T \) is a topological group, then \( 3^T \) or \( \mathcal{T} \) or \( \mathcal{J} \) denotes the class of all closed syndetic invariant subgroups of \( T \). If \( (X, T) \) is a transformation group and if \( S \subseteq T \), then \( X/S \) denotes \( \{xS \mid x \in X\} \).

1.02. Definition. Let \( (X, T) \) be a transformation group. The trace relation of \( (X, T) \), denoted \( \Lambda(X, T) \) or \( \Lambda(X) \) or \( \Lambda_X \) or \( \Lambda \), is defined to be the set of all couples \( (x, y) \in X \times X \) such that \( y \in \bigcap_{H \in \mathcal{J}} xH \). If \( x \in X \), then the trace of \( x \) under \( (X, T) \) and the trace of \( (X, T) \) at \( x \) is defined to be the subset \( x\Lambda = \bigcap_{H \in \mathcal{J}} xH \) of \( X \). The transformation group \( (X, T) \) is said to have singleton traces provided \( x \in X \) implies \( x\Lambda = \{x\} \), or equivalently \( \Lambda = \Delta \). The transformation group \( (X, T) \) is said to be totally minimal provided that \( x \in X \) implies \( x\Lambda = X \), or

Received by the editors August 14, 1962.

(1) Partially supported by Research Grant No. NSF-G11287 from the National Science Foundation.

467
1.03. Remark. Let \((X, T)\) and \((Y, T)\) be transformation groups, let \(\phi\) be a homomorphism of \((X, T)\) into \((Y, T)\), and let \(x \in X\). Then:

1. \(x \Lambda \phi \subseteq x \phi \Lambda_{Y}\).
2. If \(X\) is compact and if \(T\) is discrete, then \(x \Lambda_{X} \phi = x \phi \Lambda_{Y}\).

Proof. (1) \(x \Lambda_{X} \phi = \bigcap_{H \in \mathcal{H}} xH \phi \subseteq \bigcap_{H \in \mathcal{H}} xH \phi \subseteq \bigcap_{H \in \mathcal{H}} xH \phi = x \phi \Lambda_{Y}\).

(2) By (1), \(x \Lambda_{X} \phi \subseteq x \phi \Lambda_{Y}\). We show \(x \phi \Lambda_{Y} \subseteq x \Lambda_{X} \phi\). Let \(y \in x \phi \Lambda_{Y} = \bigcap_{H \in \mathcal{H}} xH \phi\). If \(H \in \mathcal{H}\), then \(y \in xH \phi = xH \phi = xH \phi\) and \(y \phi^{-1} \cap xH \neq \emptyset\). Since \(T\) is discrete, \(\{xH \mid H \in \mathcal{H}\}\) is a closed filter base on \(X\) by [3, 5.04]. Hence
\[
y \phi^{-1} \cap \bigcap_{H \in \mathcal{H}} xH \neq \emptyset, \quad y \in \left(\bigcap_{H \in \mathcal{H}} xH\right) \phi = x \Lambda_{X} \phi,
\]
and \(y \in x \Lambda_{X} \phi\).

1.04. Remark. Let \((X, T)\) and \((Y, T)\) be transformation groups such that \((X, T)\) is homomorphic to \((Y, T)\). Then:

1. If \((X, T)\) is minimal, then \((Y, T)\) is minimal.
2. If \((X, T)\) is totally minimal, then \((Y, T)\) is totally minimal.

Proof. (1) By hypothesis there exists a homomorphism \(\phi\) of \((X, T)\) onto \((Y, T)\). Let \(y \in Y\). There exists \(x \in X\) such that \(x \phi = y\). Then \(Y = X \phi = x T \phi \subseteq x T \phi = xT \phi \subseteq y T \subseteq Y\) and \(y T = Y\).

(2) is immediate from (1).

1.05. Definition. Let \((X, T)\) be a transformation group where \(X\) is compact. The structure relation of \((X, T)\), denoted \(\Sigma(X, T)\) or \(\Sigma(X)\) or \(\Sigma_{X}\) or \(\Sigma\), is defined to be the least invariant closed equivalence relation \(R\) in \(X\) such that the partition transformation group \((X/R, T)\) is equicontinuous; such is known to exist \([1, \text{Theorem 1}]\). The star-closed partition \(X/\Sigma = \{x\Sigma \mid x \in X\}\) of \(X\) is called the structure partition of \((X, T)\). The partition transformation group \((X/\Sigma, T)\) is called the structure transformation group of \((X, T)\). When convenient, as in the case of flows, the structure transformation group \((X/\Sigma, T)\) of \((X, T)\) may also be denoted \(\mathcal{F}(X, T)\). The canonical map of \(X\) onto \(X/\Sigma\) is a homomorphism of \((X, T)\) onto \((X/\Sigma, T)\) and is called the structure homomorphism of \((X, T)\). If \(G\) is the transition group of \((X/\Sigma, T)\), if \(\Phi\) is the group of all homeomorphisms of \(X/\Sigma\) onto \(X/\Sigma\), and if \(\Phi\) is provided with its space-index topology, then \(\Phi\) is a topological group and the structure group of \((X, T)\), denoted \(\mathcal{G}(X, T)\) or \(\mathcal{G}_{X}\) or \(\mathcal{G}\), is defined to be the closure \(\overline{G}\) of \(G\) in \(\Phi\). It is known \([3, 4.45]\) that \(\mathcal{G}(X, T)\) is a compact topological group.

1.06. Definition. Let \((X, T)\) be a transformation group where \(X\) is a uniform space. If \(x, y \in X\), then \(x\) and \(y\) are said to be proximal (the first to the second) under \((X, T)\) provided that if \(\alpha\) is an index of \(X\), then there exists \(t \in T\) such that \((xt, yt) \in \alpha\). The proximal relation of \((X, T)\), denoted \(\Pi(X, T)\) or \(\Pi(X)\) or \(\Pi_{X}\) or \(\Pi\)
is defined to be the set of all couples \((x, y) \in X \times X\) such that \(x\) is proximal to \(y\) under \((X, T)\).

1.07. Definition. Let \((X, T)\) be a transformation group and let \(R\) be a relation in \(X\). The ancestral of \(R\) under \((X, T)\), denoted \(R^*\), is defined to be the least invariant closed equivalence relation in \(X\) which contains \(R\).

1.08. Theorem. Let \((X, T)\) be a minimal transformation group where \(X\) is compact. Then:

1. \(\Lambda\) is the least invariant closed equivalence relation \(R\) in \(X\) such that the partition transformation group \((X/R, T)\) has singleton traces.
2. \((X/\Lambda, T)\) is equicontinuous.
3. \(\Lambda \supset \Sigma \supset \Pi^* \supset \Pi \supset \Delta\).
4. If \((X, T)\) is point regularly almost periodic, then \(\Lambda = \Sigma = \Pi^* = \Pi\).
5. If there exists \(x_0 \in X\) such that \(x_0 \Lambda \subset x_0 \Sigma\), then \(\Lambda = \Sigma\).
6. If \((X, T)\) is equicontinuous and if \(T\) is locally compact connected abelian, then \((X, T)\) has singleton traces and \(\Lambda = \Sigma = \Pi^* = \Pi = \Delta\).

Proof. (1) For each \(H \in \mathcal{H}\) define \(R_H\) to be the set of all couples \((x, y) \in X \times X\) such that \(xH = yH\). From [3, 2.32] it follows that for each \(H \in \mathcal{H}\), \(R_H\) is an invariant closed equivalence relation in \(X\). Hence \(\Lambda = \bigcap_{H \in \mathcal{H}} R_H\) is an invariant closed equivalence relation in \(X\).

We show \((X/\Lambda, T)\) has singleton traces. Let \(\phi\) be the canonical map of \(X\) onto \(X/\Lambda\) whence \(x \in X\) implies \(x\phi = x\Lambda\). Now \(\phi\) is a homomorphism of \((X, T)\) onto \((X/\Lambda, T)\). Let \(y_0 \in X/\Lambda\). Assume there exists an element \(y_1\) of the trace of \(y_0\) under \((X/\Lambda, T)\) such that \(y_0 \neq y_1\). Then \(y_1 \in \overline{y_0H}\) for all \(H \in \mathcal{H}\). Since \((X/\Lambda, T)\) is minimal by 1.04 (1), it follows from [3, 2.24] that \(\overline{y_0H} = \overline{y_1H}\) for all \(H \in \mathcal{H}\). Choose \(x_0 \in y_0\phi^{-1}\) and \(x_1 \in y_1\phi^{-1}\). Then \(x_0\phi = y_0\), \(x_1\phi = y_1\); and \(H \in \mathcal{H}\) implies \(x_0\phi = x_0H\phi = x_0\phi H = y_0H = y_1H = x_1H = x_1H\phi = x_1H\phi\) and \(x_0\phi = x_1\phi\). Since \(x_0\phi = x_1\phi\) have different traces \(y_0\) and \(y_1\) under \((X, T)\), there exists \(H \in \mathcal{H}\) such that \(x_0H \cap x_1H = \emptyset\); again by [3, 2.24]. Since \(x_0\phi = x_0H\phi = x_1H\phi\), there exists \(x_2 \in x_1H\) such that \(x_0\phi = x_2\phi\) whence \(x_0H = x_1H\phi\) and \(x_0\Lambda = x_2\Lambda\). Yet \(x_0\Lambda \subset x_0H\) and \(x_2\Lambda \subset x_2H = x_1H\), so that \(x_0H \cap x_1H \neq \emptyset\). This is a contradiction.

Let \(R\) be an invariant closed equivalence relation in \(X\) such that \((X/R, T)\) has singleton traces. We show \(\Lambda \subset R\). Let \(\phi\) be the canonical homomorphism of \((X, T)\) onto \((X/R, T)\) whence \(x \in X\) implies \(x\phi = xR\). If \(x \in X\), then it follows from 1.03 (1) that \((x\Lambda)\phi \subset (x\phi)\Lambda(X/R, T) = \{x\phi\} = \{xR\}\) whence \((x\Lambda)\phi = \{xR\}\). If \((x, y) \in \Lambda\), then \(y \in x\Lambda\), \(y\phi \in (x\Lambda)\phi = \{xR\}\), \(\{y\phi\} = \{xR\}\), \(\{yR\} = \{xR\}\), \(yR = xR\), and \((x, y) \in R\). Hence \(\Lambda \subset R\).

The proof of (1) is completed.

(2) It is enough by (1) to prove the following lemma:

(L) Let \((X, T)\) be a minimal transformation group with singleton traces where \(X\) is compact. Then \((X, T)\) is equicontinuous.
Proof of (L). Let \( \mathcal{U} \) be the compatible uniformity of \( X \). Consider the squared transformation group \( (X \times X, T) \) and define \( Q = \bigcap_{a \in \mathcal{A}} aT \). In order to show that \( (X, T) \) is equicontinuous it is enough to show that \( Q = \Delta_X \). Let \( x, y \in X \) such that \( x \neq y \). It is enough to show \( (x, y) \notin Q \). Since \( xA = \{x\} \neq \{y\} = yA \), then by [3, 2.24] there exists \( H \in \mathcal{H} \) such that \( xH \cap yH = \emptyset \). Choose an open symmetric index \( \beta \) of \( X \) for which \( \beta^3 \cap (xH \times yH) = \emptyset \). Define
\[
U = \bigcup \{zH \mid z \in X \quad \& \quad \overline{zH} \subset \overline{xH} \beta\}
\]
and
\[
V = \bigcup \{zH \mid z \in X \quad \& \quad \overline{zH} \subset \overline{yH} \beta\}.
\]
Since \( \{zH \mid z \in X\} \) is a star-closed partition of \( X \) by [3, 2.32], it follows that \( U \) and \( V \) are open neighborhoods of \( x \) and \( y \). Choose a compact subset \( K \) of \( T \) such that \( T = HK \). There exists an index \( \alpha \) of \( X \) such that \( \alpha K^{-1} \subset \beta \). In order to show that \( (x, y) \notin Q \), it is enough to show there do not exist \( x_t \in U, y_t \in V, \) and \( t \in T \) such that \( (x_t, y_t) \in \alpha \). Let \( x_t \in U, y_t \in V, \) and \( t \in T \). Assume \( (x_t, y_t) \in \alpha \). Now \( t = hk \) for some \( h \in H \) and some \( k \in K \). We have
\[
(x_1 h, y_1 h) = (x_1 h k, y_1 h k) k^{-1} = (x_t, y_t) k^{-1} \in \alpha K^{-1} \subset \beta
\]
and \( (x_1 h, y_1 h) \in \beta \). Since \( x_1 h \in xH \subset xH \beta \) and \( y_1 h \in yH \subset yH \beta \), it follows that \( x_1 h \in x_2 \beta \) for some \( x_2 \in xH \) and \( y_1 h \in y_2 \beta \) for some \( y_2 \in yH \). Hence
\[
(x_2, y_2) = (x_2, x_1 h)(y_1 h, y_2) \in \beta^3, \quad (x_2, y_2) \in \overline{xH \times yH}
\]
and \( (x_2, y_2) \in \beta^3 \cap (\overline{xH \times yH}) \neq \emptyset \). This is a contradiction. The proof of (L) is completed.

The proof of (2) is completed.

(3) This follows immediately from (2). Also see [1].

(4) This is immediate from (3) and [3, 10.05]. The hypothesis of (4) means by definition that some point of \( X \) is regularly almost periodic under \( (X, T) \).

(5) Let \( \phi \) be the canonical homomorphism of \( (X, T) \) onto \( (X/\Sigma, T) \).

We show \( x \in X \) implies \( x\Lambda \phi = \{x\phi\} \). Let \( x \in X \) and let \( \alpha \) be an index of \( X/\Sigma \). It is enough to show \( x\Lambda \phi \times x\Lambda \phi \subset \alpha \). Since \( (X/\Sigma, T) \) is uniformly equicontinuous, there exists an index \( \beta \) of \( X/\Sigma \) such that \( \beta T \subset \alpha \). Choose a neighborhood \( U \) of \( x_0 \phi \) so that \( U \times U \subset \beta \). Since \( U^{-1} \phi \) is a neighborhood of \( x_0 \phi \phi^{-1} = x_0 \Sigma = x_0 \Lambda \) and since \( X/\Lambda \) is star-closed by (1), there exists a neighborhood \( V \) of \( x_0 \phi \phi^{-1} \) such that \( V \Lambda \subset U^{-1} \phi^{-1} \). Choose \( t \in T \) so that \( xt \in V \). Then \( xt \Lambda \subset V \Lambda \subset U^{-1} \phi^{-1} \), \( x\Lambda \phi t = x\Lambda t \phi = xt \Lambda \phi \subset U, x\Lambda \phi \subset U t^{-1}, \) and
\[
x\Lambda \phi \times x\Lambda \phi \subset U t^{-1} \times U t^{-1} = (U \times U) t^{-1} \subset \beta T \subset \alpha.
\]

If \( (x, y) \in \Lambda \), then \( y \in x \Lambda \), \( y \phi \in x \Lambda \phi = \{x \phi\} \), \( y \phi = x \phi \), \( y \Sigma = x \Sigma \), and \( (x, y) \in \Sigma \). Thus \( \Lambda \subset \Sigma \). Since \( \Lambda \supset \Sigma \) by (3), it follows that \( \Lambda = \Sigma \).
Let $x, y \in X$ such that $x \neq y$. By [3, 4.48] there exists an abelian group structure in $X$ which makes $X$ a topological group with $x$ the identity element and which makes $\pi$, a continuous group homomorphism of $T$ into $X$, where $\pi$ is the action of $(X, T)$. Let $C$ be the circle group. Since $X$ is a compact abelian topological group, there exists a continuous group homomorphism $\phi$ of $X$ into $C$ such that $1 = x\phi \neq y\phi$. Since $X\phi$ is a nondegenerate connected subgroup of $C$, it follows that $X\phi = C$. Since $T\pi_x\phi = (xT)\phi$ is a dense connected subgroup of $C$, it follows that $T\pi_x\phi = (xT)\phi = C$. Define $H = 1(\pi_x\phi)^{-1} = 1\phi^{-1}\pi_x^{-1}$. Now $H$ is a closed invariant subgroup of $T$. Since $T$ is locally compact and $T/H$ is isomorphic to the compact group $C$, it follows that $H$ is a syndetic subgroup of $T$. Clearly $xH = H\pi_x \subset 1\phi^{-1}$ whence $xH \subset 1\phi^{-1}$. From $y \notin 1\phi^{-1}$ we conclude that $y \notin xH$ and $(x, y) \notin \Lambda$. The proof of (6) is completed by use of (3).

1.09. Definition. Let $(X, T)$ be a minimal transformation group where $X$ is compact. The star-closed partition $X/A = \{xA | x \in X\}$ of $X$ is called the trace partition of $(X, T)$. The partition transformation group $(X/A, T)$ is called the trace transformation group of $(X, T)$. The canonical map of $X$ onto $X/A$ is a homomorphism of $(X, T)$ onto $(X/A, T)$ and is called the trace homomorphism of $(X, T)$.

1.10. Remark. Let $(X, T)$ be a transformation group where $X$ is compact and $T$ is discrete, and let $x \in X$. Then the following statements are equivalent:

1. $x\Lambda$ is a singleton.
2. $x$ is a regularly almost periodic point under $(X, T)$.

Proof. By [3, 5.04], $H$ is a filter base on $T$ and therefore $\{xH | H \in \mathcal{H}\}$ is a filter base on $X$.

Of course, (2) implies (1) even when $X$ is not assumed compact and $T$ is not assumed discrete. However, (1) does not imply (2) when $T$ is not assumed discrete. For an example, consider a "straight line" minimal continuous flow on the 2-torus. Compare 1.08 (6).

1.11. Remark. Let $(X, T)$ be a transformation group where $X$ is compact and $T$ is discrete. Then the following statements are equivalent:

1. $(X, T)$ has singleton traces.
2. $(X, T)$ is pointwise regularly almost periodic.

Proof. Use 1.10.

The last paragraph of 1.10 applies here word for word.

1.12. Remark. Let $(X, T)$ be an equicontinuous transformation group where $X$ is compact, and let $\mathcal{E}$ be a partition of $X$ into open-closed subsets of $X$. Then there exists $H \in \mathcal{H}$ such that $E \in \mathcal{E}$ implies $EH = E$.

Proof. We assume without loss of generality that $T$ is discrete.

Define $\alpha = \bigcup_{E \in \mathcal{E}} E \times E$. Now $\alpha$ is an index of $X$ such that $E \in \mathcal{E}$ implies $E\alpha = E$. By [3, 4.38] there exists a (left) syndetic subset $A$ of $T$ such that $x \in X$ implies $xA \subset x\alpha$. Then $EA \subset E\alpha = E$ for all $E \in \mathcal{E}$. If $t \in A$, then $\{Et | E \in \mathcal{E}\}$ is a partition of $X$ such that $E \in \mathcal{E}$ implies $Et \subset E$. Therefore, if $t \in A$ and if $E \in \mathcal{E}$,
then $Et = E$ and $Et^{-1} = E$. Thus $E \in \mathcal{E}$ implies $E(A \cup A^{-1}) = E$. Define $H_0$ to be the subgroup of $T$ generated by $A$ whence $H_0$ is a syndetic subgroup of $T$ and $EH_0 = E$ for all $E \in \mathcal{E}$. By [3, 5.04] there exists $H \in \mathcal{H}$ such that $H \subset H_0$. Hence $E \in \mathcal{E}$ implies $EH = E$.

1.13. Remark. Let $(X, T)$ be a transformation group where $X$ is compact zero-dimensional. Then $(X, T)$ is regularly almost periodic if and only if $(X, T)$ is equicontinuous.

**Proof.** Use 1.12 and [3, 4.38].

1.14. Remark. Let $(X, T)$ be an equicontinuous totally minimal transformation group where $X$ is compact. Then $X$ is connected.

**Proof.** Use 1.12.

1.15. Theorem. Let $(X, T)$ be a minimal transformation group where $X$ is compact and $T$ is discrete. Then:

1. $\Lambda$ is the least invariant closed equivalence relation $R$ in $X$ such that the partition transformation group $(X/R, T)$ is pointwise regularly almost periodic.
2. $\Lambda$ is the least invariant closed equivalence relation $R$ in $X$ such that the partition transformation group $(X/R, T)$ is regularly almost periodic.
3. $\Lambda = \Sigma$ if and only if $X/\Sigma$ is zero-dimensional.
4. $\Lambda = X \times X$ if and only if $X/\Sigma$ is connected.
5. If there exists $x_0 \in X$ such that $x_0 \Lambda \subset x_0\Pi^*$, then $\Lambda = \Sigma = \Pi^*$.
6. If $\mathcal{F}$ is a filter base of syndetic invariant subgroups of $T$, and if there exists $x_0 \in X$ such that $x_0 \Lambda \supset \bigcap_{F \in \mathcal{F}} x_0F$, then $x \in X$ implies $x\Lambda = \bigcap_{F \in \mathcal{F}} xF$.

**Proof.** (1) Use 1.08 (1) and 1.11.

(2) Use (1) and [3, 5.18].

(3) Suppose $X/\Sigma$ is zero-dimensional. By 1.13, $(X/\Sigma, T)$ is regularly almost periodic. By (2), $\Lambda \subset \Sigma$. Since $\Lambda \supset \Sigma$ by 1.08 (3), we have that $\Lambda = \Sigma$.

Suppose $\Lambda = \Sigma$. By (2), $(X/\Lambda, T)$ is regularly almost periodic. Since $(X/\Lambda, T)$ is minimal, it follows from [3, 5.08] that $X/\Sigma = X/\Lambda$ is zero-dimensional.

(4) Suppose $X/\Sigma$ is connected. Since $\Sigma \subset \Lambda$ by 1.08 (3), $X/\Lambda$ is a continuous image of $X/\Sigma$ and $X/\Lambda$ is therefore connected. Since $(X/\Lambda, T)$ is minimal, it follows from [3, 2.28] that $(X/\Lambda, T)$ is totally minimal so that $X/\Lambda$ is a trace (indeed, the only trace) of $(X/\Lambda, T)$. By 1.08 (1), $(X/\Lambda, T)$ has singleton traces whence the cardinal of $X/\Lambda$ is 1 and $\Lambda = X \times X$.

Suppose $\Lambda = X \times X$, that is, $(X, T)$ is totally minimal. By 1.04 (2), $(X/\Sigma, T)$ is totally minimal. By 1.14, $X/\Sigma$ is connected.

(5) Let $\phi$ be the canonical homomorphism of $(X, T)$ onto the partition transformation group $(X/\Pi^*, T)$. By [1, Theorem 2], $(X/\Pi^*, T)$ is distal whence $\Pi(X/\Pi^*, T) = \Delta(X/\Pi^*)$. By 1.03 (2), $x_0\Lambda \phi = x_0\phi \Lambda (X/\Pi^*, T)$. Since $x_0\Lambda \subset x_0\Pi^*$ and $x_0\Pi^*\phi$ is a singleton in $X/\Pi^*$, it follows that $x_0\Lambda \phi$ is a singleton in $X/\Pi^*$ and therefore the trace of $x_0\phi$ under $(X/\Pi^*, T)$ is a singleton.
By 1.10, \((X/\Pi^*, T)\) is regularly almost periodic at \(x_0\phi\). By 1.08 (4), 
\(\Sigma(X/\Pi^*, T) = \Pi(X/\Pi^*, T) = \Delta(X/\Pi^*)\) and thus \((X/\Pi^*, T)\) is equicontinuous. 
Hence, using 1.08 (3.5), \(\Lambda = \Sigma \subset \Pi^* \subset \Lambda\) and \(\Lambda = \Sigma = \Pi^*\).

(6) By [3, 2.25], \(H \in \mathcal{H}\) implies \(X/H\) is a finite partition of \(X\). Let \(x \in X\).

Let \(H \in \mathcal{H}\). Now \(\bigcap_{F \in \mathcal{F}} x_0F = x_0H\), \(\{x_0F | F \in \mathcal{F}\}\) is a filter base on \(X\), and \(x_0H\) is an open subset of \(X\). Hence there exists \(F \in \mathcal{F}\) such that \(x_0F \subset x_0H\). 
Since \(x_0F\) is an open subset of \(X\), there exists \(t \in T\) such that \(xt^{-1} \subset x_0F\). Hence \(x \in x_0tF \subset x_0tH\), \(xF = x_0tF\), \(xH = x_0tH\), and \(xF \subset xH\), making use of [3, 2.24].

Therefore, if \(H \in \mathcal{H}\), then there exists \(F \in \mathcal{F}\) such that \(xF \subset xH\). Hence 
\[\bigcap_{H \in \mathcal{H}} xH \supset \bigcap_{F \in \mathcal{F}} xF.\] Since \(\mathcal{H} \supset \mathcal{F}\), it follows that 
\[\bigcap_{H \in \mathcal{H}} xH \supset \bigcap_{F \in \mathcal{F}} xF.\]
Thus \(x\Lambda = \bigcap_{H \in \mathcal{H}} xH = \bigcap_{F \in \mathcal{F}} xF.\)

The proof is completed.

2. \(\psi\)-adic transformation groups.

2.01. Notation. Let \(I\) denote the set of all integers. For \(n \in I\) let \(I_n\) denote \(\{i \mid i \in I \& n \leq i\}\). If \(A\) is a set, then \(\text{card } A\) denotes the cardinal of \(A\).

2.02. Remark and definition. Let \(T\) be a discrete abelian group and let \(\psi\) be a group isomorphism of \(T\) into \(T\) such that \(T\psi\) is syndetic in \(T\) and \(T\psi \neq T\). Then:

(0) \(T \supset T\psi \supset T\psi^2 \supset \cdots\)
(1) For each \(i \in I_1\), \(T\psi^i\) is a syndetic subgroup of \(T\) whence \(T/T\psi^i\) is a finite abelian group.

Let \(K\) be a choice set of the quotient group \(T/T\psi\) whence \(2 \leq \text{card } (T/T\psi)\) = \(\text{card } K < \aleph_0\). Then for each \(i \in I_1\):

(2) \(\prod_{j=0}^{i-1} K\psi^j\) is a choice set of \(T/T\psi^i\).
(3) \(\text{card } (T/T\psi^i) = (\text{card } K)^i\).

For each \(i \in I_1\), let \(T/T\psi^i\) be provided with its discrete topology. Let coordinate-wise multiplication and the product topology be introduced in \(\times_{i \in I_1} T/T\psi^i\). Then:

(4) \(\times_{i \in I_1} T/T\psi^i\) is an abelian multiplicative topological group which is homeomorphic to the Cantor discontinuum.

For each \(i \in I_1\) let \(h_i\) be the canonical group homomorphism of \(T/T\psi^{i+1}\) onto \(T/T\psi^i\) whence \(\tau \in T\) implies \((T\psi^{i+1} \cdot \tau)h_i = T\psi^i \cdot \tau\). Then for each \(i \in I_1\):

(5) If \(\tau \in T\), then 
\[(T\psi^i \cdot \tau)h_i^{-1} = \{T\psi^{i+1} \cdot k\psi^i \cdot \tau \mid k \in K\}.\]

(6) \(h_i\) is exactly \((\text{card } K)\)-to-1.

Let 
\[T/\psi = \text{inv lim}_{i \in I_1} (T/T\psi^i \leftarrow_{h_i} T/T\psi^{i+1}).\]

That is to say, \(T/\psi\) is the set of all sequences \((a_i | i \in I_1) \in \times_{i \in I_1} T/T\psi^i\) such that \(a_i = a_{i+1}h_i\) for all \(i \in I_1\). Then:
(7) $T/\psi$ is the set of all sequences $(T\psi^i \cdot \tau| i \in I_1)$ such that $\tau_i \in T$ for all $i \in I_1$ and $\tau_i^{-1} \in T\psi^i$ for all $i \in I_1$.

(8) $T/\psi$ is the closure in $\times_{i \in I_1} T/T\psi^i$ of the image of $T$ under the canonical homomorphism of $T$ into $\times_{i \in I_1} T/T\psi^i$, which maps each $\tau \in T$ into $(T\psi^i \cdot \tau| i \in I_1)$.

(9) $T/\psi$ is a closed subgroup of $\times_{i \in I_1} T/T\psi^i$.

(10) $T/\psi$ is an abelian multiplicative topological group which is homeomorphic to the Cantor discontinuum.

The topological group $T/\psi$ is called the $\psi$-adic group.

For each $i \in I_1$ let $(T/T\psi^i, T)$ be the coset transformation group on $T/T\psi^i$ under $T$ whence $\tau, \tau \in T$ implies $(T\psi^i \cdot \tau t = T\psi^i \cdot (tt))$. Let $(\times_{i \in I_1} T/T\psi^i, T)$ be the product transformation group of the sequence $((T/T\psi^i, T)| i \in I_1)$ whence $(a_i| i \in I_1) \in \times_{i \in I_1} T/T\psi^i$ and $t \in T$ implies $(a_i| i \in I_1) t = (a_i| i \in I_1)$. Then:

(11) $(\times_{i \in I_1} T/T\psi^i, T)$ is also the homomorphism transformation group on $\times_{i \in I_1} T/T\psi^i$ under $T$ induced by the canonical homomorphism of $T$ into $\times_{i \in I_1} T/T\psi^i$, which maps each $\tau \in T$ into $(T\psi^i \cdot \tau| i \in I_1)$.

(12) $T/\psi$ is the orbit-closure of the identity element of the group $\times_{i \in I_1} T/T\psi^i$ under the regularly almost periodic product transformation group

$$(\times_{i \in I_1} T/T\psi^i, T).$$

(13) The subset transformation group $(T/\psi, T)$ of the regularity almost periodic product transformation group $(\times_{i \in I_1} T/T\psi^i, T)$ is a regularly almost periodic minimal transformation group.

(14) $(T/\psi, T)$ is also the homomorphism transformation group on $T/\psi$ under $T$ induced by the canonical homomorphism of $T$ into $T/\psi$, which maps each $\tau \in T$ into $(T\psi^i \cdot \tau| i \in I_1)$.

(15) For each $i \in I_1$, $h_i$ is a homomorphism of the transformation group $(T/T\psi^i+1, T)$ onto the transformation group $(T/T\psi^i, T)$.

(16) $(T/\psi, T)$ is also the inverse limit transformation group

$$\text{inv lim}_{i \in I_1}((T/T\psi^i, T) \xleftarrow{h_i} (T/T\psi^{i+1}, T)).$$

The regularly almost periodic minimal transformation group $(T/\psi, T)$ is called the $\psi$-adic transformation group.

2.03. DEFINITION. Let $n \in I_1$, let $T$ be the discrete additive group $I$ of all integers and let $\psi$ be the group isomorphism of $T$ into $T$ such that $t \in T$ implies $t\psi = nt$.

Then, the $\psi$-adic group $T/\psi$ is familiarly called the $n$-adic group and is denoted $D_n$; the $\psi$-adic transformation group $(T/\psi, T)$ is called the $n$-adic minimal set and is denoted $(D_n, \tau_n)$ where $\tau_n$ is the transition of $(T/\psi, T)$ corresponding to $1 \in T$.

2.04. NOTATION. Let $(X, T)$ and $(Y, T)$ be transformation groups. The statement that $(X, T)$ is isomorphic to $(Y, T)$ is denoted $(X, T) \simeq (Y, T)$. The statement
that \((X, T)\) is homomorphic to \((Y, T)\), that is, that there exists a homomorphism of \((X, T)\) onto \((Y, T)\), is denoted \((X, T) \cong (Y, T)\).

2.05. Theorem. Let \((X, T)\) be a minimal transformation group where \(X\) is compact and \(T\) is discrete abelian, let \(\psi\) be a group isomorphism of \(T\) into \(T\) such that \(T\psi\) is syndetic in \(T\) and \(T\psi \neq T\), let \(\text{crd}(X, T) = \text{crd}(T/T\psi)\), let \(\phi\) be a homeomorphism of \(X\) into \(X\) such that \((x, t) \in X \times T\) implies \(x\phi = (x\phi)(t\psi)\), and define \(Y = \bigcap_{i=10}^{1} X\phi^i\). Then:

1. \((X, T) \cong (T/\psi, T)\).

2. If \(Y \times Y \subset \Lambda, \text{then } (X, T) \cong (T/\psi, T)\) and \(yt\Lambda = Yt\) for all \(y \in Y\) and all \(t \in T\).

3. If \(Y \times Y \subset \Sigma, \text{then } \Lambda = \Sigma, (X, T) \cong (T/\psi, T), \text{and } \mathcal{B}(X, T) \cong T/\psi\).

4. If \(Y \times Y \subset \Pi^*, \text{then } \Lambda = \Sigma = \Pi^*\).

Proof. Let \(K\) be a choice set of \(T/T\psi\) and for each \(i \in I_1\) let \(K_i = \prod_{j=0}^{i-1} K_j\). We prove:

(i) For each \(i \in I_1, X/T\psi = \{X\phi^i t \mid t \in T\} = \{X\phi^i k \mid k \in K_i\}\) is a partition of \(X\).

Let \(i \in I_1\). By 2.02 (1) and [3, 2.24], \(X/T\psi^i\) is a partition of \(X\). If \(x \in X\), then \(X\phi^i = xT\phi^i = [(xT)\phi^i]^- = [x\phi^i(T\psi^i)]^-\). Hence \(X\phi^i \subset X/T\psi^i\). Clearly then

\[\{X\phi^i k \mid k \in K_i\} \subset \{X\phi^i t \mid t \in T\} \subset X/T\psi^i.\]

By 2.02 (2), \(T = T\psi^i \cdot K_i\). It follows from [3, 1.18 (6)] that if \(x \in X\), then \(X = [x\phi^i T]^- = [x\phi^i(T\psi^i)K_i^-]^- = [x\phi^i(T\psi^i)]^-K_i = xT\phi^i K_i = X\phi^i K_i\).

The proof of (i) is completed.

We prove by induction:

(ii) For each \(i \in I_1, (X\phi^i k \mid k \in K_i)\) is a partition of \(X\).

By hypothesis and (i), \(\text{crd} K = \text{crd}(T/T\psi) = \text{crd}(X/T\psi) = \text{crd} \{X\phi k \mid k \in K\}\) whence \((X\phi k \mid k \in K)\) is a partition of \(X\). Let \(i \in I_1\) and assume \((X\phi^i k_0 \mid k_0 \in K_i)\) is a partition of \(X\). Since \((X\phi k_1 \mid k_1 \in K)\) is a partition of \(X\) and \(\phi^i\) is one-to-one, it follows that \((X\phi k_1 \phi^i \mid k_1 \in K) = (X\phi^{i+1}(k_1 \phi^i) \mid k_1 \in K)\) is a partition of \(X\phi^i\). Thus if \(k_0 \in K_i\), then \((X\phi^{i+1} k_0 \phi^i \mid k_0 \in K_i)\) is a partition of \(X\phi^i k_0\). Since \((X\phi^i k_0 \mid k_0 \in K_i)\) is a partition of \(X\) by the induction assumption it follows that

\[(X\phi^{i+1} k_0 (k_1 \phi^i) \mid (k_0, k_1) \in K_i \times K)\]

is a partition of \(X\) and therefore \((X\phi^{i+1} k \mid k \in K_{i+1})\) is a partition of \(X\). The proof of (ii) is completed.

We prove:

(iii) If \(t_1, t_2 \in T\) and if \(i \in I_1\), then \(X\phi t_1 = X\phi t_2\) if and only if \(T\psi^i \cdot t_1 = T\psi^i \cdot t_2\).

If \(T\psi^i \cdot t_1 = T\psi^i \cdot t_2\) and if \(x \in X\), then

\[X\phi^i t_1 = \overline{xT\phi^i t_1} = [(xT)\phi^i]^- t_1 = [(x\phi^i(T\psi^i)]^- t_1 = [(x\phi^i)(T\psi^i)t_1]^- = [(x\phi^i)(T\psi^i)t_2]^- = [(x\phi^i)(T\psi^i)]^- t_2 = \overline{(xT)\phi^i t_2} = X\phi^i t_2.\]
Now suppose $X_\phi t_1 = X_\phi t_2$. Since $T = T_\psi \cdot K_1$ by 2.02 (2), there exists $s_1, s_2 \in T_\psi$ and $k_1, k_2 \in K_1$ such that $t_1 = s_1 k_1$ and $t_2 = s_2 k_2$. Since $X_\phi t_1 \in X / T_\psi$, it follows that $X_\phi k_1 = X_\phi s_1 k_1 = X_\phi t_1 = X_\phi s_2 k_2 = X_\phi k_2$. By (ii), $k_1 = k_2$ whence $T_\psi t_1 = T_\psi s_1 k_1 = T_\psi k_1 = T_\psi s_2 k_2 = T_\psi t_2$. The proof of (iii) is completed.

It is immediate from (ii) and (i) that:

(iv) If $x \in X$ and if $i \in I_1$, then there exists a unique $f(x, i) \in K_1$ such that $x \in X_\phi f(x, i)$, whence $X f(x, i) = X_\phi f(x, i)$.

It is immediate from 2.02 (2), (iii) and (iv) that:

(v) If $x, y \in X$ and if $i \in I_1$, then the following statements are pairwise equivalent:

$$f(x, i) = f(y, i), T_\psi f(x, i) = T_\psi f(y, i),$$

$$X_\phi f(x, i) = X_\phi f(y, i), \overline{X T_\psi} = \overline{y T_\psi}.$$

We prove:

(vi) There exists a unique map $\xi$ of $X$ into the $\psi$-adic group $T / \psi$ such that $x \in X$ implies $x \xi = (T_\psi \cdot f(x, i) | i \in I_1)$.

Let $x \in X$. It is enough to show $(T_\psi \cdot f(x, i) | i \in I_1) \in T / \psi$. If $i \in I_1$, then $x \in X_\phi f(x, i)$ and $x \in X_\phi f(x, i + 1) \subset X_\phi f(x, i + 1)$ whence $X_\phi f(x, i) = X_\phi f(x, i + 1)$ by (i) and $(f(x, i) f(x, i + 1))^{-1} \in T_\psi$ by (iii). The proof of (vi) is completed.

We prove:

(vii) $\xi$ is a homomorphism of the transformation group $(X, T)$ onto the $\psi$-adic transformation group $(T / \psi, T)$.

The continuity of $\xi$ follows from (v) and the fact that for each $i \in I_1$, $X / T_\psi$ is a finite partition of $X$ into closed subsets of $X$.

We show $\xi$ is onto $T / \psi$. Let $(\tau_i | i \in I_1)$ be a sequence in $T$ such that $(T_\psi \cdot \tau_i | i \in I_1) \in T / \psi$, that is, $\tau_i \tau_{i+1} \in T_\psi$ for all $i \in I_1$. Then it follows from (iii) that for each $i \in I_1$, $X_\phi \tau_i = X_\phi \tau_{i+1} \supset X_\phi \tau_{i+1} \tau_{i+1}$. Hence there exists $x \in \bigcap_{i \in I_1} X_\phi \tau_i$. It follows from (i), (ii) and (iv) that for each $i \in I_1$, $x \in X_\phi f(x, i)$, $X_\phi \tau_i = X_\phi f(x, i)$, and $T_\psi \tau_i = T_\psi f(x, i)$. Thus $x \xi = (T_\psi \cdot f(x, i) | i \in I_1) = (T_\psi \cdot \tau_i | i \in I_1)$.

Let $x \in X$ and let $t \in T$. If $i \in I_1$, then $x \in X_\phi f(x, i)$, $x t = X_\phi f(x, i) t = X_\phi f(x, i) t = X_\phi f(x, i) t = T_\psi f(x, i) t = T_\psi f(x, i) t$ by (iii). Now

$$(x t) \xi = T_\psi f(x, i) t \in I_1)$$

$$= (T_\psi f(x, i) t \in I_1)$$

The proof of (vii) is completed.
We prove:

(viii) If \( x \in X \), then \( xA \subset \bigcap_{i \in I_1} xT\psi^i = x\xi^\xi^{-1} \).

The inclusion in (viii) is obvious from 2.02 (1). If \( y \in X \), then

\[
y \in \bigcap_{i \in I_1} xT\psi^i \iff \forall \; i \in I_1 \cdot y \in xT\psi^i \iff \forall \; i \in I_1 \cdot yT\psi^i = xT\psi^i \text{ by (i)} \iff \forall \; i \in I_1 \cdot yT\psi^i \cdot f(y, i) = T\psi^i \cdot f(x, i) \text{ by (v)} \iff (T\psi^i \cdot f(y, i) \mid i \in I_1) = (T\psi^i \cdot f(x, i) \mid i \in I_1) \iff y\xi = x\xi \iff y \in x\xi^{-1}.
\]

The proof of (viii) is completed.

From (vii) and (viii) it follows immediately that:

(ix) There exists a unique map \( \eta \) of the trace partition \( X/A \) into the \( \psi \)-adic group \( T/\psi \) such that \( x \in X \) implies \( (xA)\eta = x\xi \); \( \eta \) is a homomorphism of the trace transformation group \( (X/A, T) \) onto the \( \psi \)-adic transformation group \( (T/\psi, T) \).

The proof of (1) is completed.

We prove:

(x) If \( y \in Y \), then \( Y = \bigcap_{i \in I_1} yT\psi^i \).

If \( i \in I_1 \), then there exists \( x \in X \) such that \( x\phi^i = y \) whence

\[
(xT)\phi^i = (x\phi^i)(T\psi^i) = yT\psi^i \text{ and } X\phi^i = xT\phi^i = (xT)\phi^i = yT\psi^i.
\]

Hence, \( Y = \bigcap_{i \in I_1} x\phi^i = \bigcap_{i \in I_1} yT\psi^i \). The proof of (x) is completed.

We (2). Suppose \( Y \times Y \subset A \).

Let \( y \in Y \). By (x), \( yA \supset y(Y \times Y) = Y = \bigcap_{i \in I_1} yT\psi^i \supset yA \). Hence \( yA = Y \) and \( yT\Lambda = y\Lambda = Yt \) for all \( t \in T \). By 1.15 (6) and (viii), \( x \in X \) implies \( xA = \bigcap_{i \in I_1} xT\psi^i = x\xi^{-1} \). Hence \( \eta \) is one-to-one and (2) is proved.

We prove (3). Suppose \( Y \times Y \subset \Sigma \). Choose \( y \in Y \). By (x), \( yA \subset Y = y(Y \times Y) \subset y\Sigma \).

By 1.08 (5), \( \Lambda = \Sigma \) and (3) follows from (2).

We prove (4). Suppose \( Y \times Y \subset \Pi^* \). Choose \( y \in Y \). By (x), \( yA \subset Y = (Y \times Y) \subset y\Pi^* \). By 1.15 (5), \( \Lambda = \Sigma = \Pi^* \) and (4) is proved.

The proof of the theorem is completed.


3.01. Notation. For this paragraph let \( S \) be a partially ordered set. For \( a, b \in S \), let

\[
S[a, b] = \{ x \mid x \in S \& a \leq x \leq b \},
\]

let

\[
S[a, \rightarrow] = \{ x \mid x \in S \& a \leq x \},
\]
and let

\[ S[\leftarrow, b] = \{x \mid x \in S \& x \leq b\}. \]

For \( a \in S \) we also denote \( S[a, \rightarrow] \) by \( S_a \).

Let \( I \) denote the set of all integers. An interval of \( I \) is defined to be a nonvacuous subset \( E \) of \( I \) such that \( i, j \in E \) implies \( I[i,j] \subset E \). Evidently, an interval of \( I \) is a finite interval \( I[a,b] \) for unique \( a, b \in I \) with \( a \leq b \), or a right ray \( I[a,\rightarrow] \) for unique \( a \in I \), or a left ray \( I[\leftarrow,b] \) for unique \( b \in I \), or \( I \) itself.

The cardinal of a set \( E \) is denoted \( \text{crd} E \). The domain of a function \( f \) is denoted \( \text{dmn} f \), and the range of \( f \) is denoted \( \text{rng} f \). If \( f \) is a function and if \( x \in \text{dmn} f \), then the value of \( f \) at \( x \) is variously denoted by \( f(x) \) or \( f x \) or \( f_x \) or \( xf \). If \( f \) is a function and if \( E \subset \text{dmn} f \), then \( f|E \) denotes the restriction of \( f \) to \( E \).

3.02. Standing Hypothesis 1. Let \( m \in I \) and let \( P \) be a set such that \( \text{crd} P = m \).

3.03. Definition (Syntax). Since only the cardinal of \( P \) is of significance here, a convenient choice for \( P \) is \( P = I[0, m-1] \). The set \( P \) may be called the symbol set, and an element of \( P \) may be called a symbol.

A string over \( P \) is defined to be a function on an interval of \( I \) to \( P \). Let \( \mathcal{S} \) or \( \mathcal{S}(P) \) denote the set of all strings over \( P \). If \( A \in \mathcal{S} \) and if \( i \in \text{dmn} A \), then the value of \( A \) at \( i \) is denoted \( A_i \) or \( A(i) \). If \( A \in \mathcal{S} \), then the length of \( A \) is defined to be \( \text{crd} A \) or equally \( \text{crd} \text{dmn} A \). If \( k \in I_1 \), then a \( k \)-string over \( P \) is defined to be a string over \( P \) whose length is \( k \). A string \( A \) over \( P \) is said to be finite or infinite according as the length of \( A \) is finite or infinite, that is, according as the set \( A \) is finite or infinite.

If \( k \in I_1 \), if \( p_i \in P \) for each \( i \in I[1,k] \), if \( \kappa \in I[1,k] \), and if \( \omega \in I \), then the \( (\kappa, \omega) \)-indexed juxtaposition (or concatenation) of \( p_1, \ldots, p_k \), denoted \( p_1 \cdots p_k(\kappa^\omega) \cdots p_k \) (2) is defined to be the \( k \)-string

\[
\{((\omega - \kappa + 1, p_1), \ldots, (\omega, p_\kappa), \ldots, (\omega - \kappa + k, p_k))\}
\]

over \( P \); in case \( \omega = 0 \), we may write \( p_1 \cdots p_\kappa \) for \( p_1 \cdots p_k(\kappa^0) \cdots p_k \); in case \( \kappa = 1 \) and \( \omega = 0 \), we may write \( p_1 \cdots p_k \) for \( p_1 \cdots p_k(\kappa^0) \cdots p_k \). If \( A \) is a finite string over \( P \) and if \( \text{dmn} A = I[c,d] \) where \( c, d \in I \) with \( c \leq d \), then we note that \( A = A_c(\kappa c)A_{c+1} \cdots A_d = A_c \cdots A_{d-1} A_d(\kappa d) \).

If \( k \in I_1 \), if \( A^i \) is a finite string over \( P \) for each \( i \in I[1,k] \), if \( \kappa \in I[1,k] \), and if \( \omega \in I \), then the \( (\kappa, \omega) \)-indexed juxtaposition (or concatenation) of \( A^1, \ldots, A^k \), denoted \( A^1 \cdots A^k(\kappa^\omega) \cdots A^k \), is defined to be the \( (\text{crd} A^1 + \cdots + \text{crd} A^k) \)-string \( A^1_{c_1} \cdots A^1_{d_1} \cdots A^k_{c_k} \cdots A^k_{d_k}(\kappa^\omega) \cdots A^k_{c_k} \cdots A^k_{d_k} \) over \( P \); in case \( \omega = 0 \), we may write \( A^1 \cdots A^k \) for \( A^1 \cdots A^k(\kappa^0) \cdots A^k \); in case \( \kappa = 1 \) and \( \omega = 0 \), we may write \( A^1 \cdots A^k \) for \( A^1 \cdots A^k \).

(2) In the notation \( p_k(\kappa^\omega) \), (\kappa^\omega) means \( \omega \) should be written directly above the \( p \).
Extensions of the above notation to certain (perhaps infinite) families of symbols and (perhaps infinite) strings over \( P \) are clear.

For this paragraph let \( A, B, C \in \mathcal{S} \). If \( k \in I \), then the \( k \)-translate of \( A \) or the translate of \( A \) by \( k \), denoted \( k + A \) or \( A + k \), is defined to be the string \( \{(k + i, p) \mid (i, p) \in A\} \) over \( P \), whence \( \text{dmn}(k + A) = k + \text{dmn} A \) and \( i \in \text{dmn} A \) implies \( (k + A)_{k+i} = A_i \). The strings \( A \) and \( B \) are said to be congruent (the first to the second), this statement being denoted \( A \sim B \), provided there exists \( k \in I \) such that \( k + A = B \). Congruence \( \sim \) of strings over \( P \) is clearly an equivalence relation in \( \mathcal{S} \). A substring of \( A \) is defined to be a string over \( P \) which is a subset of \( A \). Thus the following statements are pairwise equivalent: (i) \( A \) is a substring of \( B \); (ii) \( A \subset B \); (iii) \( \text{dmn} A \subset \text{dmn} B \) and \( A = B \mid \text{dmn} A \). The string \( A \) is said to occur in \( B \), this statement being denoted \( A \preceq B \), provided that \( A \) is congruent to a substring of \( B \). In the present context the sign \( \preceq \) is to be read "occurs in." An occurrence of \( A \) in \( B \) is defined to be a substring of \( B \) which is congruent to \( A \). Thus \( A \) occurs in \( B \) if and only if there exists an occurrence of \( A \) in \( B \). If \( a = \text{crd} A < N_0 \), if \( b = \text{crd} B \), and if \( a \preceq b \), then the number of occurrences of \( A \) in \( B \) is at most \( b - a + 1 \). Clearly,

\[
\begin{align*}
(1) & \quad A \preceq A, \\
(2) & \quad A \preceq B \land B \preceq C \Rightarrow A \preceq C, \\
(3) & \quad A \sim B \Rightarrow A \preceq B \land B \preceq A, \\
(4) & \quad \text{crd} A < N_0 \Rightarrow (A \preceq B \land B \preceq A) \Rightarrow A \sim B).
\end{align*}
\]

Thus the relation \( \preceq \) is reflexive transitive (but not antisymmetric) in \( \mathcal{S} \).

A string \( A \) over \( P \) is said to be medial provided that \( 0 \in \text{dmn} A \). Let \( \mathcal{S}_0 \) or \( \mathcal{S}_0(P) \) denote the set of all medial strings over \( P \). A block over \( P \) is defined to be a string \( A \) over \( P \) such that \( \text{dmn} A = I[0, k-1] \) for some \( k \in I_1 \). Thus a block over \( P \) is exactly a finite medial string over \( P \) whose domain is contained in \( I_0 \). If \( k \in I_1 \), then a \( k \)-block over \( P \) is defined to be a \( k \)-string over \( P \) which is also a block over \( P \), that is, a function on \( I[0, k-1] \) to \( P \). For each \( k \in I_1 \) let \( P^k \) denote the set of all \( k \)-blocks over \( P \). Thus if \( k \in I_1 \), then \( P^k \) is the set of all functions on \( k \) to \( P \) where \( k = \text{crd}\{0,1,\ldots,k-1\} \) according to a familiar definitional scheme of the natural numbers. If \( k \in I_1 \) and if \( p_i \in P \) for each \( i \in I[1,k] \), then according to previous notational agreement \( p_1 \cdots p_k \) denotes the \( k \)-block

\[
\hat{p}_1 \cdots p_k = \{(0, p_1), \ldots, (k-1, p_k)\}
\]

over \( P \). We observe that if \( k \in I_1 \) and if \( A \in P^k \), then \( A = A_0 \cdots A_{k-1} \). Let \( P^+ \) denote \( \bigcup_{i \in I_1} P^i \) whence \( P^+ \) is the set of all blocks over \( P \). We note that the relation \( \preceq \) (occurs in) is reflexive antisymmetric transitive in \( P^+ \) and thus, restricted to \( P^+ \), is a partial ordering in \( P^+ \). If \( A \in P^+ \), then a subblock of \( A \) is defined to be a block over \( P \) which occurs in \( A \).

If \( A \in \mathcal{S} \), if \( i,j \in I \) with \( i \leq j \), and if \( I[i,j] \subset \text{dmn} A \), then \( A|I[i,j] \) denotes the substring \( A|I[i,j] \) of \( A \), and \( A[i,j] \) denotes the \((j-i+1)\)-block \( A_i A_{i+1} \cdots A_j \) over \( P \).
A bisequence over $P$ is defined to be a string over $P$ whose domain is $I$, that is a function on $I$ to $P$. Let $X$ or $X(P)$ denote the set of all bisequences over $P$. The set $X$ is called the bisequence set over $P$. In a customary notation $X = P^I$, namely, the set of all functions on $I$ to $P$.

3.04. **Definition.** Let $P^*$ denote $\bigcup_{i \in I^2} P^i$ whence $P^*$ is the set of all blocks over $P$ which are of length 2 or more. A substitution over $P$ is defined to be a map of $P$ into $P^*$. If $\theta$ is a substitution over $P$, then the length of $\theta$ is defined to be the family $(\text{crd } (\theta(p)))_{p \in P}$.

3.05. **Standing hypothesis 2.** Let $m \in I^1$, let $P$ be a set such that $\text{crd } P = m$, let $\theta$ be a substitution over $P$, and let $(n_p | p \in P)$ be the length of $\theta$. (For convenience we repeat all earlier standing hypotheses in each standing hypothesis.)

3.06. **Notation.** Let $X$ or $X(\theta)$ denote the map of $P^2$ into $P^2$ such that $pq \in P^2$ implies $pq \lambda = (\theta(p))_{n_p} (\theta(q))_0$; here $(\theta(p))_{n_p}$ is the last symbol in $\theta(p)$ and $(\theta(q))_0$ is the first symbol in $\theta(q)$. Let $L$ or $L(\theta)$ denote the set of all $pq \in P^2$ such that there exists $i \in I$ such that $pq \lambda^i = pq$.

3.07. **Remark.** The following statements are valid:

1. $L \neq \emptyset$.
2. $L$ coincides with the set of all points of $P^2$ which are periodic under $\lambda$.
3. $\lambda \mid L$ is a one-to-one map of $L$ onto $L$.
4. $\lambda \mid L$ is periodic, that is, there exists $i \in I$ such that $pq \in L$ implies $pq \lambda^i = pq$.
5. $P^2 \supset P^2 \lambda \supset P^2 \lambda^2 \supset P^2 \lambda^3 \supset \cdots, L = \bigcap_{i \in I} P^2 \lambda^i$, and there exists $k \in I$ such that $i \in I$ with $i \geq k$ implies $P^2 \lambda^i = L$.

**Proof.** The above statements all follow easily from the fact that $P^2$ is finite whence $pq \in P^2$ implies the semi-orbit $\{pq \lambda^i | i \in I\}$ of $pq$ under $\lambda$ is finite.

3.08. **Notation.** Let $\mu$ or $\mu(\theta)$ denote the period of $\lambda \mid L$, that is, $\mu$ is the least $i \in I$ such that $pq \in L$ implies $pq \lambda^i = pq$. Let $\nu$ or $\nu(\theta)$ denote the least $i \in I$ such that $pq \in P^2$ implies $pq \lambda^i \in L$.

3.09. **Remark.** If for each $pq \in L$ we let $\mu_{pq}$ denote the least $i \in I$ such that $pq \lambda^i = pq$, then $\mu = \text{lcm } \{\mu_{pq} | pq \in L\}$. Here lcm means "least common multiple." If for each $pq \in P^2$ we let $\nu_{pq}$ denote the least $i \in I$ such that $pq \lambda^i \in L$, then $\nu = \max \{\nu_{pq} | pq \in P^2\}$.

3.10. **Notation.** If $A \in \mathcal{A}_0$, then $A \theta$ denotes the medial string $\cdots (A_{-1} \theta) (A_0 \theta) (A_1 \theta) \cdots$ over $P$.

If $i \in I$, then $\theta^i$ is defined to be the map of $P$ into $P^*$ which is determined recursively as follows:

1. If $p \in P$, then $p \theta^1 = p \theta$.
2. If $p \in P$ and if $j \in I^2$, then $p \theta^j = (p \theta^{j-1}) \theta$.

Thus if $i \in I$, then $\theta^i$ is again a substitution over $P$. For the sake of technical convenience let $\theta^0$ be the identity map of $P$.

If $i \in I$, then $\theta^i$ is a substitution over $P$ and $A \theta^i$ is now defined as a medial string over $P$ for every medial string $A$ over $P$. If $A$ is a medial string over $P$, then $A \theta^0$ denotes $A$ by convention.
3.11. Remark. The following statements are valid:
(1) If $A \in S_0$ and if $i, j \in I_0$, then $(A^i)^j = A^{i+j}$ and $A^i = A^j$.
(2) If $A, B \in S_0$, $A \subseteq B$, and if $i \in I_0$, then $A^i \subseteq B^i$.
(3) If $A, B \in S_0$, if $A \subseteq B$, and if $i \in I_0$, then $A^{i+1} \subseteq B^i$.
(4) If $A \in S_0$, if $i \in I_0$, and if $A \subseteq A^i$, then
$$A \subseteq A^i \subseteq A^2 \subseteq A^3 \subseteq \cdots.$$ 
(5) If $A \in S_0$, if $i \in I_0$, and if $A \subseteq A^i$, then
$$A \subseteq A^i \subseteq A^{2i} \subseteq A^{3i} \subseteq \cdots.$$ 
(6) If $A \in P^+$, then $A^i \in P^* \subseteq P^+$.
(7) If $pq, rs \in P^2$, and if $i \in I_0$, then $pq \lambda^i = rs$ if and only if $pq^i \supseteq r^j$.
(8) If $pq \in P^2$ and if $i \in I_0$, then $pq \lambda^i \subseteq pq \theta^i$.

3.12. Notation. If $pq \in L$, then $pq \lambda^i = pq$ whence $pq \in pq \theta^i \subseteq pq \theta^{i+1} \subseteq pq \theta^{2i} \subseteq \cdots$
and we may define $w_{pq}$ or $w_{pq}(\theta)$ to be $\bigcup_{i \in I_0} pq \theta^i \in X$. Let $L_0$ or $L\theta(\theta)$ denote the
set of all $pq \in L$ such that for each $r \in \operatorname{rng} w_{pq}$ there exists $i \in I_1$ such that $pq \leq r^i$. Let $W$ or $W(\theta)$ denote
$\{w_{pq} \mid pq \in L_0\}$. Let $W_0$ or $W_\theta(\theta)$ denote $\{w_{pq} \mid pq \in L_0\}$.

3.13. Remark. If $pq, rs \in L$, then $w_{pq} = w_{rs}$ if and only if $pq = rs$. Thus
$$\operatorname{crd} W = \operatorname{crd} L \text{ and } \operatorname{crd} W_0 = \operatorname{crd} L_0.$$ 

3.14. Lemma. If $pq \in L_0$ and if $r \in \operatorname{rng} w_{pq}$, then there exists $k \in I_1$ such that
$$pq \leq r^i \text{ and } r \geq r^k.$$ 

Proof. There exists $i \in I_1$ such that $pq \leq r^i$. Since $w_{pq} = \bigcup_{j \in I_1} pq \theta^j$, there
exists $j \in I_1$ such that $r \leq pq \theta^j$. Thus $pq \leq pq \theta^j \leq r \theta^j$ and $r \leq pq \theta^j \leq r \theta^k$.
Defining $k = i + j \in I_1$ it follows that $pq \leq r^k$ and $r \leq r^k$. The proof is completed.

3.15. Lemma. If $pq \in L_0$, then there exists $k \in I_1$ such that $r \in \operatorname{rng} w_{pq}$ and
$$j \in I_1 \text{ implies } pq \leq r^j \text{ and } r \leq r^k.$$ 

Proof. By 3.14 for each $r \in \operatorname{rng} w_{pq}$ there exists $k_r \in I_1$ such that $pq \leq r^{k_r}$
and $r \leq r^{k_r}$. Define $k = \operatorname{lcm} \{k_r \mid r \in \operatorname{rng} w_{pq}\}$. Here \operatorname{lcm} means “least common
multiple.” Then $r \in \operatorname{rng} w_{pq}$ implies $pq \leq r^k$ and $r \leq r^k$. The conclusion follows.

3.16. Remark. If $a \in I_1$, then:
(0) $\theta^a$ is a substitution over $P$.
(1) $\lambda(\theta^a) = \lambda(\theta)^a$.
(2) $L(\theta^a) = L(\theta)$.
(3) $\mu(\theta^a)$ divides $\mu(\theta)$.
(4) If $pq \in L(\theta^a) = L(\theta)$, then $w_{pq}(\theta^a) = w_{pq}(\theta)$.
(5) $L_\theta(\theta^a) = L_\theta(\theta)$.
(6) $W(\theta^a) = W(\theta)$.
(7) $W_\theta(\theta^a) = W_\theta(\theta)$. 

3.17. Remark. If $Q$ is a nonvacuous subset of $P$ such that $Q \theta \subseteq Q^*$, then:

(0) $\theta \mid Q$ is a substitution over $Q$.

(1) $\lambda(\theta \mid Q) = \lambda(\theta \mid Q^2)$.

(2) $L(\theta \mid Q) = L(\theta) \cap Q^2$.

(3) $\mu(\theta \mid Q)$ divides $\mu(\theta)$.

(4) If $pq \in L(\theta \mid Q) = L(\theta) \cap Q^2$, then $w_{pq}(\theta \mid Q) = w_{pq}(\theta)$.

(5) $L_0(\theta \mid Q) = L_0(\theta) \cap Q^2$.

(6) $W(\theta \mid Q) = W(\theta) \cap Q^I$.

(7) $W_0(\theta \mid Q) = W_0(\theta) \cap Q^I$.

3.18. Definition. We shall find the following notions useful:

(1) $\theta$ is said to be reflective provided that $p \in P$ implies $p \leq p\theta$.

(2) $\theta$ is said to be laced (that is, interlaced) provided that $p, q \in P$ implies $p \wedge q \theta = p \wedge q$.

(3) $\theta$ is said to be intersective provided that $\exists \lambda(p \in \text{rng}(p\theta) \neq \emptyset$.

(4) $\theta$ is said to be irreducible provided that if $Q$ is a nonvacuous subset of $P$ such that $Q \theta \subseteq Q^*$, then $Q = P$.

3.19. Lemma. If $A \in P^*$, then there exists $pq \in L$ such that $pq \leq A\theta^*$.

Proof. There exists $rs \in P^2$ such that $rs \leq A$. Let $pq = ds \in P$.

3.20. Remark. If $\theta$ is reflective and if $i, j \in I_0$ with $i \leq j$, then $p \in P$ implies $p \theta^i \leq p\theta^j$.

3.21. Lemma. If $\theta$ is intersective, then there exists $pq \in L$ such that $r \in P$ implies $pq \leq r\theta + 2$.

Proof. There exists $s \in \cap_{r \in P} \text{rng}(r\theta)$. By 3.19 there exists $pq \in L$ such that $pq \leq (s\theta)^*\theta\theta^\omega$. If $r \in P$, then $s \leq r\theta$ and $pq \leq (s\theta)^*\theta^\omega = s\theta + 1 \leq (r\theta)^{\theta + 1} = r\theta^{\theta + 2}$ whence $pq \leq r\theta^{\theta + 2}$. The proof is completed.

3.22. Lemma. If $\theta$ is intersective, then $L_0 \neq \emptyset$.

Proof. Use 3.21.

3.23. Lemma. If $\theta$ is irreducible, then for each $pq \in P^2$ there exists $i \in I_1$ such that $p \leq q\theta^i$.

Proof. Let $pq \in P^2$. Let $Q = \text{rng}(q\theta^i)$. It follows that $\emptyset \neq Q \subseteq P$ and $Q \theta \subseteq Q^*$ whence $Q = P$, $p \in Q$, and $p \leq q\theta^i$ for some $i \in I_1$. The proof is completed.

3.24. Lemma. If $\theta$ is reflective irreducible, then there exists $k \in I_1$ such that $\theta^k$ is laced and therefore intersective.

Proof. By 3.23 for each $pq \in P^2$ there exists $k_{pq} \in I_1$ such that $p \leq q\theta^{k_{pq}}$. Let $k = \max \{k_{pq} \mid pq \in P^2\}$. If $pq \in P^2$, then $q\theta^{k_{pq}} \leq q\theta^k$ by 3.20. Hence $pq \in P^2$ implies $p \leq q\theta^k$. The proof is completed.
3.25. Lemma. If \( \theta \) is reflective irreducible then \( L_0 \neq \emptyset \).

Proof. By 3.24 there exists \( k \in I_1 \) such that \( \theta^k \) is intersective. By 3.22, \( L_0(\theta^k) \neq \emptyset \). Since \( L_0(\theta^k) = L_0(\theta) \) by 3.16 (5), it follows that \( L_0(\theta) \neq \emptyset \). The proof is completed.

3.26. Lemma. There exists a nonvacuous subset \( Q \) of \( P \) such that \( Q\theta \subset Q^* \) and \( \theta\mid Q \) is an irreducible substitution over \( Q \).

Proof. Let \( \mathcal{A} \) be the class of all nonvacuous subsets \( Q \) of \( P \) such that \( Q\theta \subset Q^* \). Now \( \mathcal{A} \neq \emptyset \) since \( P \in \mathcal{A} \). Let \( \mathcal{A} \) be partially ordered by subset inclusion. Since \( P \) is finite, it follows that \( \mathcal{A} \) is finite. Since \( \mathcal{A} \) is finite nonvacuous, it follows there exists a minimal element \( Q_0 \) of \( \mathcal{A} \). The set \( Q_0 \) has the desired properties. The proof is completed.

3.27. Lemma. If \( \theta \) is reflective, then \( L_0 \neq \emptyset \).

Proof. By 3.26 there exists a nonvacuous subset \( Q \) of \( P \) such that \( Q\theta \subset Q^* \) and \( \theta\mid Q \) is an irreducible substitution over \( Q \). Clearly \( \theta\mid Q \) is a reflective substitution over \( Q \). By 3.25, \( L_0(\theta\mid Q) \neq \emptyset \). Since \( L_0(\theta\mid Q) = L_0(\theta) \cap Q^2 \) by 3.17, it follows that \( L_0(\theta) \neq \emptyset \). The proof is completed.

3.28. Lemma. If \( \theta \) is irreducible, then there exists \( k \in I_1 \) such that \( \theta^k \) is reflective.

Proof. By 3.23 for each \( p \in P \) there exists \( k_p \in I_1 \) such that \( p \leq p^{\theta^k} \). Let \( k = \text{lcm}\{k_p \mid p \in P\} \). Here lcm means "least common multiple." By 3.11 (5), \( p \in P \) implies \( p^{\theta^{k_p}} \leq p^{\theta^k} \). Hence \( p \in P \) implies \( p \leq p^{\theta^k} \); and \( \theta^k \) is reflective. The proof is completed.

3.29. Lemma. If \( \theta \) is irreducible, then \( L_0 \neq \emptyset \).

Proof. By 3.28 there exists \( k \in I_1 \) such that \( \theta^k \) is reflective. By 3.27, \( L_0(\theta^k) \neq \emptyset \). Since \( L_0(\theta^k) = L_0(\theta) \) by 3.16 (5), it follows that \( L_0(\theta) \neq \emptyset \). The proof is completed.

3.30. Lemma. \( L_0 \neq \emptyset \).

Proof. By 3.26 there exists a nonvacuous subset \( Q \) of \( P \) such that \( Q\theta \subset Q^* \) and \( \theta\mid Q \) is an irreducible substitution over \( Q \). By 3.29, \( L_0(\theta\mid Q) \neq \emptyset \). Since \( L_0(\theta\mid Q) = L_0(\theta) \cap Q^2 \) by 3.17 (5), it follows that \( L_0(\theta) \neq \emptyset \). The proof is completed.

3.31. Notation. Let \( \phi \) or \( \phi(\theta) \) denote the map of \( X \) into \( X \) such that \( x \in X \) implies \( x_\phi = x_\theta \).

For each \( pq \in P^2 \) let

\[
X_{pq} = \{ x \mid x \in X \land x_{-1} = p \land x_0 = q \} = \{ x \mid x \in X \land x[-1,0] = pq \}.
\]
3.32. Remark. The following statements are valid:

1. \((X_{pq} \mid pq \in P^2)\) is a partition of \(X\).
2. If \(pq \in P^2\) and if \(i \in I_0\), then \(X_{pq} \phi^i = X_{pq2^i}\).
3. If \(pq \in L\), then:
   i. \(X_{pq} \supset X_{pq} \phi^1 \supset X_{pq} \phi^2 \supset \cdots\).
   ii. \(\bigcap_{i \in I_0} X_{pq} \phi^i = \{w_{pq}\}\).
4. If \(pq \in L\) and if \(i \in I_0\), then \(w_{pq} \phi^i = w_{pq2^i}\).
5. \(X = X_{pq} \supset X_{pq2^i} = \cdots\).
6. \(X = X_{pq} \supset X_{pq} \phi^1 \supset X_{pq} \phi^2 \supset \cdots\).
7. \(w = \bigcup_{i \in I_0} X_{pq} \phi^i = \{x \mid x \in X \& x^\phi^i = x\} = \{x \mid x \in X \& \exists i \in I_1 \cdot x^\phi^i = x\}\).

3.33. Notation. Let \(\mathbb{R}\) denote the set of all real numbers. Let \(d\) or \(d(P)\) denote the function on \(X \times X\) to \(\mathbb{R}\) such that:

1. If \(x \in X\), then \(d(x, x) = 0\).
2. If \(x, y \in X\) with \(x \neq y\), then \(d(x, y) = (1 + k)^{-1}\) where \(k\) is the least non-negative integer such that \(x^k \neq y^k\) or \(x^{-k} \neq y^{-k}\). It may be verified that \(d\) is a metric of \(X\) such that the metric topology of \(X\) induced by \(d\) coincides with the product topology of \(X = P^L\) induced by the discrete topology of \(P\). Let \(X\) be provided with its metric \(d\) so that \(X\) is a metric space. The metric space \(X\) is easily seen to be homeomorphic to the Cantor discontinuum if \(m = \text{card} P \geq 2\). Of course, if \(m = 1\), then \(X = \{1\}\). The metric space \(X\) is called the bisequence space over \(P\).

3.34. Remark. If \(pq \in P^2\), then \(X_{pq}\) is a nonvacuous open-closed subset of \(X\).

3.35. Notation. Let \(\sigma\) or \(\sigma(P)\) denote the map of \(X\) into \(X\) such that \(x \in X\) implies \(x\sigma = (x_{i+1} \mid i \in I)\). It is readily proved that \(\sigma\) is a homeomorphism of \(X\) onto \(X\). The map \(\sigma\) is called the shift (transformation) of \(X\). The discrete flow \((X, \sigma)\) is called the symbolic flow over \(P\) or the \(m\)-ary symbolic flow. It is well known that \((X, \sigma)\) is expansive mixing.

3.36. Remark. If \(x \in X\) and if \(i, j, k \in I\) with \(i \leq j\), then
\[
    x^\sigma[i, j] = x[i + k, j + k].
\]

3.37. Definition. If \((E, h)\) is a discrete flow, that is, if \(E\) is a topological space and if \(h\) is a homeomorphism of \(E\) onto \(E\), and if \(x \in E\), then the orbit of \(x\) under \(h\), denoted \(O(x, h)\), is defined to be \(\{xh^i \mid i \in I\}\), and the orbit-closure of \(x\) under \(h\), denoted \(\overline{O}(x, h)\), is defined to be the closure \(\{xh^i \mid i \in I\}^{-}\) of the orbit of \(x\) under \(h\).

3.38. Theorem. The following statements are valid:

1. \(W_0\) coincides with the set of all \(x \in W\) such that \(x\) is almost periodic under \(\sigma\).
2. \(W_0 \neq \emptyset\).
3. If \(w \in W_0\), then \(\overline{O}(w, \sigma)\) is minimal under \(\sigma\).

Prove. (1) Let \(pq \in L_0\). Since \(w_{pq} \phi^i = w_{pq}\), in order to show that \(w_{pq}\) is almost periodic under \(\sigma\) it is enough to show that there exists \(k \in I_1\) such that \(r \in \text{rng} w_{pq}\) implies \(pq \leq r\phi^k\). Now use 3.15.
Let \( pq \in L \) such that \( w_{pq} \) is almost periodic under \( \sigma \). Since \( pq \leq w_{pq} \), it follows readily that for each \( r \in \text{rng} w_{pq} \) there exists \( i \in I_1 \) such that \( pq \leq r \theta^i \). Hence \( pq \in L_0 \) and \( w_{pq} \in W_0 \).

(2) Use 3.30.

(3) Use (1) and [3, 4.07].

3.39. Definition. A substitution minimal set over \( P \) generated by \( \theta \) is defined to be a minimal set \( \tilde{\sigma}(w_{pq}, \sigma) \) under \( \sigma \) where \( pq \in L_0 \).

3.40. Remark. The class of all substitution minimal sets over \( P = I[0, m - 1] \) generated by \( \theta \), for all \( m \) and \( \theta \), is denumerable.

3.41. Remark. The following statements are valid:

(1) If \( x \in X \) and if \( x \) is almost periodic under \( \sigma \), then \( x \phi \) is almost periodic under \( \sigma \).

(2) \( L_0 \lambda = L_0 \) and \( \lambda | L_0 \) is a one-to-one map of \( L_0 \) onto \( L_0 \).

(3) \( W_0 \phi = W_0 \) and \( \phi | W_0 \) is a one-to-one map of \( W_0 \) onto \( W_0 \).

3.42. Remark. Let \( pq, rs \in L \). Then:

(1) If \( p = r \), the \( w_{pq} \) is negatively-asymptotic to \( w_{rs} \).

(2) If \( q = s \), then \( w_{pq} \) is positively-asymptotic to \( w_{rs} \).

3.43. Standing Hypothesis 3. Let \( m \in I_1 \), let \( P \) be a set such that \( \text{crd} P = m \), let \( n \in I_2 \), and let \( \theta \) be a substitution over \( P \) of constant length \( (n | p \in P) \) whence \( \theta : P \to P^n \).

3.44. Remark. The following statements are valid:

(1) If \( x \in X \) and if \( i \in I \), then
\[
x \phi [ni, n(i + 1) - 1] = x_i \theta.
\]

(2) If \( x, y \in X \), if \( i, j \in I \) with \( i \leq j \), and if \( x[i, j] = y[i, j] \), then
\[
x \phi [ni, n(j + 1) - 1] = y \phi [ni, n(j + 1) - 1].
\]

(3) \( \phi \) is a homomorphism of \((X, \sigma)\) into \((X, \sigma^n)\); that is, \( \phi \) is a continuous map of \( X \) into \( X \) such that \( \sigma \phi = \phi \sigma^n \).

(4) If \( i \in I \) and if \( j \in I_0 \), then \( \sigma^i \phi^j = \phi^j \sigma^{inj} \).

(5) If \( pq \in P^2 \), if \( x, y \in X_{pq} \), and if \( i \in I_0 \), then \( x \phi^i [-n^i, n^i - 1] = y \phi^i [-n^i, n^i - 1] \) and \( d(x \phi^i, y \phi^i) \leq (1 + n^i)^{-1} \).

(6) If \( pq \in P^2 \) and if \( i \in I_0 \), then
\[
diam X_{pq} \phi^i < n^{-i}.
\]

(7) If \( pq \in P^2 \), if \( i \in I_0 \), if \( j \in I \), and if \( 0 \leq j < n^i \), then
\[
diam X_{pq} \phi^i \sigma^j < (n^i - j)^{-1}.
\]

3.45. Remark. Let \( pq, rs \in L \). Then:

(1) \( w_{pq} \) is negatively-proximal to \( w_{rs} \) if and only if there exists \( i \in I_0 \) such that \( p \theta^i \cap r \theta^i = \emptyset \). If \( p \theta \cap r \theta \neq \emptyset \), then \( w_{pq} \) is negatively-proximal to \( w_{rs} \).
486 W. H. GOTTSCHALK

(2) \( w_{pq} \) is positively-proximal to \( w_{rs} \) if and only if there exists \( i \in I_0 \) such that \( q^i \cap s^i \neq \emptyset \). If \( q^i \cap s^i = \emptyset \), then \( w_{pq} \) is positively-proximal to \( w_{rs} \).

(3) If \( p \neq r \), and if \( t, u \in P \) with \( t \neq u \) implies \( t \cap u = \emptyset \), then \( w_{pq} \) is negatively-distal from \( w_{rs} \).

(4) If \( q \neq s \), and if \( t, u \in P \) with \( t \neq u \) implies \( t \cap u = \emptyset \), then \( w_{pq} \) is positively-distal from \( w_{rs} \).

3.46. **STANDING HYPOTHESIS 4.** Let \( m \in I_1 \), let \( P \) be a set such that \( \text{card } P = m \), let \( n \in I_2 \), and let \( \theta \) be a one-to-one substitution over \( P \) of constant length \( (n \mid p \in P) \) whence \( \theta: P \to P^n \) is one-to-one.

3.47. **Remark.** Let \( pq, rs \in L \). Then:

(1) \( w_{pq} \) is negatively-asymptotic to \( w_{rs} \) if and only if

\[
p^0 [0, n^n - 2] = r^0 [0, n^n - 2].
\]

(2) \( w_{pq} \) is positively-asymptotic to \( w_{rs} \) if and only if

\[
q^0 [1, n^n - 1] = s^0 [1, n^n - 1].
\]

3.48. **Remark.** The map \( \theta \) is an isomorphism of \((X, \sigma)\) into \((X, \sigma^n)\); that is \( \theta \) is a homeomorphism of \( X \) into \( X \) such that \( \sigma \phi = \phi \sigma^n \).

3.49. **Theorem.** Let \( \Theta(w, \sigma) \) be one and the same subset \( M \) of \( X \) for all \( w \in W_0 \), let \( \Lambda, \Sigma, \Pi, \Pi^* \) refer to \((M, \sigma)\), and let \( W_0 \times W_0 \subset \Pi^* \). Then:

(1) \((M, \sigma)\) is minimal.

(2) \( \Lambda = \Sigma = \Pi^* \).

(3) If \( w \in W_0 \) and if \( i \in I \), then \( \sigma^i \Lambda = W_0 \sigma^i \).

(4) If \( \text{card } M/\sigma^n = n \), then \( \mathcal{F}(M, \sigma) \simeq (D_n, r_n) \) and \( \mathcal{F}(M, \sigma) \simeq D_n \).

(5) If \( M \) is infinite and if \( n \) is prime, then \( \text{card } M/\sigma^n = n \).

(6) If \( M \) is infinite and if \( m = \text{card } P \) is less than or equal to the least prime factor of \( n \), then \( \text{card } M/\sigma^n = n \).

(7) If \( M \) is infinite and if \( m = \text{card } P = 2 \), then \( M \) is infinite and \( \text{card } M/\sigma^n = n \).

(8) If \( \text{card } L_0 \geq 2 \), then \( M \) is infinite and therefore \( M \) is homeomorphic to the Cantor discontinuum.

**Proof.** It is immediate from 3.38 (3) that \((M, \sigma)\) is minimal.

If \( w \in W_0 \), then \( M\phi = \Theta(w, \sigma)\phi = \Theta(w\phi, \sigma^n) \subset \Theta(w, \sigma) \subset M \) by 3.41 (3). Thus \( M \supset M\phi \supset M\phi^2 \supset \ldots \).

By 3.32 (7), \( \bigcap_{i \in I_0} M\phi^i \subset \bigcap_{i \in I_0} X\phi^i = W \). By 3.41 (3) and 3.38 (1), \( W_0 \subset \bigcap_{i \in I_0} M\phi^i \subset M \cap W = W_0 \). Hence

\[
W_0 = \bigcap_{i \in I_0} M\phi^i.
\]

Statements (2), (3) and (4) are now immediate from 1.15 and 2.05.

We prove (5). Suppose \( n \) is prime and \( \text{card } M/\sigma^n \neq n \). Let \( w \in W_0 \). Let \( a \) be the
least positive integer such that $\bar{O}(w^a, \sigma^n) = \bar{O}(w, \sigma^n)$. Since $1 \leq a < n$ and $a$ divides $n$, it follows that $a = 1$ and

$$\bar{O}(w, \sigma^n) = \bar{O}(w, \sigma^n) = \bar{O}(w, \sigma^n) = \cdots = \bar{O}(w, \sigma^n) = \bar{O}(w, \sigma^n)$$

whence

$$M = M\phi = M\phi^2 = \cdots, W_0 = \bigcap_{i \in I_0} M\phi^i = M,$$

and $M$ is finite. This proves (5).

We prove (6). Suppose $m = \text{crd} P$ is less than or equal to the least prime factor of $n$ and $\text{crd} M/\sigma^n \neq n$. Let $x \in M$. Let $a$ be the least positive integer such that $\bar{O}(x, \sigma^n, \sigma^n) = \bar{O}(x, \sigma^n, \sigma^n)$. Now $1 \leq a < n$ and $a$ divides $n$. Define $b = n/a$ whence $b \in I$ and $2 \leq b < n$. We have $M\phi = M\phi^a = M\phi^2a = \cdots$. Hence $x\phi$ can be partitioned into $a$-strings over $P$ to yield a new bisequence whose $b$-blocks are at most $m$ in number. Since $m \leq b$, it follows from [5, Theorem 7.4] that $x\phi$ is periodic, $x$ is periodic, and $M$ is finite. This proves (6).

Statement (7) is immediate from (6).

If $M$ is finite, then $x$ is periodic on $M$, $\Pi = \Delta_M$, $\Pi^* = \Delta_M$, $W_0 \times W_0 \subset \Delta_M$, $\text{crd} W_0 = 1$, and $\text{crd} L_0 = 1$. This proves (8).

The writer has a strong suspicion that this theorem (3.49) can be considerably improved.

4. Examples.

4.01. Notation. We adopt the notation of §3 here in so far as applicable. If $a \in I_1$, then $p^a$ denotes the constant block $pp\cdots p$ of length $a$.

4.02. Example 1. Let $P = \{0, 1\}$ and let $\theta: 0 \to 00, 1 \to 11$. Then:

(1) The hypotheses of 3.46 with $m = 2$ and $n = 2$ are satisfied.
(2) $\lambda$ is the identity map of $P^2$.
(3) $L = P^2$, $L_0 = \{00, 11\}$, $\mu = 1$,

$$W = \{w_{pq} \mid pq \in P^2\}, W_0 = \{w_{00}, w_{11}\}.$$

(4) The hypotheses of 3.49 are not satisfied since

$$\bar{O}(w_{00}, \sigma) = \{w_{00}\} \neq \{w_{11}\} = \bar{O}(w_{11}, \sigma).$$

(5) $w_{00}$ and $w_{11}$ are each fixed under $\sigma$; $w_{01} \downarrow w_{00}; w_{01} \uparrow w_{11}; w_{10} \downarrow w_{11}; w_{10} \uparrow w_{00}$. Here $\downarrow$ means "negatively-asymptotic to" and $\uparrow$ means "positively-asymptotic to."

4.03. Example 2. Let $P = \{0, 1\}$ and let $\theta: 0 \to 01, 1 \to 01$. Then:

(1) The hypotheses of 3.46 with $m = 2$ and $n = 2$ are satisfied.
(2) $\lambda: 00 \to 10, 01 \to 11, 10 \to 00, 11 \to 01$.
(3) $L = L_0 = P^2$, $\mu = 2$, $W = W_0 = \{w_{pq} \mid pq \in P^2\}$.
(4) The hypotheses of 3.49 are satisfied and $\text{crd} M/\sigma^2 = 2$. 
4.04. Example 3. Let $P = \{0, 1\}$ and let $\theta: 0 \rightarrow 01, 1 \rightarrow 00$. Then:

1. The hypotheses of 3.46 with $m = 2$ and $n = 2$ are satisfied.
2. $\lambda: 00 \rightarrow 10, 01 \rightarrow 10, 10 \rightarrow 00, 11 \rightarrow 00$.
3. $L = L_0 = \{00, 10\}, \mu = 2, W = W_0 = \{w_{00}, w_{10}\}$.
4. The hypotheses of 3.49 are satisfied and $\text{crd } M/\sigma^2 = 2$.
5. $\mathcal{S}(M, \sigma) \simeq (D_4, \tau_2)$.
6. $\mathcal{H}(M, \sigma) \simeq D_2$.
7. The minimal set $(M, \sigma)$ was constructed in a different way in [3, 12.52]. Actually, the right-hand sequence in $w_{00}$ and $w_{10}$ may be constructed recursively in many ways. For example, write down a column with 0 and 1 alternating, and at any level both prefix and suffix the previous block. Again, construct the sequence $010201030102010\cdots$ by suffixing the next integer and repeating the previous block, and then reduce modulo 2.

4.05. Example 4. Let $P = \{0, 1\}$ and let $\theta: 0 \rightarrow 0010, 1 \rightarrow 1011$. Then:

1. The hypotheses of 3.46 with $m = 2$ and $n = 4$ are satisfied.
2. $\lambda$ is the identity map of $P^2$.
3. $L = L_0 = P^2, \mu = 1, W = W_0 = \{w_{pq} \mid pq \in P^2\}$.
4. The hypotheses of 3.49 are satisfied and $\text{crd } M/\sigma^4 = 4$.
5. $\mathcal{S}(M, \sigma) \simeq (D_4, \tau_4) \simeq (D_2, \tau_2)$.
6. $\mathcal{H}(M, \sigma) \simeq D_4 \simeq D_2$.
7. The points $w_{01}$ and $w_{10}$ of $M$ are bilaterally proximal under $\sigma$ but not unilaterally asymptotic under $\sigma$.
8. If $\rho: X \rightarrow X$ is the map such that $x \in X$ implies $x\rho = (x^{\omega})_i \mid i \in I$, then:
   (i) $\rho$ is a reversor of $(X, \sigma)$, that is, $\rho$ is a homeomorphism of $X$ onto $X$ such that $x \in X$ implies $x\rho = x\sigma^{-1}$;
   (ii) $M\rho = M$ whence $\rho | M$ is a reversor of $(M, \sigma)$ and $(M, \sigma)$ is reversible.
9. No bisecuence in $M$ is symmetric. A bisecuence $x \in X$ is said to be symmetric provided there exists $a \in I$ such that $i \in I$ implies $x(i) = x(-i + a)$. If a bisecuence $x \in X(P)$ is symmetric for any $P$, then its orbit-closure $O(x, \sigma)$ is reversible. Examples 2, 3, 8 contain symmetric bisecuences and are therefore reversible.

We prove directly that $\text{crd } M/\sigma^4 = 4$. This will illustrate a certain technique. For $pq \in P^2$ define $M_{pq} = M \cap X_{pq}$. Then:

$$(-1)M = M_{00} \cup M_{01} \cup M_{10} \cup M_{11},$$
$$M_{00} \neq \emptyset, M_{01} \neq \emptyset, M_{10} \neq \emptyset, M_{11} \neq \emptyset$$

$(M_{00}, M_{01}, M_{10}, M_{11})$ is a partition of $M$.

0. $M_{00} \phi \subset M_{00}, M_{01} \phi \subset M_{01}, M_{10} \phi \subset M_{10}, M_{11} \phi \subset M_{11}$
1963] SUBSTITUTION MINIMAL SETS 489

\[ M\phi \subset M_{00} \cup M_{01} \cup M_{10} \cup M_{11} \]

\[ M\phi \cap M_{00} \neq \emptyset, \ M\phi \cap M_{01} \neq \emptyset, \ M\phi \cap M_{10} \neq \emptyset, \ M\phi \cap M_{11} \neq \emptyset \]

(1) \( M_{00}\phi \sigma \subset M_{00}, \ M_{01}\phi \sigma \subset M_{10}, \ M_{10}\phi \sigma \subset M_{00}, \ M_{11}\phi \sigma \subset M_{10} \)

\[ M\phi \sigma \subset M_{00} \cup M_{10} \]

\[ M\phi \sigma \cap M_{00} \neq \emptyset, \ M\phi \sigma \cap M_{10} \neq \emptyset \]

(2) \( M_{00}\phi \sigma^2 \subset M_{01}, \ M_{01}\phi \sigma^2 \subset M_{01}, \ M_{10}\phi \sigma^2 \subset M_{01}, \ M_{11}\phi \sigma^2 \subset M_{01} \)

\[ M\phi \sigma^2 \subset M_{01} \]

\[ M\phi \sigma^2 \cap M_{01} \neq \emptyset \]

(3) \( M_{00}\phi \sigma^3 \subset M_{10}, \ M_{01}\phi \sigma^3 \subset M_{11}, \ M_{10}\phi \sigma^3 \subset M_{10}, \ M_{11}\phi \sigma^3 \subset M_{11} \)

\[ M\phi \sigma^3 \subset M_{10} \cup M_{11} \]

\[ M\phi \sigma^3 \cap M_{10} \neq \emptyset, \ M\phi \sigma^3 \cap M_{11} \neq \emptyset \]

Therefore \( M\phi, M\phi \sigma, M\phi \sigma^2, M\phi \sigma^3 \) are pairwise different and consequently disjoint. Since \( M/\sigma^4 = \{M\phi, M\phi \sigma, M\phi \sigma^2, M\phi \sigma^3\} \), we conclude that \( \text{crd } M/\sigma^4 = 4 \). Actually it is enough to recognize that \( M\phi \neq M\phi \sigma \) and \( M\phi \neq M\phi \sigma^2 \) since 1 and 2 are the only positive divisors of 4 which are less than 4.

4.06. Example 5. Let \( P = \{0,1\} \) and let \( \theta: 0 \to 0010, 1 \to 1101 \). Then:

(1) The hypotheses of 3.46 with \( m = 2 \) and \( n = 4 \) are satisfied.
(2) \( \lambda \) is the identity map of \( P^2 \).
(3) \( L = L_0 = P^2, \mu = 1, W = W_0 = \{w_{pq} \mid pq \in P^2\} \).
(4) The hypotheses of 3.49 are satisfied and \( \text{crd } M/\sigma^4 = 4 \).
(5) \( \mathcal{S}(M, \sigma) \simeq (D_4, \tau_4) \simeq (D_2, \tau_2) \).
(6) \( \mathcal{S}(M, \sigma) \simeq D_4 \simeq D_2 \).
(7) The minimal set \((M, \sigma)\) is irreversible [2].

4.07. Example 6. Let \( P = \{0,1\} \), let \( a \in I_1 \), and let \( \theta: 0 \to 00^a1, 1 \to 01^a1 \). Then:

(1) The hypotheses of 3.46 with \( m = 2 \) and \( n = a + 2 \) are satisfied.
(2) If \( pq \in P^2 \), then \( pq\lambda = 10 \).
(3) \( L = L_0 = \{10\}, \mu = 1, W = W_0 = \{w_{10}\} \).
(4) The hypotheses of 3.49 are satisfied and \( \text{crd } M/\sigma^{a+2} = a + 2 \).
(5) \( \mathcal{S}(M, \sigma) \simeq (D_{a+2}, \tau_{a+2}) \).
(6) \( \mathcal{S}(M, \sigma) \simeq D_{a+2} \).
(7) The point \( w_{10} \) of \( M \) is regularly almost periodic under \( \sigma \) but not periodic under \( \sigma \).
(8) To reverse the minimal set \((M, \sigma)\), dualize and reflect the bisequences between the indices \(-1\) and \(0\).

4.08. Example 7. Let \( a \in I_2 \), let \( P = I[0, a - 1] \), and let \( \theta: 0 \to 012 \cdots (a-2)(a-1), 1 \to 123 \cdots (a-1)0, \cdots, (a-1) \to (a-1)01 \cdots (a-3)(a-2) \).
Then:

1. The hypotheses of 3.46 with \( m = a \) and \( n = a \) are satisfied.
2. If \( pq \in P^2 \), then \( pq \lambda = rq \) where \( r \in P \) such that \( r \equiv p - 1 \) (mod \( a \)).
3. \( L = L_0 = P^2, \mu = a, W = W_0 = \{ w_{pq} \mid pq \in P^2 \} \).
4. The hypotheses of 3.49 are satisfied and \( \text{crd} \, M/\sigma^a = a \).
5. \( \mathcal{S}(M, \sigma) \simeq (D_a, \tau_a) \).
6. \( \mathcal{G}(M, \sigma) \simeq D_a \).
7. If \( \rho : M \to M \) is the map defined by leaving 0 unchanged but interchanging 1 and \( a - 1 \), 2 and \( a - 2 \), and so on, and subsequently reflecting between \(-1\) and 0, then \( \rho \) is a reversor of \((M, \sigma)\) whence \((M, \sigma)\) is reversible.
8. The minimal set \((M, \sigma)\) may be called the \textit{a-cyclic} (substitution) minimal set and may be regarded as a generalization of Example 2 to any integer greater than 1.

4.09. Example 8. Let \( P = \{0, 1, 2\} \) and let

\[ \theta : 0 \to 0121021201210, 1 \to 1202102012021, 2 \to 20102120120102. \]

Then:

1. The hypotheses of 3.46 with \( n = 3 \) and \( n = 13 \) are satisfied.
2. \( \lambda \) is the identity map of \( P^2 \).
3. \( L = P^2, \quad L_0 = P^2 - \{00, 11, 22\} = \{01, 02, 10, 12, 20, 21\}, \quad \mu = 1, \quad W = \{ w_{pq} \mid pq \in P^2 \}, \quad W_0 = \{ w_{01}, w_{02}, w_{10}, w_{12}, w_{20}, w_{21} \}. \)
4. The hypotheses of 3.49 are satisfied and \( \text{crd} \, M/\sigma^{13} = 13 \).
5. \( \mathcal{S}(M, \sigma) \simeq (D_{13}, \tau_{13}) \).
6. \( \mathcal{G}(M, \sigma) \simeq D_{13} \).
7. No bisequence in \( M \) contains a substring congruent to a block of the form \( AA \) where \( A \in P^+ \) [4].

4.10. Example 9. Let \( P = \{0, 1, 2\} \) and let

\[ \theta : 0 \to 012102120121, 1 \to 12021020121, 2 \to 20102120121. \]

Then:

1. The hypotheses of 3.46 with \( m = 3 \) and \( n = 12 \) are satisfied.
2. If \( pq \in P^2 \), then \( pq \lambda = 10 \).
3. \( L = L_0 = \{10\}, \mu = 1, W = W_0 = \{ w_{10} \}. \)
4. The hypotheses of 3.49 are satisfied and \( \text{crd} \, M/\sigma^{12} = 12 \).
5. \( \mathcal{S}(M, \sigma) \simeq (D_{12}, \tau_{12}) \).
6. \( \mathcal{G}(M, \sigma) \simeq D_{12} \simeq D_2 \oplus D_3 \).
7. No bisequence in \( M = \hat{O}(w_{10}, \sigma) \) contains a substring congruent to a block of the form \( AA \) where \( A \in P^+ \) [6].
8. The point \( w_{10} \) of \( M \) is regularly almost periodic under \( \sigma \) but not periodic under \( \sigma \).

4.11. Example 10. Let \( P = \{0, 1, 2\} \) and let \( \theta : 0 \to 02, 1 \to 0121, 2 \to 012021. \)
Then:

(1) The hypotheses of 3.05, but not of 3.43, are satisfied.
(2) \( \lambda : 00 \to 20, 01 \to 20, 02 \to 20, 10 \to 10, 11 \to 10, 12 \to 10, 20 \to 10, 21 \to 10, 22 \to 10. \)

(3) \( L = L_0 = \{10\}, \mu = 2, W = W_0 = \{w_{10}\}. \)

(4) No bisequence in \( \bar{O}(w_{10}, \sigma) \) contains a substring congruent to a block of the form \( AA \) where \( A \in P^+ \). [The writer learned of this substitution and this property of it from Marshall Hall.]

4.12. Remark. In at least certain particular instances of substitution minimal sets, their traces may be calculated in the manner of [2].

References


University of Pennsylvania, Philadelphia, Pennsylvania, Yale University, New Haven, Connecticut