WEAKLY WANDERING SETS AND INVARIANT MEASURES

BY

ARSHAG B. HAJIAN AND SHIZUO KAKUTANI

1. **Introduction.** Let $X = \{x\}$ be an abstract set, and let $\mathscr{B} = \{B\}$ be a σ-field of subsets of X. The system $(X, \mathscr{B})$ is called a *measurable space.* A subset B of X is said to be *measurable* if it belongs to $\mathscr{B}$. A one-to-one mapping T of X onto itself is said to be *measurability preserving* if $T(B)$ and $T^{-1}(B)$ are measurable for any measurable set B.

Let $(X, \mathscr{B})$ be a measurable space, and let m be a countably additive set function defined on $\mathscr{B}$. The values of m are either non-negative real numbers or $+\infty$. m is called a *measure,* and the system $(X, \mathscr{B}, m)$ is called a *measure space.* If $m(X) < +\infty$, then we say that m is a *finite measure* and that $(X, \mathscr{B}, m)$ is a *finite measure space.* We also consider the case when $m(X) = +\infty$, but we always assume that there exists a sequence $\{B_n | \ n = 1, 2, \cdots\}$ of measurable sets such that $X = \bigcup_{n=1}^{\infty} B_n$ and $0 < m(B_n) < +\infty$, $n = 1, 2, \cdots$. In this case, we say that m is a *σ-finite measure* and that $(X, \mathscr{B}, m)$ is a *σ-finite measure space.*

Let $(X, \mathscr{B}, m)$ be a finite or σ-finite measure space, and let T be a measurability preserving transformation of X onto itself. T is said to be *nonsingular* if $m(T(B)) = m(T^{-1}(B)) = 0$ for any measurable set B with $m(B) = 0$. T is said to be *measure preserving* if $m(T(B)) = m(T^{-1}(B)) = m(B)$ for any measurable set B. We also say that m is *invariant* under T in this case.

Let $(X, \mathscr{B})$ be a measurable space, and let m and m' be two finite or σ-finite measures defined on $\mathscr{B}$. m is said to be *equivalent* with m' (notation: $m \sim m'$) if $m'(B) = 0$ for any measurable set B with $m(B) = 0$ and conversely $m(B) = 0$ for any measurable set B with $m'(B) = 0$.

We can now state our problem: Let $(X, \mathscr{B}, m)$ be a finiet or a σ-finite measure space, and let T be a nonsingular measurability preserving transformation of X onto itself. Under what conditions on T does there exist a finite measure on $\mathscr{B}$ which is equivalent with m and is invariant under T? This problem has been discussed by many authors, and many necessary and sufficient conditions have been obtained. The main purpose of this paper is to obtain some more necessary and sufficient conditions, and to show, by using a systematic method, that all of these conditions are mutually equivalent.

We introduce the following notion: A measurable subset W of X is called a *weakly wandering set* for a measurability preserving transformation T if there

Received by the editors January 22, 1963.

exists an infinite sequence $\{p_i \mid i = 0, 1, 2, \cdots\}$ of positive or negative integers or 0 such that the image sets $T^{p_i}(W)$, $i = 0, 1, 2, \cdots$ are all mutually disjoint. It is obvious that there is no weakly wandering set of positive measure if there exists a finite measure which is equivalent with m and is invariant under T. It will be shown that the nonexistence of a weakly wandering set of positive measure is also a sufficient condition for the existence of a finite equivalent invariant measure.

The first interesting result concerning the existence of a finite, equivalent, invariant measure was obtained in 1932 by E. Hopf [4] who introduced the notion of incompressibility and proved that there exists a finite, equivalent, invariant measure if and only if the whole space is incompressible.

In 1955, Y. N. Dowker [2] proved, by using the mean ergodic theorem, that $(\mathrm{I})_+$ is a necessary and sufficient condition for the problem. (As for the condition $(\mathrm{I})_+$ and other similar conditions, see §3 where these conditions are defined.) In the same year, A. P. Calderón [1] showed, by using a different method, that the conditions $(\mathrm{I})_+$ and $(\mathrm{II})_+$ are necessary and sufficient. Later, in 1956, Y. N. Dowker [3] showed that the condition $(\mathrm{III})_+$ is equivalent with the condition $(\mathrm{II})_+$ and hence $(\mathrm{III})_+$ is still another necessary and sufficient condition. It is obvious from the definitions that $(\mathrm{I})_+$ implies $(\mathrm{II})_+$ and $(\mathrm{III})_+$, but it does not seem to be so easy to show that conversely $(\mathrm{II})_+$ or $(\mathrm{III})_+$ implies $(\mathrm{I})_+$ directly (i.e., without using the fact that $(\mathrm{II})_+$ or $(\mathrm{III})_+$ implies the existence of a finite, equivalent, invariant measure).

Let us now consider the conditions $(\mathrm{I})_-$, $(\mathrm{II})_-$, $(\mathrm{III})_-$ concerning the inverse transformation T^{-1} which correspond to the conditions $(\mathrm{I})_+$, $(\mathrm{II})_+$, $(\mathrm{III})_+$, and also the conditions $(\mathrm{I})^*_+$, $(\mathrm{II})^*_+$, $(\mathrm{III})^*_+$, $(\mathrm{I})^*_-$, $(\mathrm{II})^*_-$, $(\mathrm{III})^*_-$, where the condition with * may be considered as a qualitative analogue of the corresponding condition without *. It is interesting to observe that each condition with * immediately implies the corresponding condition without *, while the converse is not so obvious in general. [It should be noted that the converse implication can be proved by using Lemma 1 in two places, namely, $(\mathrm{III})_+ \to (\mathrm{III})^*_+$ and $(\mathrm{III})_- \to (\mathrm{III})^*_-$. We also observe that there is no obvious relation between two corresponding conditions with $+$ and $-$, i.e., between two correponding conditions on T and T^{-1}.]

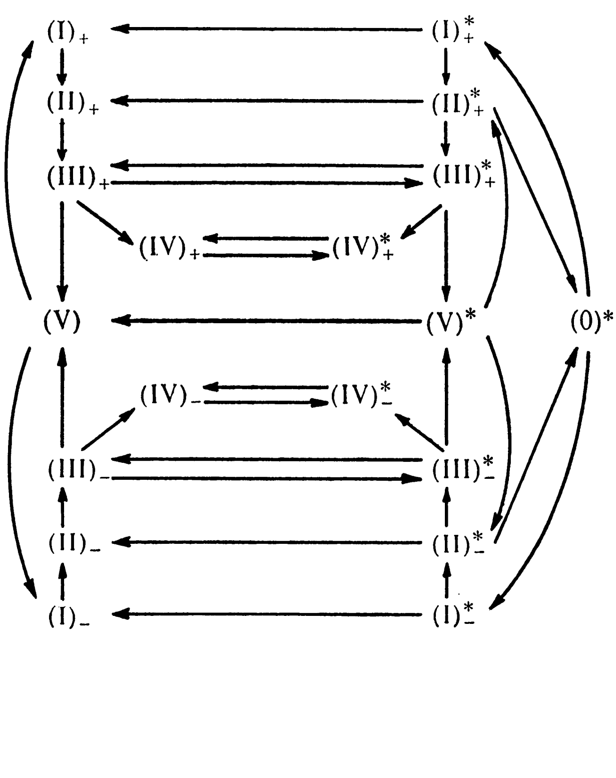
We will show in §4 that all of these conditions are equivalent. For this purpose, we introduce the condition (V) which is based on the notion of weakly wandering sets, and its quantitative analogue (V)*. We also introduce the condition (0)* which is based on the notion of equi-uniform absolute continuity of measures. This condition (0)* is quantitative in nature, and there is no qualitative analogue to (0)*. We will prove, by direct and elementary arguments, that the conditions (0)*, $(\mathrm{I})_+$, $(\mathrm{II})_+$, $(\mathrm{III})_+$, $(\mathrm{I})_-$, $(\mathrm{II})_-$, $(\mathrm{III})_-$, $(\mathrm{I})^*_+$, $(\mathrm{II})^*_+$, $(\mathrm{III})^*_+$, $(\mathrm{I})^*_-$, $(\mathrm{II})^*_-$, $(\mathrm{III})^*_-$, (V), (V)* are all mutually equivalent. (First half of Theorem 1.)

The way our proof is carried out is best described by the diagram inserted at the end of §1. We note that the implications $(\mathrm{V}) \to (\mathrm{I})_+$ and $(\mathrm{V}) \to (\mathrm{I})_-$ can be proved in one step, while the proof of the implications (V)* → (I)$^*_+$ and (V)* → (I)$^*_-$ requires three steps (V)* → (II)$^*_+$ → (0)* → (I)$^*_+$ and (V)* → (II)$^*_-$ → (0)* → (I)$^*_-$. We also note that the conditions (V), (V)* and (0)* are symmetric in T and T^{-1} and serve as a link between the conditions with + and those with −.

It is then an easy matter to observe, by using the technique of Banach limit, that the condition (0)* is equivalent with the existence of a finite, equivalent, invariant measure (second half of Theorem 1).

The conditions (IV)$_+$, (IV)$_-$, (IV)$^*_+$, (IV)$^*_-$ which appear in the diagram are obviously necessary conditions for our problem. We note that the implications (IV)$^*_+$ → (IV)$_+$ and (IV)$^*_-$ → (IV)$_-$ are obvious, while the converse implications (IV)$_+$ → (IV)$^*_+$ and (IV)$_-$ → (IV)$^*_-$ are again the consequences of Lemma 1. It turns out that these conditions are not sufficient for our problem. In fact, it is possible to show that there exists an ergodic measurability preserving transformation T defined on a finite measure space which satisfies the conditions (IV)$_+$, (IV)$_-$, (IV)$^*_+$ and (IV)$^*_-$ and which admits a σ-finite, equivalent, invariant measure (and hence, since T is ergodic, T does not admit any finite, equivalent, invariant measure).

The study of such examples leads us to the problem of classifying ergodic measure preserving transformations defined on a σ-finite measure space. We will show in §5 that every ergodic measure preserving transformation defined on a σ-finite measure space admits a weakly wandering set of positive measure (Theorem 2). This result makes it possible to investigate the properties of ergodic measure preserving transformations defined on a σ-finite measure space in further detail.



The discussion of the above cited examples and other related problems will be left to a subsequent paper.

2. **Two lemmas on set functions.** In this section we prove two lemmas on set functions which we need in §4.

Let $(X,\mathscr{B})$ be a measurable space, and let λ be a real-valued non-negative set function defined on $\mathscr{B}$. [This means that λ takes only finite real non-negative values. This assumption of finiteness of λ is essential in Lemma 2, while Lemma 1 holds even when λ takes the value $+\infty$.] λ is said to be *monotonic* if $\lambda(A) \leqq \lambda(B)$ for any two measurable sets A, B with $A \subset B$. λ is said to be *subadditive* if $\lambda(A \cup B) \leqq \lambda(A) + \lambda(B)$ for any two measurable sets A, B, and *superadditive* if $\lambda(A \cup B) \geqq \lambda(A) + \lambda(B)$ for any two disjoint measurable sets A, B. [We observe that if λ is non-negative and superadditive then λ is monotonic.]

Let λ, μ be two real-valued non-negative monotonic set functions defined on $\mathscr{B}$. μ is said to be *absolutely continuous* with respect to λ if $\mu(B) = 0$ for any measurable set B with $\lambda(B) = 0$. μ is said to be *uniformly absolutely continuous* with respect to λ if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\mu(B) < \varepsilon$ for any measurable set B with $\lambda(B) < \delta$. It is obvious that the uniform absolute continuity implies the absolute continuity, while the converse is not always true. It is known, however, that the converse is true if both λ and μ are finite measures defined on $\mathscr{B}$. The following lemma may be considered as a generalization of this fact:

LEMMA 1. *Let $(X,\mathscr{B},m)$ be a finite measure space. Let λ be a finite measure on $\mathscr{B}$, or more generally, a real-valued, non-negative, monotonic and subadditive set function defined on $\mathscr{B}$. If m is absolutely continuous with respect to λ, then m is uniformly absolutely continuous with respect to λ.*

Proof. Assume that Lemma 1 is not true. Then there would exist an $\varepsilon > 0$ and a sequence $\{B_n \mid n=1,2,\cdots\}$ of measurable sets such that $\lambda(B_n) < 1/2^n$, and $m(B_n) \geqq \varepsilon$, $n = 1,2,\cdots$. For $n = 1,2,\cdots$, let p_n be a positive integer such that $p_n \geqq n$ and

$$m(\Delta_n) < \frac{\varepsilon}{2^{n+1}}, \tag{2.1}$$

where

$$\Delta_n = \bigcup_{k=n}^{\infty} B_k - \bigcup_{k=n}^{p_n} B_k. \tag{2.2}$$

This is possible since $m(X) < \infty$ by assumption. Let us put

$$B^{**} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k (\equiv \limsup_{n\to\infty} B_n), \tag{2.3}$$

$$B^{*} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{p_n} B_k. \tag{2.4}$$

Then

$$B^* = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} B_k - \Delta_n\right) \supset \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} B_k - \bigcup_{=1}^{\infty} \Delta_n \tag{2.5}$$
$$= B^{**} - \bigcup_{n=1}^{\infty} \Delta_n,$$

and hence

$$m(B^*) \geqq m(B^{**}) - \sum_{n=1}^{\infty} m(\Delta_n) \tag{2.6}$$
$$> \varepsilon - \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}.$$

On the other hand

$$\lambda(B^*) \leqq \lambda\left(\bigcup_{k=n}^{p_n} B_k\right) \leqq \sum_{k=n}^{p_n} \lambda(B_k) \tag{2.7}$$
$$< \sum_{k=n}^{p_n} \frac{1}{2^k} < \frac{1}{2^{n-1}} \to 0$$

as $n \to \infty$, and hence $\lambda(B^*) = 0$. This is a contradiction.

The following simple lemma is also useful:

LEMMA 2. *Let $(X,\mathscr{B})$ be a measurable space. Let λ be a real-valued, nonnegative, superadditive (and hence monotonic) set function defined on $\mathscr{B}$. If $\{B_n \mid n=1,2,\cdots\}$ is a decreasing sequence of measurable sets, then, for any $\varepsilon > 0$, there exists a positive integer n_0 such that $\lambda(B_{n_0}-B_n)< \varepsilon$ for any $n > n_0$.*

Proof. From the monotonicity of λ follows that $\lim_{n\to\infty} \lambda(B_n) = \beta \geqq 0$ exists and that $\lambda(B_n) \geqq \beta$ for $n = 1,2,\cdots$. For any $\varepsilon > 0$, let n_0 be a positive number such that $\lambda(B_{n_0}) < \beta + \varepsilon$. Then $\lambda(B_{n_0} - B_n) + \lambda(B_n) \leqq \lambda(B_{n_0}) < \beta + \varepsilon$ for $n > n_0$ and hence $\lambda(B_{n_0} - B_n) < \lambda(B_{n_0}) - \lambda(B_n) < \varepsilon$.

3. **Statement of the main theorem.** Let $(X,\mathscr{B},m)$ be a finite measure space, and let T be a nonsingular, measurability preserving transformation of X onto itself. We put

$$\rho_n(B) = m(T^n B), \qquad n = 0, \pm 1, \pm 2, \cdots, \tag{3.1}$$

$$\sigma_n(B) = \frac{1}{n}\sum_{k=0}^{n-1} \rho_k(B) = \frac{1}{n}\sum_{k=0}^{n-1} m(T^k B), \qquad n = 1,2,\cdots, \tag{3.2$_+$}$$

$$\sigma_{-n}(B) = \frac{1}{n}\sum_{k=0}^{n-1} \rho_{-k}(B) = \frac{1}{n}\sum_{k=0}^{n-1} m(T^{-k} B), \qquad n = 1,2,\cdots. \tag{3.2$_-$}$$

It is clear that ρ_n $(n = 0, \pm 1, \pm 2,\cdots)$ and σ_n $(n = \pm 1, \pm 2,\cdots)$ are finite measures defined on $\mathscr{B}$. From the nonsingularity of T follows that each of them is equivalent with m, and hence any two of them are equivalent to each other.

From Lemma 1 follows that any one of them is uniformly absolutely continuous with respect to any other of them.

Next we put

$$(3.3)_+ \qquad \rho_{*+}(B) = \liminf_{n\to\infty} \rho_n(B),$$

$$(3.3)_- \qquad \rho_{*-}(B) = \liminf_{n\to\infty} \rho_{-n}(B),$$

$$(3.4)_+ \qquad \rho^*_+(B) = \limsup_{n\to\infty} \rho_n(B),$$

$$(3.4)_- \qquad \rho^*_-(B) = \limsup_{n\to\infty} \rho_{-n}(B),$$

$$(3.5)_+ \qquad \sigma_{*+}(B) = \liminf_{n\to\infty} \sigma_n(B),$$

$$(3.5)_- \qquad \sigma_{*-}(B) = \liminf_{n\to\infty} \sigma_{-n}(B),$$

$$(3.6)_+ \qquad \sigma^*_+(B) = \limsup_{n\to\infty} \sigma_n(B),$$

$$(3.6)_- \qquad \sigma^*_-(B) = \limsup_{n\to\infty} \sigma_{-n}(B).$$

It is clear that the set functions $\rho_{*+}, \rho_{*-}, \rho^*_+, \rho^*_-, \sigma_{*+}, \sigma_{*-}, \sigma^*_+, \sigma^*_-$ are all real-valued, non-negative and monotonic. We observe that $\rho_{*+}, \rho_{*-}, \sigma_{*+}, \sigma_{*-}$ are superadditive and that $\rho^*_+, \rho^*_-, \sigma^*_+, \sigma^*_-$ are subadditive, and further that all of them are invariant under T.

It is also clear that

$$(3.7)_+ \qquad 0 \leqq \rho_{*+}(B) \leqq \sigma_{*+}(B) \leqq \sigma^*_+(B) \leqq \rho^*_+(B) \leqq m(X) < +\infty,$$

$$(3.7)_- \qquad 0 \leqq \rho_{*-}(B) \leqq \sigma_{*-}(B) \leqq \sigma^*_-(B) \leqq \rho^*_-(B) \leqq m(X) < +\infty.$$

The following result concerning the asymptotic behavior of $\sigma_n(B)$ as $n \to \infty$ is needed in §4:

LEMMA 3. *For any measurable set B and for any finite set* $\{p_i \mid i=0,1,\cdots,r-1\}$ *of integers, we have*

$$(3.8) \qquad \limsup_{n\to\infty} \sum_{i=0}^{r-1} \sigma_n(T^{p_i}B) = r\sigma^*_+(B),$$

$$(3.9) \qquad \liminf_{n\to\infty} \sum_{i=0}^{r-1} \sigma_n(T^{p_i}B) = r\sigma_{*+}(B).$$

In particular, if the sets $T^{p_i}B$, $i = 0, 1, \cdots, r-1$, *are mutually disjoint, then we have*

$$\sigma_+^*\left(\bigcup_{i=0}^{r-1} T^{p_i}B\right) = r\sigma_+^*(B), \tag{3.10}$$

$$\sigma_+^*\left(\bigcup_{i=0}^{r-1} T^{p_i}B\right) = r\sigma_{*+}(B). \tag{3.11}$$

Proof. We first observe that

$$\begin{aligned} |\sigma_n(T^pB) - \sigma_n(B)| &= \left|\frac{1}{n}\sum_{k=p}^{p+n-1} m(T^kB) - \frac{1}{n}\sum_{k=0}^{n-1} m(T^kB)\right| \\ &\leqq \frac{2|p|}{n} m(X) \end{aligned} \tag{3.12}$$

for any integer $p = 0, \pm 1, \pm 2, \cdots$ and $n = 1, 2, \cdots$. From this follows that

$$\begin{aligned} \left|\sum_{i=0}^{r-1} \sigma_n(T^{p_i}B) - r\sigma_n(B)\right| &\leqq \sum_{i=0}^{r-1} |\sigma_n(T^{p_i}B) - \sigma_n(B)| \\ &\leqq \sum_{i=0}^{r-1} \frac{2|p_i|}{n} m(X) \to 0 \end{aligned} \tag{3.13}$$

as $n \to \infty$. The relations (3.8) and (3.9) follow from (3.13); and (3.10) and (3.11) are consequences of (3.8) and (3.9), respectively.

Let us consider the following conditions:

$(\mathrm{I})_+$ $\quad m(B) > 0$ *implies* $\rho_{*+}(B) > 0$,

$(\mathrm{II})_+$ $\quad m(B) > 0$ *implies* $\sigma_{*+}(B) > 0$,

$(\mathrm{III})_+$ $\quad m(B) > 0$ *implies* $\sigma_+^*(B) > 0$,

$(\mathrm{IV})_+$ $\quad m(B) > 0$ *implies* $\rho_+^*(B) > 0$,

$(\mathrm{I})_-$ $\quad m(B) > 0$ *implies* $\rho_{*-}(B) > 0$,

$(\mathrm{II})_-$ $\quad m(B) > 0$ *implies* $\sigma_{*-}(B) > 0$,

$(\mathrm{III})_-$ $\quad m(B) > 0$ *implies* $\sigma_-^*(B) > 0$,

$(\mathrm{IV})_-$ $\quad m(B) > 0$ *implies* $\rho_-^*(B) > 0$.

All of these conditions may be interpreted as saying that m is absolutely continuous with respect to ρ_{*+}, σ_{*+}, etc.

We observe that the conditions $(\mathrm{I})_+ - (\mathrm{IV})_+$ and $(\mathrm{I})_- - (\mathrm{IV})_-$ give a qualitative description of the different limits involved. The corresponding quantitative conditions may be stated as follows:

(I)$_+^*$ *for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $m(B) \geqq \varepsilon$ *implies* $\rho_{*+}(B) \geqq \delta$,

(II)$_+^*$ *for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $m(B) \geqq \varepsilon$ *implies* $\sigma_{*+}(B) \geqq \delta$,

(III)$_+^*$ *for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $m(B) \geqq \varepsilon$ *implies* $\sigma_+^*(B) \geqq \delta$,

(IV)$_+^*$ *for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $m(B) \geqq \varepsilon$ *implies* $\rho_+^*(B) \geqq \delta$,

(I)$_-^*$ *for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $m(B) \geqq \varepsilon$ *implies* $\rho_{*-}(B) \geqq \delta$,

(II)$_-^*$ *for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $m(B) \geqq \varepsilon$ *implies* $\sigma_{*-}(B) \geqq \delta$,

(III)$_-^*$ *for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $m(B) \geqq \varepsilon$ *implies* $\sigma_-^*(B) \geqq \delta$,

(IV)$_-^*$ *for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $m(B) \geqq \varepsilon$ *implies* $\rho_-^*(B) \geqq \delta$.

All of these conditions may be interpreted as saying that m is uniformly absolutely continuous with respect to ρ_{*+}, σ_{*+}, etc.

A measurable subset W of X is called a *weakly wandering set* if there exists a sequence $\{p_i \mid i = 0,1,2,\cdots\}$ of integers (positive, negative or zero) such that the image sets $T^{p_i}W$, $i = 0,1,2,\cdots$, are all mutually disjoint. By using this notion of weakly wandering set, we can introduce the following condition:

(V) *There is no weakly wandering set of positive measure.*

It is easy to see that the condition (V) is symmetric in T and T^{-1}. In fact, if the image sets $T^{p_i}B$, $i = 0,1,2,\cdots$, are mutually disjoint, then the sets $T^{-p_i}B$, $i = 0,1,2,\cdots$, are also mutually disjoint.

The qualitative analogue to the condition (V) may be stated as follows:

(V)* *for any* $\varepsilon > 0$, *there exists a positive integer* k *such that, if* B *is a measurable set with* $m(B) \geqq \varepsilon$, *then at most* k *images of* B *by powers of* T *are mutually disjoint.*

Again this condition is symmetric in T and T^1.

Let $\{m_n \mid n = 1,2,\cdots\}$ be a sequence of finite measures defined on the same σ-field $\mathscr{B}$. $\{m_n \mid n = 1,2,\cdots\}$ is said to be *equi-uniformly absolutely continuous* with respect to a finite measure m defined on $\mathscr{B}$ if, for any $\varepsilon > 0$, there exists a

$\delta > 0$ such that $m_n(B) < \varepsilon$, $n = 1, 2, \cdots$ for any measurable set B with $m(B) < \delta$. The last condition of this section can now be stated as follows:

(0)* $\{\rho_n \mid n = 0, \pm 1, \pm 2, \cdots\}$ *is equi-uniformly, absolutely continuous with respect to m, i.e., for any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $m(B) < \delta$ *implies* $\rho_n(B) < \varepsilon$ *for* $n = 0, \pm 1, \pm 2, \cdots$.

We now state our main result in the following form:

THEOREM 1. *Let* $(X, \mathscr{B}, m)$ *be a finite measure space, and let T be a non-singular, measurability preserving transformation of X onto itself. The following conditions* (0)*, $(\mathrm{I})_+$, $(\mathrm{II})_+$, $(\mathrm{III})_+$, $(\mathrm{I})_-$, $(\mathrm{II})_-$, $(\mathrm{III})_-$, $(\mathrm{I})^*_+$, $(\mathrm{II})^*_+$, $(\mathrm{III})^*_+$, $(\mathrm{I})^*_-$, $(\mathrm{II})^*_-$, $(\mathrm{III})^*_-$, (V), (V)* *are mutually equivalent, and each of them is a necessary and sufficient condition for the existence of a finite measure defined on* $\mathscr{B}$ *which is equivalent to m and is invariant under T.*

The proof of this theorem will be given in §4.

4. **Proof of the main theorem.** We prove Theorem 1 by following the way described in §1 and indicated by arrows in the diagram at the end of §1. We first list obvious implications:

PROPOSITION 1. *The following implications hold*:

$(4.1)_+$ $\quad (\mathrm{I})_+ \rightarrow (\mathrm{II})_+ \rightarrow (\mathrm{III})_+ \rightarrow (\mathrm{IV})_+,$

$(4.1)_-$ $\quad (\mathrm{I})_- \rightarrow (\mathrm{II})_- \rightarrow (\mathrm{III})_- \rightarrow (\mathrm{IV})_-,$

$(4.2)_+$ $\quad (\mathrm{I})^*_+ \rightarrow (\mathrm{II})^*_+ \rightarrow (\mathrm{III})^*_+ \rightarrow (\mathrm{IV})^*_+,$

$(4.2)_-$ $\quad (\mathrm{I})^*_- \rightarrow (\mathrm{II})^*_- \rightarrow (\mathrm{III})^*_- \rightarrow (\mathrm{IV})^*_-,$

$(4.3)_+$ $\quad (\mathrm{I})^*_+ \rightarrow (\mathrm{I})_+,$

$(4.3)_-$ $\quad (\mathrm{I})^*_- \rightarrow (\mathrm{I})_-,$

$(4.4)_+$ $\quad (\mathrm{II})^*_+ \rightarrow (\mathrm{II})_+,$

$(4.4)_-$ $\quad (\mathrm{II})^*_- \rightarrow (\mathrm{II})_-,$

$(4.5)_+$ $\quad (\mathrm{III})^*_+ \rightarrow (\mathrm{III})_+,$

$(4.5)_-$ $\quad (\mathrm{III})^*_- \rightarrow (\mathrm{III})_-,$

$(4.6)_+$ $\quad (\mathrm{IV})^*_+ \rightarrow (\mathrm{IV})_+,$

$(4.6)_-$ $\quad (\mathrm{IV})^*_- \rightarrow (\mathrm{IV})_-,$

(4.7) $\quad (\mathrm{V})^* \rightarrow (\mathrm{V}).$

Proof. Follows immediately from definitions.

PROPOSITION 2. *The following implications hold*

(4.8)$_+$ $\quad$ $(\mathrm{III})_+ \rightarrow (\mathrm{III})_+^*$,

(4.8)$_-$ $\quad$ $(\mathrm{III})_- \rightarrow (\mathrm{III})_-^*$,

(4.9)$_+$ $\quad$ $(\mathrm{IV})_+ \rightarrow (\mathrm{IV})_+^*$,

(4.9)$_-$ $\quad$ $(\mathrm{IV})_- \rightarrow (\mathrm{IV})_-^*$.

Proof. It follows from Lemma 1 and the fact that the set functions σ_+^*, σ_-^*, ρ_+^* and ρ_-^* are subadditive functions.

PROPOSITION 3. *The following implications hold*:

(4.10)$_+$ $\quad$ $(\mathrm{III})_+ \rightarrow (\mathrm{V})$,

(4.10)$_-$ $\quad$ $(\mathrm{III})_- \rightarrow (\mathrm{V})$.

Proof. We prove only (4.10)$_+$ since (4.10)$_-$ can be proved in exactly the same way. Assume that the condition (III)$_+$ holds but the condition (V) does not hold. Then there would exist a measurable set W of positive measure and a sequence $\{p_i \mid i = 0, 1, 2, \cdots\}$ of integers such that the sets $T^{p_i}W$, $i = 0, 1, 2, \cdots$, are mutually disjoint. From (3.10) of Lemma 3 of §3 follows that

$$r\sigma_+^*(W) = \sigma_+^*\left(\bigcup_{i=0}^{r-1} T^{p_i}W\right) \leqq m(X) < \infty \text{ for } r = 1, 2, \cdots. \tag{4.11}$$

This implies that $\sigma_+^*(W) = 0$, which is a contradiction to (III)$_+$.

PROPOSITION 4. *The following implications hold:*

(4.12)$_+$ $\quad$ $(\mathrm{III})_+^* \rightarrow (\mathrm{V})^*$,

(4.12)$_-$ $\quad$ $(\mathrm{III})_-^* \rightarrow (\mathrm{V})^*$.

Proof. We prove only (4.12)$_+$ since (4.12)$_-$ can be proved exactly in the same way. Assume that the condition (III)$_+^*$ holds but the condition (V)* does not hold. Then there would exist a positive number $\varepsilon > 0$ such that, for any positive integer r, there exists a measurable set B with $m(B) > \varepsilon$ and a set $\{p_i \mid i = 0, 1, \cdots, r-1\}$ of r integers such that the sets $T^{p_i}B$, $i = 0, 1, \cdots, r-1$, are mutually disjoint. From (3.11) of Lemma 3 of §3 follows the relation (4.11). This is a contradiction since $\varepsilon > 0$ is fixed and r is an arbitrary positive integer.

PROPOSITION 5. *The following implications hold*:

(4.13)$_+$ $\quad$ $(\mathrm{V}) \rightarrow (\mathrm{I})_+$,

(4.13)$_-$ $\quad$ $(\mathrm{V}) \rightarrow (\mathrm{I})_-$.

Proof. We prove only (4.13)$_+$ since (4.13)$_-$ can be proved exactly in the same way. In order to prove (4.13)$_+$ it is sufficient to prove the following:

LEMMA 4. *Let A be a measurable set with $m(A) = \alpha > 0$ and assume that*

(4.14) $$\rho_{*+}(A) = \liminf_{n\to\infty} m(T^n A) = 0.$$

Then, for any ε with $0 < \varepsilon < \alpha$, there exists a measurable subset A' of A such that $m(A') < \varepsilon$ and $W = A - A'$ is a weakly wandering set.

Proof. Let $p_0 = 0$, and put $\varepsilon_i = \varepsilon / i\, 2^i$, $i = 1, 2, \cdots$. Choose a positive integer p_1 such that $m(T^{p_1}A) < \varepsilon_1$. This is possible because of (4.14). Assume now that the integers $0 = p_0 < p_1 < \cdots < p_{i-1}$ are already chosen. We choose a positive integer p_i such that $p_i > p_{i-1}$ and

(4.15) $$m(T^{p_i - p_j}A) < \varepsilon_i$$

for $j = 0, 1, \cdots, i-1$. This is possible because of (4.14) and because the measures ρ_{-p_j}, $j = 0, 1, \cdots, i-1$, are uniformly absolutely continuous with respect to m. In fact, we first choose a positive number $\delta_i > 0$ such that $m(B) < \delta_i$ implies $\rho_{-p}(B) = \rho(T^{-p_j}B) < \varepsilon_i$, $j = 0, 1, \cdots, i-1$. Then choose a positive integer p_i such that $p_i > p_{i-1}$ and $m(T^{p_i}A) < \delta_i$. This implies that $m(T^{p_i - p_j}A) < \varepsilon_i$, $j = 0, 1, \cdots, i-1$.

In this way we can choose an increasing sequence of integers $\{p_i \mid i = 0, 1, 2, \cdots\}$ with $p_0 = 0$ such that (4.15) is satisfied for $j = 0, 1, \cdots, i-1$ and $i = 1, 2, \cdots$. We put

(4.16) $$A' = \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i-1} T^{p_i - p_j}A \cap A.$$

Then A' is a measurable subset of A with

(4.17) $$\begin{aligned} m(A') &\leqq \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} m(T^{p_i - p_j}A) \\ &< \sum_{i=1}^{\infty} i\varepsilon_i = \varepsilon, \end{aligned}$$

and it is easy to see that $W = A - A'$ is a weakly wandering set. In fact, $T^{p_i}W$, $i = 0, 1, 2, \cdots$ are mutually disjoint since

(4.18) $$T^{p_i - p_j}W \cap W \subset T^{p_i - p_j}A \cap (A - A') = \emptyset$$

for $j = 0, 1, \cdots, i-1$ and $i = 1, 2, \cdots$.

PROPOSITION 6. *The following implications hold*:

$(4.19)_+$ $$(\mathrm{V})^* \;\rightarrow\; (\mathrm{II})^*_+,$$

$(4.19)_-$ $$(\mathrm{V})^* \;\rightarrow\; (\mathrm{II})^*_-.$$

Proof. We prove only $(4.19)_+$ since $(4.19)_-$ can be proved exactly in the same way. Assume that the condition $(\mathrm{V})^*$ holds but the condition $(\mathrm{II})^*_+$ does not hold. Then there would exist a positive number $\varepsilon > 0$ such that for any positive number

$\delta > 0$ there exists a measurable set A with $m(A) \geqq \varepsilon$ and $\sigma_{*+}(A) < \delta$. In order to complete the proof it is sufficient to prove the following:

LEMMA 5. *Let A be a measurable set with $m(A) = \alpha > 0$ and assume that*

$$\sigma_{*+}(A) = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} m(T^k A) < \delta. \tag{4.20}$$

Let k be a positive integer such that $(k(k+1)/2)\delta < \alpha$. Then there exists a measurable subset A' of A with $m(A') < (k(k+1)/2)\delta$ such that $W = A - A'$ has $k+1$ disjoint images by powers of T, i.e., there exist $k+1$ integers $p_0, p_1, \cdots, p_k$, such that $T^{p_i}W$, $i = 0, 1, \cdots, k$ are mutually disjoint.

Proof. Let $p_0 = 0$. Choose a positive integer p_1 such that $m(T^{p_1}A) < \delta$. This is possible since (4.20) is satisfied by A. Let $1 < i \leqq k$ and assume that the integers $0 = p_0 < p_1 < \cdots < p_{i-1}$ are already chosen. We choose a positive integer p such that $p_i > p_{i-1}$ and

$$m\left(\bigcup_{j=0}^{i-1} T^{p_i - p_j} A\right) < i\delta. \tag{4.21}$$

This is possible since

$$\begin{aligned} \sigma_{*+}\left(\bigcup_{j=0}^{i-1} T^{-p_j} A\right) &= \liminf_{n \to \infty} \sigma_n \left(\bigcup_{j=0}^{i-1} T^{-p_j} A\right) \\ &\leqq \liminf_{n \to \infty} \sum_{j=0}^{i-1} \sigma_n(T^{-p_j} A) = i\sigma_{*+}(A) < i\delta \end{aligned} \tag{4.22}$$

by (3.9) of Lemma 3.

In this way we get $k+1$ integers $0 = p_0 < p_1 < \cdots < p_k$ such that (4.21) holds for $i = 1, \cdots, k$. Let us put

$$A' = \bigcup_{i=1}^{k} \bigcup_{j=0}^{i-1} T^{p_i - p_j} A. \tag{4.23}$$

Then

$$\begin{aligned} m(A') &\leqq \sum_{i=1}^{k} m\left(\bigcup_{j=0}^{i-1} T^{p_i - p_j} A\right) \\ &< \sum_{i=1}^{k} i\delta = \frac{k(k+1)}{2}\delta \end{aligned} \tag{4.24}$$

and if we put $B = A - A'$ then it is easy to see that the sets $T^{p_i}B$, $i = 0, 1, \cdots, k$ are mutually disjoint. In fact

$$T^{p_i - p_j} B \cap B \subset T^{p_i - p_j} A \cap (A - A') = \emptyset \tag{4.25}$$

for $0 \leqq j < i \leqq k$.

PROPOSITION 6. *The following implications hold*:

$$(4.26)_+ \qquad (\mathrm{II})^*_+ \text{ implies } (0)^*,$$

$$(4.26)_- \qquad (\mathrm{II})^*_- \text{ implies } (0)^*.$$

Proof. We prove only $(4.26)_+$ since $(4.26)_-$ can be proved exactly in the same way. Assume that the condition $(\mathrm{II})^*_+$ holds but $(0)^*$ does not hold. Then there exists a positive number $\varepsilon > 0$, a sequence of measurable sets $\{A_i \mid i = 1,2,\cdots\}$ and a sequence of integers $\{p_i \mid i = 1,2,\cdots\}$ such that $m(A_i) < 1/2^i$ and $m(T^{p_i}A_i) \geqq 2\varepsilon$ for $i = 1,2,\cdots$. Let $B_k = \bigcup_{i=k}^{\infty} A_i$, $k = 1,2,\cdots$. Then $\{B_k \mid k = 1,2,\cdots\}$ is a decreasing sequence of measurable sets with $\lim_{k\to\infty} m(B_k) = 0$ such that

$$(4.27) \qquad m(T^{p_k}B_k) \geqq 2\varepsilon$$

for $k = 1,2,\cdots$. On the other hand, from condition $(\mathrm{II})^*_+$ follows that there exists a positive number $\delta > 0$ such that $\sigma_{*+}(B) \geqq \delta$ for any measurable set B with $m(B) \geqq$ a. We apply Lemma 2 of §2 to $\lambda = \sigma_{*+}$. Then there exists a positive integer k_0 such that

$$(4.28) \qquad \sigma^*{}_+(B_{k_0} - B_k) < \delta$$

for any integer $k > k_0$. If we put $k = k_0$ in (4.27), we have $m(T^{p_{k_0}}B_{k_0}) \geqq 2\varepsilon$. Choose a positive integer k_1 so large that $k_1 > k_0$ and $m(T^{p_{k_0}}B_{k_1}) < \varepsilon$, and put $B^* = T^{p_{k_0}}(B_{k_0} - B_{k_1})$. Then $m(B^*) = m(T^{n_{k_0}}B_{k_0}) - m(T^{n_{k_0}}B_{k_1}) > 2\varepsilon - \varepsilon = \varepsilon$ and $\sigma_{*+}(B^*) = \sigma_{*+}(B_{k_0} - B_{k_1}) < \delta$ by (4.28). This is a contradiction to $(\mathrm{II})^*_+$.

PROPOSITION 7. *The following implications hold*:

$$(4.29)_+ \qquad (0)^* \rightarrow (\mathrm{I})^*_+,$$

$$(4.29)_- \qquad (0)^* \rightarrow (\mathrm{I})^*_-.$$

Proof. We prove only $(4.29)_+$ since $(4.29)_-$ can be proved exactly in the same way. Assume that the condition $(0)^*$ holds, but $(\mathrm{I})^*_+$ does not hold. From $(0)^*$ follows that for any $\varepsilon > 0$ there exists a positive number $\delta > 0$ such that $m(B) < \delta$ implies $\rho_{-n}(B) = m(T^{-n}B) < \varepsilon$ for $n = 0, \pm 1, \pm 2, \cdots$. If condition $(\mathrm{I})^*_+$ does hold then there would exist a positive number $\varepsilon > 0$ such that for any positive number $\delta > 0$ there exists a measurable set A with $m(A) \geqq \varepsilon$ and $\rho_{*+}(A) < \delta$. From this follows that there exists a positive integer n such that $\rho_n(A) = m(T^nA) < \delta$. This is a contradiction since $m(T^nA) < \delta$ would imply $m(A) = \rho_{-n}(T^nA) < \varepsilon$.

PROPOSITION 8. *In order that there exist a finite measure defined on $\mathscr{B}$ which is equivalent with m and is invariant under T, it is necessary and sufficient that the condition* $(0)^*$ *is satisfied.*

Proof. Assume that there exists a finite measure μ defined on $\mathscr{B}$ which is equivalent with m and is invariant under T. From Lemma 1 of §2 follows that m is uniformly absolutely continuous with respect to μ. Since μ is invariant under T, this implies that $\{\rho_n \mid n = 0, \pm 1, \pm 2, \cdots\}$ is equi-uniformly absolutely continuous with respect to μ. On the other hand, again from Lemma 1 follows that μ is uniformly absolutely continuous with respect to m. From these statements we conclude that $\{\rho_n \mid n = 0, \pm 1, \pm 2, \cdots\}$ is equi-uniformly absolutely continuous with respect to m. Thus the condition (0)* is satisfied.

Conversely, assume that the condition (0)* is satisfied. Let us put

$$\mu(B) = \operatorname{Lim} \rho_n(B), \tag{4.30}$$

where Lim denotes a Banach limit. It is obvious that $\mu(B)$ is invariant and finitely additive on $\mathscr{B}$. From the equi-uniform absolute continuity of $\{\rho_n \mid n = 0, \pm 1, \pm 2, \cdots\}$ follows that μ is countably additive and equivalent with m. In fact, for any $\varepsilon > 0$, choose a positive number $\delta > 0$ such that $m(B) < \delta$ implies $\rho_n(B) < \varepsilon$ for all $n = 0, \pm 1, \pm 2, \cdots$. From this follows that $m(B) < \delta$ implies $\mu(B) = \operatorname{Lim} \rho_n(B) \leqq \varepsilon$. This shows that μ is countably additive and uniformly absolutely continuous with respect to m. On the other hand, $\mu(B) < \delta$ implies that $m(T^n B) = \rho_n(B) < \delta$ for at least one n, and hence $m(B) = \rho_{-n}(T^n B) < \varepsilon$. This shows that m is uniformly absoutely continuous with respect to μ.

Propositions 1–8 together give all the implications denoted by arrows in the diagram at the end of §1. This completes the proof of Theorem 1.

5. **Application to σ-finite measure space.** Let $(X, \mathscr{B}, m)$ be a finite or σ-finite measure space, and let T be a nonsingular, measurability preserving transformation of X onto itself. A measurable subset A of X is said to be *invariant* under T if $TA = A$, and T is said to be *ergodic* if $m(A) = 0$ or $m(X - A) = 0$ for any invariant measurable subset A of X.

LEMMA 6. *Let T be an ergodic, nonsingular, measurability preserving transformation defined on a finite or σ-finite measure space $(X, \mathscr{B}, m)$. Let λ and μ be two finite or σ-finite nonzero measures defined on $\mathscr{B}$ which are equivalent with m and are invariant under T. Then there exists a positive constant $c > 0$ such that $\lambda = c\mu$, i.e., λ and μ are either both finite or both σ-finite, and $\lambda(B) = c\mu(B)$ for any measurable subset B of X.*

Proof. Since λ and μ are equivalent, there exists (by Radon-Nikodym's theorem) a positive measurable function $f(x)$ defined on X such that

$$\lambda(B) = \int_B f(x)\, \mu(dx) \tag{5.1}$$

for any measurable subset B of X. Since both λ and μ are invariant under T, it follows that $f(Tx) = f(x)$ almost everywhere on X. Since T is ergodic, it follows that $f(x)$ is equal to a constant c almost everywhere on X. It is easy to see that $c > 0$ and that (5.1) holds for any measurable subset B of X.

THEOREM 2. *Let T be an ergodic measure preserving transformation defined on a σ-finite measure space $(X,\mathscr{B},\mu)$. Then T admits a weakly wandering set of positive measure, i.e., there exist a measurable subset W of X with $\mu(W) > 0$ and a sequence $\{p_i \mid i = 1,2,\cdots\}$ of positive integers such that the sets $T^{p_i}W$, $i = 1,2,\cdots$, are mutually disjoint.*

Proof. Let $\{B_n \mid n = 1,2,\cdots\}$ be a sequence of mutually disjoint measurable subsets of X such that $X = \bigcup_{n=1}^{\infty} B_n$ and $0 < \mu(B_n) < \infty$, $n = 1,2,\cdots$. If we put

$$m(B) = \sum_{n=1}^{\infty} \frac{\mu(B \cap B_n)}{2^n \mu(B_n)} \tag{5.2}$$

for any measurable subset B of X, then m is a finite measure defined on $\mathscr{B}$ and is equivalent to μ. T may be considered as an ergodic, nonsingular, measurability preserving transformation defined on the finite measure space $(X,\mathscr{B},m)$. From Lemma 6 above follows that there is no finite measure defined on $\mathscr{B}$ which is equivalent to m and is invariant under T. Thus none of the conditions of Theorem 1 is satisfied. In particular, condition (V) does not hold. Thus there exists a weakly wandering set W with $m(W) > 0$, and hence $\mu(W) > 0$. This completes the proof of Theorem 2.

Let again T be an ergodic measure preserving transformation defined on a σ-finite measure space $(X,\mathscr{B},\mu)$. From the individual ergodic theorem follows easily that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(T^k A \cap A) = 0 \tag{5.3}$$

for any measurable subset A of X with $\mu(A) < \infty$. This, however, does not imply that

$$\lim_{n\to\infty} \mu(T^n A \cap A) = 0. \tag{5.4}$$

It is not difficult to show that either (5.4) holds for any measurable subset A of X with $\mu(A) < \infty$, or

$$\limsup_{n\to\infty} \mu(T^n A \cap A) > 0 \tag{5.5}$$

for any measurable subset A of X with $\mu(A) > 0$. In the first case T is said to be *of zero type* and in the second case T is said to be *of positive type*. It is possible to show that ergodic measure preserving transformations of both types exist on a suitable σ-finite measure space (for example, the Lebesgue measure space on the real line $X = (-\infty,\infty)$). Ergodic measure preserving transformations of positive type are interesting in connection with the conditions $(\mathrm{IV})_+$, $(\mathrm{IV})_+^*$, $(\mathrm{IV})_-$, $(\mathrm{IV})_-^*$ introduced in §3.

Let m be a finite measure defined on $\mathscr{B}$ which is equivalent to μ. As was observed earlier in this section, T may be considered as an ergodic, nonsingular, measurability preserving transformation defined on a finite measure space $(X, \mathscr{B}, m)$, and there is no finite measure defined on $\mathscr{B}$ which is equivalent with m and invariant under T. It is easy to see that T is of positive type on $(X, \mathscr{B}, \mu)$ if and only if T satisfies the condition $(\mathrm{IV})_+$ of §3. Thus we see that the condition $(\mathrm{IV})_+$, which is obviously a necessary condition for the existence of a finite, equivalent, invariant measure, is not a sufficient condition.

The existence of ergodic measure preserving transformations of zero and positive types on a σ-finite Lebesgue measure space will be shown in a subsequent paper where more detailed classification of ergodic measure preserving transformations on a σ-finite measure space is discussed.

Bibliography

1. A. P. Calderón, *Sur les mesures invariantes*, C. R. Acad. Sci. Paris **240** (1955), 1960–1962.

2. Y. N. Dowker, *On measurable transformations in finite measure spaces*, Ann. of Math. (2) **62** (1955), 504–516.

3. ———, *Sur les applications mésurables*, C. R. Acad. Sci. Paris **242** (1956), 329–331.

4. E. Hopf, *Theory of measures and invariant integrals*, Trans. Amer. Math. Soc. **34** (1932), 373–393.

Cornell University,
Ithaca, New York
Yale University,
New Haven, Connecticut