ON THE EXISTENCE AND CHARACTERIZATION OF
BEST NONLINEAR TCHEBYCHEFF APPROXIMATIONS

BY
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1. Introduction. This paper is concerned with the following approximation problem:

Tchebycheff approximation problem. Let \( f(x) \) be continuous on \([0,1]\) and let \( F(A, x) \) be a continuous approximating function depending on \( n \) parameters, \( A = (a_1, a_2, \ldots, a_n) \). Denote by \( P \) the domain of the parameters. Given \( f(x) \) determine \( A^* \in P \) such that

\[
\max_{x \in [0,1]} |F(A^*, x) - f(x)| \leq \max_{x \in [0,1]} |F(A, x) - f(x)|
\]

for all \( A \in P \). Such an \( F(A^*, x) \) is a best approximation to \( f(x) \).

Relative to this problem there are three principal statements to be investigated. They are

Statement A. \( f(x) \) possesses a best approximation.

Statement B. Best approximations are characterized as those \( F(A^*, x) \) for which \( F(A^*, x) - f(x) \) alternates at least \( n \) times on \([0,1]\).

Statement C. The best approximation is unique.

Statement B gives the usual characteristic property of best Tchebycheff approximations. The function \( F(A^*, x) - f(x) \) is said to alternate \( n \) times if there are \( n + 1 \) points

\[
0 \leq x_1 < x_2 < \cdots < x_{n+1} \leq 1
\]

such that

\[
F(A^*, x_j) - f(x_j) = -[F(A^*, x_{j+1}) - f(x_{j+1})] = \pm \max_{x \in [0,1]} |F(A^*, x) - f(x)|.
\]

All maxima in this paper are taken over \( x \in [0,1] \) unless otherwise stated.

The usual effort on a problem of this type is to be given a particular \( F(A, x) \) and then to attempt to establish one or more of Statements A, B and C. The effort presented in this paper is one of a more general and partially converse nature. This study concerns the following question: What conditions on \( F(A, x) \) are both necessary and sufficient for a certain combination of Statements A, B and C to be valid for all continuous functions?


88
Such a question was first posed and answered by Haar [2] where, for linear approximating functions

\[ F(A, x) = \sum_{i=1}^{n} a_i \phi_i(x), \]

he asked what are necessary and sufficient conditions for Statement C to be valid for all continuous functions. More recently Rice [4] has posed and answered the question (for linear approximating functions) for Statement A and for Statements A and B. Also Rice [5] has answered this question for Statement B and for a continuous nonlinear approximating function.

In this paper conditions on a nonlinear approximating function \( F(A, x) \) are found which are necessary and sufficient for both Statements A and B to be valid for all continuous functions.

The conditions found to be equivalent to Statements A and B are closure under pointwise convergence (Definition 4) and local unisolvence (Definition 3).

In the final section it is noted that if Statement B is valid for all continuous functions, then so is Statement C. This implies that the condition found here for Statements A and B are also the conditions for Statements A, B and C.

2. The theorem and proof. In order to describe the properties of \( F \) that characterize Statements A and B, and in order to facilitate the discussion, the following definitions are made.

**Definition 1.** \( F \) has Property Z if \( A^* \neq A \) implies that

\[ F(A^*, x) - F(A, x) \]

has at most \( n - 1 \) zeros in \([0,1]\).

**Definition 2.** \( F \) is said to be locally solvent if given \( 0 \leq x_1 < x_2 < \cdots < x_n \leq 1, A^* \in P \) and \( \epsilon > 0 \) then there is a \( \delta(A^*, \epsilon, x_1, x_2, \cdots, x_n) > 0 \) such that

\[ |F(A^*, x_j) - y_j| < \delta \]

implies the existence of a solution \( A \in P \) to

\[ (2.1) \quad F(A, x_j) = y_j \]

with \( \max |F(A, x) - F(A^*, x)| < \epsilon \).

**Definition 3.** \( F \) is said to be locally unisolvent if \( F \) is locally solvent and has Property Z.

The final property is closely related to unisolvence [3]. In both cases Property Z is present. For \( F \) unisolvent one is assured of solving (2.1) for any set of values \( \{y_j\} \), the present definition only assures a solution of (2.1) if the points \( \{(x_j, y_j) | j = 1, 2, \cdots, n\} \) lie in a neighborhood of some curve \( F(A^*, x) \).

The next definition describes a property to be associated with Statement A.

**Definition 4.** \( F \) is said to be closed if \( P \) is arcwise connected and if \( F \) is closed under pointwise limits, i.e.,
(2.2) \[ \lim_{k \to \infty} F(A_k, x) = G(x), \quad x \in [0, 1], \quad |F(A_k, x)| \leq M \]

implies there is an \( A_0 \in P \) such that

(2.3) \[ F(A_0, x) = G(x). \]

At this point it is appropriate to make a remark on the topology of the parameter space \( P \). Since \( F(A, x) \) depends on \( n \) parameters, one naturally associates the parameters \( A \) with a point in Euclidean \( n \)-space \( E_n \). However the topology of \( E_n \) may not be suitable for \( P \) and indeed it may be extremely difficult to imbed \( P \) in \( E_n \) in such a way that the \( E_n \) topology has any meaning at all for \( F \). Thus \( P \) is considered to be an abstract space with its topology derived from the uniform norm on the set of functions \( \{F(A, x)\} \). In this way the statement

(2.4) \[ \lim_{k \to \infty} A_k = A^* \]

is defined to be equivalent to

(2.5) \[ \lim_{k \to \infty} \max_{x \in [0, 1]} |F(A_k, x) - F(A^*, x)| = 0. \]

If \( F \) is locally solvent then one may show that pointwise closure (2.2) becomes uniform closure (2.5).

The above definitions were originally made in [5]. The definition of closure here has been made slightly less restrictive. For the application of the results of [5] in this paper, the definitions are equivalent.

The main theorem of this paper answers the question posed in the introduction, namely, what does the validity of Statements A and B for every continuous \( f(x) \) imply about the properties of the approximating function \( F(A, x) \), and vice versa.

**Theorem 1.** Statements A and B are valid for every continuous function if and only if \( F \) is closed and locally unisolvent.

A portion of the proof of this theorem is found in [5]. There are two points which remain to be established. The first and simplest is that the closure of \( F \) (along with Property Z) implies the existence of best approximations for every continuous function. The second is that the validity of Statements A and B imply that \( F \) is closed.

**Lemma 1.** If \( F \) is closed and has Property Z then Statement A is valid for every continuous function.

**Proof.** Let \( f(x) \) be a given function continuous on \([0, 1]\) and \( F(A', x) \) an approximating function. Denote by \( P' \) the parameters

\[ P' = \{A | \max_{x \in [0, 1]} |F(A, x) - f(x)| \leq \max_{x \in [0, 1]} |F(A', x) - f(x)| \}. \]
This set is not empty since it contains \( A' \). Furthermore it is a bounded set, i.e., there exists an \( M < \infty \) such that \( |F(A, x)| \leq M \) for \( A \in P' \).

There is a sequence \( \{A_k\} \) in \( P' \) such that
\[
\lim_{k \to \infty} \max A_k, x) - f(x) = \inf \max A \in P \, |F(A, x) - f(x)|.
\]

It is known \(^5, \text{Theorem 2}\) that Property Z implies the existence of a pointwise convergent subsequence of every infinite sequence in \( P' \). If \( P' \) contains only a finite number of parameter sets then Statement A is clearly valid for \( f(x) \). Thus the sequence \( \{F(A_k, x)\} \) contains a pointwise convergent subsequence and, by the hypothesis of closure, this subsequence possesses a limit \( F(A_0, x) \) which is a best approximation to \( f(x) \).

In order to establish the second point, one would like to construct for \( G(x) \) a continuous function \( f(x) \) such that \( f(x) - G(x) \) alternates \( n \) times at \( n + 1 \) specified points in \([0, 1]\). This would imply (after some arguments) that \( G(x) \) is a best approximation to \( f(x) \) and hence (by Statement A) that \( G(x) \equiv F(A_0, x) \). However it is not possible at this point to construct such an \( f(x) \) since \( G(x) \) is an unknown and possibly highly discontinuous function. This difficulty is circumvented in Lemma 2 where two functions associated with \( G(x) \) are introduced for which one may construct the required continuous function \( f(x) \). These two functions are

\[
G^+(x) = \max \left[ G(x), \limsup_{|x - y| \to 0} G(y) \right],
\]
\[
G^-(x) = \min \left[ G(x), \liminf_{|x - y| \to 0} G(y) \right].
\]

Since \( G(x) \) is a bounded function, both of these functions are well defined.

**Lemma 2.** Given \( G(x) \) bounded on \([0, 1]\), \( M > 0 \), \( \delta_0 > 0 \) and \( x_0 \in [0, 1] \) there exists a continuous function \( f(x) \) such that
\[
f(x) - G^+(x)
\]
has a minimum \(-M\) at \( x_0 \) in the interval \( |x - x_0| \leq \delta_0 \). Further
\[
f(x_0 \pm \delta_0) - G^+(x_0 \pm \delta_0) = 0.
\]

**Proof.** Set
\[
\omega^+(\delta) = \sup \left[ G^+(y) - G^+(x_0) \right], \quad 0 < |x_0 - y| < \delta.
\]

This is an “upper modulus of continuity” of \( G^+(x) \) at \( x_0 \). It is also the upper-semicontinuous function \( u(x) \) (upper boundary function) described in \([6, \text{Chapter 7}]\). If
\[
G(x_0) = G^+(x_0) > \limsup_{|x_0 - y| \to 0} G(x)
\]
then clearly
\[ \lim_{\delta \to 0} \omega^+(\delta) < 0. \]

When (2.9) does not hold then we have the following assertion: If
\[ G(x_0) \leq G^+(x_0) \]
then
\[ \lim_{\delta \to 0} \omega^+(\delta) = 0. \tag{2.10} \]

The basic reason that this assertion is true is that \( G^+(x) \) itself is upper-semicontinuous
Assume (2.10) to be false, then there is an \( \varepsilon > 0 \) and a sequence \( \{x_i\} \) tending to \( x_0 \) such that
\[ G^+(x_0) < G^+(x_i) - \varepsilon, \quad i = 1, 2, \ldots. \]
This contradicts the fact
\[ G^+(x_0) \geq \limsup_{|x_0-y| \to 0} G^+(y) \]
which may be established by a straightforward argument.

A construction is now given to establish the following
**Assertion.** There exists a continuous function \( \omega(\delta) \) such that for \( 0 \leq \delta \leq \delta_0 \)
\[ \omega^+(\delta) \leq \omega(\delta) \tag{2.11} \]
and if (2.10) holds then \( \omega(0) = 0. \)

Note that \( \omega^+(\delta) \) is a monotonic nondecreasing function. Define
\[ \omega(\delta) = \frac{1}{\delta} \int_{\delta}^{2\delta} \omega^+(x) \, dx, \quad \delta > 0, \]
\[ \omega(0) = \limsup_{|x_0-y| \to 0} G(y) - G^+(x_0). \]
It follows immediately from the mean value theorem that
\[ \omega^+(\delta) \leq \omega(\delta) \leq \omega^+(2\delta), \quad \delta > 0. \]
It is clear that \( \omega(\delta) \) is a continuous function and further if (2.10) holds then \( \omega(0) = 0. \)

The function \( f(x) \) required in this proof is now constructed. If \( \omega(\delta_0) \leq 0 \) set
\[ f(x) = G^+(x_0) - M + 2|x - x_0|. \]
Then if \( |x - x_0| \leq \delta_0 \) one has
\[ f(x) - G^+(x) = G^+(x_0) - G^+(x) - M + 2|x - x_0| \geq -M + 2|x - x_0|. \]
If \( \omega(\delta_0) > 0 \) set \( \omega'(\delta) = \max[\omega(\delta), 0] \) and
\[ f(x) = G^+(x_0) - M + 2\omega'(|x - x_0|). \]
Then

\[ f(x) - G^+(x) = G^+(x_0) - G^+(x) - M + 2\omega'(|x - x_0|) \]

\[ \geq -M + 2\omega'(|x - x_0|) - \omega^+(|x - x_0|) \]

\[ \geq -M + \omega'(|x - x_0|) \geq -M. \]

This construction gives an \( f(x) \) satisfying the minimum requirement of the lemma. The construction may be easily modified so that

\[ f(x_0 \pm \delta_0) = G^+(x_0 \pm \delta_0). \]

It is clear that the same type of construction is applicable to the

**Corollary.** Given \( G(x) \) bounded on \([0, 1]\), \( M > 0 \), \( \delta_0 > 0 \) and \( x_0 \in [0, 1] \) then there exists a continuous function \( f(x) \) such that

\[ f(x) - G^+(x) \]

has a local maximum of \( M \) at \( x_0 \) in the interval \( |x - x_0| \leq \delta_0 \). Further

\[ f(x_0 \pm \delta_0) - G^+(x_0 \pm \delta_0) = 0. \]

The next lemma is required to establish the second point of the proof of Theorem 1. This lemma is a restatement of some results in [5], particularly Lemma 4 and Theorem 1.

**Lemma 3.** Assume Statement B is valid for every continuous function. If

\[ \max_{i \neq j} |F(A, x_i) - f(x_i)| \leq 2M, \quad i = 1, 2, \cdots, n + 1 \]

then either (i) \( F(A, x) - f(x) \) alternates \( n \) times on \( \{x_i\} \) with deviation \( 2M \) or (ii) there is an \( A_0 \in \mathcal{P} \) such that

\[ |F(A_0, x_i) - f(x_i)| < 2M, \quad i = 1, 2, \cdots, n + 1. \]

The next lemma establishes the second point required for the proof of Theorem 1.

**Lemma 4.** If Statements A and B are valid for every continuous function then \( F \) is closed.

**Proof.** Assume that

\[ \lim_{k \to \infty} F(A_k, x) = G(x), \quad x \in [0, 1] \]

with

\[ |F(A_k, x)| \leq M. \]

Let \( n + 1 \) points be given

\[ 0 \leq x_1 < x_2 < \cdots < x_{n+1} \leq 1 \]

and set \( \delta_0 = \frac{1}{n} \min |x_j - x_{j+1}|. \)
By Lemma 2 a continuous function \( f_1(x) \) may be defined by (2.12) so that
\[
(2.14) \quad f_1(x_j) - G^-(x_j) = +2M, \quad j = 1, 3, 5, \ldots, \\
f_1(x_j) - G^+(x_j) = -2M, \quad j = 2, 4, \ldots
\]

and these points are local extrema of \( f_1(x) - G^+(x) \) and \( f_1(x) - G^-(x) \) in \([x_j - \delta, x_j + \delta] \). Further, the definition of \( f_1(x) \) may be extended to the remainder of \([0, 1]\) so that
\[
(2.15) \quad |f_1(x) - G(x)| \leq M, \quad |x - x_j| > \delta.
\]

Since \( f_1(x) \) is continuous, the assumption that Statement A is valid implies that \( f_1(x) \) possesses a best approximation \( F(A^1, x) \). The assumption that Statement B is valid implies \([5,\text{Lemma 3}]\) that \( F \) has Property Z. This fact is used essentially in the remainder of the proof.

The following assertion is now established:

**Assertion 1.**
\[
(2.16) \quad \max |f_1(x) - F(A^1, x)| \leq 2M.
\]

Since Statement B is valid there are \( n + 1 \) points \( \{y_j \mid y_j < y_{j+1}\} \) such that
\[
F(A^1, y_j) - f_1(y_j) = (-1)^j K,
\]
where \( K = \pm \max |f_1(x) - F(A^1, x)| \). If \( |K| > 2M \) then since \( |G(y_j) - f_1(y_j)| \leq 2M < |K| \) one has
\[
[F(A^1, y_j) - G(y_j)](-1)^j \text{sgn}[K] \geq |K| - 2M > 0.
\]
For \( k \) sufficiently large one has
\[
|F(A_k, y_j) - G(y_j)| < \frac{1}{2}(|K| - 2M)
\]
and hence
\[
\text{sgn}[F(A_k^1, y_j) - F(A_k, y_j)] = (-1)^j \text{sgn}[K].
\]
This implies that \( F \) does not have Property Z which contradicts the assumption that Statement B is valid for all continuous functions. This establishes the assertion (2.16).

We now establish

**Assertion 2.**
\[
(2.17) \quad |F(A^1, x_j) - f_1(x_j)| = 2M, \quad j = 1, 2, \ldots, n + 1
\]

It follows from the first assertion that
\[
(2.18) \quad |F(A^1, x_j) - f_1(x_j)| \leq 2M, \quad j = 1, 2, \ldots, n + 1.
\]

Lemma 3 implies that either (2.17) follows from (2.18) or there is an \( A_0 \in P \) such that
(2.19) \[ |F(A_0, x_j) - f_1(x_j)| < 2M, \quad j = 1, 2, \ldots, n + 1. \]

It is now shown that the alternative (2.19) leads to a contradiction.

Set
\[ \varepsilon = \min_{j} [2M - |F(A_0, x_j) - f_1(x_j)|] > 0. \]

There is an \( \eta > 0 \) such that \( |x - x_j| < \eta \) implies
\[ |F(A_0, x_j) - F(A_0, x)| < \frac{\varepsilon}{3}, \quad j = 1, 2, \ldots, n + 1. \]

Further, there exists a \( y_j \) with \( |y_j - x_j| < \eta \) so
\[ |G(y_j) - G^+(x_j)| < \frac{\varepsilon}{3}, \quad j = 1, 2, \ldots, n + 1. \]

One may choose \( k \) so large that
\[ |G(y_j) - F(A_k, y_j)| < \frac{\varepsilon}{3}, \quad j = 1, 2, \ldots, n + 1. \]

With alternative (2.19) it follows from these estimates that
\[ F(A_0, y_j) > F(A_k, y_j), \quad j \text{ odd}, \]
\[ F(A_0, y_j) < F(A_k, y_j), \quad j \text{ even}. \]

This implies that \( F(A_0, x) - F(A_k, x) \) has \( n \) zeros which is impossible. This establishes the assertion.

These two assertions imply that
\[ F(A^1, x_j) = G^-(x_j), \quad j \text{ odd}, \]
(2.20)
\[ F(A^1, x_j) = G^+(x_j), \quad j \text{ even}. \]

A similar construction of a continuous function \( f_2(x) \) and an analysis of a best approximation \( F(A^2, x) \) to it leads to
\[ F(A^2, x_j) = G^+(x_j), \quad j \text{ odd}, \]
(2.21)
\[ F(A^2, x_j) = G^-(x_j), \quad j \text{ even}. \]

It follows from (2.20) and (2.21) that
\[ (2.22) \quad [F(A^2, x_j) - F(A^1, x_j)](-1)^{j+1} \geq 0, \quad j = 1, 2, \ldots, n + 1. \]

Since \( F \) must have Property Z, this implies that
\[ (2.23) \quad F(A^2, x_j) = F(A^1, x_j). \]

Since
\[ G^+(x) \geq G(x) \geq G^-(x) \]

it follows that
(2.24) \[ G(x_j) = F(A^1, x_j), \quad j = 1, 2, \cdots, n + 1. \]

One may fix \( n \) distinct points and let the \((n + 1)\)st point be variable. Then (2.24) is valid with \( A^1 \) replaced by a new parameter set \( A_0 \). However, on the \( n \) fixed points one has \( F(A^1, x_j) = F(A_0, x_j) \), \( j = 1, 2, \cdots, n \) which implies that \( F(A^1, x) = F(A_0, x) \). Thus one has for any \( x \),

\[ G(x) = F(A^1, x). \]

This is the approximating function required in this lemma and concludes the proof.

**Proof of Theorem 1.** There are two implications to be established: (i) local unisolvence and closure imply Statements A and B and (ii) Statements A and B imply local unisolvence and closure.

For the first implication we have shown (Lemma 1) that local unisolvence (which includes Property Z) and closure imply Statement A. It is known \([5, \text{Theorem 3}]\) that local unisolvence and closure imply Statement B.

For the second implication it is known \([5, \text{Lemma 3}]\) that Statement B implies Property Z. We have shown (Lemma 4) that Statements A and B imply closure. It is known \([5, \text{Theorem 3}]\) that Statement B and closure (and hence Statements A and B) imply local unisolvence. This concludes the proof.

3. **Example.** For linear approximating functions, it is known \([4]\) that the classical\(^{(1)}\) approximating functions are the only ones for which Statements A, B and C are all valid for all continuous functions. The simplest and most "classical" nonlinear approximating functions are the unisolvent functions \([3]\). It is known that Statements A, B and C are valid for these approximating functions. One might conjecture then that these are the only approximating functions for which Statements A, B and C are all valid. That this is not true is seen by the simple example

\[ F(A, x) = \frac{a}{1 + ax}, \quad -1 < a < +1, \quad -1 \leq x \leq +1. \]

One may easily verify that the three Statements A, B and C are valid for all continuous functions. The range of this approximating function is shown in Figure 1.

4. **Remark on Haar's problem and uniqueness.** The following results are known \([5, \text{Lemma 3, and classical}]\):

1. If Statement B is valid for every continuous function then \( F \) has Property Z.
2. If \( F \) has Property Z then Statement C is valid for every continuous function. These results have the obvious corollary, which has not been explicitly stated previously.

\footnote{\(^{(1)}\) The classical linear approximating functions are \( F(A, x) = \Sigma_{t=1}^n a_t \varphi_t(x) \), where \( P = E_n \) and \( \{ \varphi_t(x) \} \) is a Tchebycheff set, i.e., \( F \) has Property Z.}
Figure 1. The range of $F(A, x) = \frac{a}{1 + ax}$

**Theorem 2.** If Statement B is valid for every continuous function then Statement C is valid for every continuous function.

Thus Haar's problem [1; 2] which is the study of the implications of Statement C is subsumed by the study of the implications of Statement B.

**References**