

# BOUNDARY IMAGES OF MEROMORPHIC FUNCTIONS

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## 1. Introduction.

1.1. Let  $D$  be the open unit disk,  $\text{bdy } D$  its bounding circle, and  $S$  the unit sphere. If  $f: \bar{D} \rightarrow S$  is continuous, then  $f(\text{bdy } D)$  is a Peano continuum. (We remind the reader that a Peano continuum is a locally connected [metric] continuum, and that it is characterized as the continuous image of a closed interval.)

It has been shown (by G. R. MacLane [8]; Piranian, Titus, and Young [12]; Salem and Zygmund [16]; and Schaeffer [17]) that there exist functions  $f$ , continuous on the closed disk  $\bar{D}$  and analytic on  $D$ , for which  $f(\text{bdy } D)$  is a solid square. Indeed Rudin [14] has proved the following theorem: Suppose (a)  $E$  is a closed subset of the circle  $\text{bdy } D$ ,  $E$  having Lebesgue measure zero; (b)  $\phi$  is a continuous function on  $E$  into the complex plane; (c)  $T$  is a two-cell such that  $\phi(E) \subset T$ . Then there exists a function  $f$  analytic on  $D$  and continuous on  $\bar{D}$  such that (i)  $f(z) = \phi(z)$  for all  $z$  in  $E$ ; (ii)  $f(\bar{D}) \subset T$ . Thus, given any compact set  $C$  in the complex plane, there is a continuous function  $\phi$  of the Cantor set  $E \subset \text{bdy } D$  onto  $C$  [25, p. 35, (4.7)], and  $C \subset f(\text{bdy } D)$  for the  $f$  given above.

The author [4] has shown that there is a Peano continuum in the plane which is not the image of the circle  $\text{bdy } D$  for any function meromorphic in  $D$  and continuous in  $\bar{D}$ . A purely topological characterization of those Peano continua which are boundary images is given here.

**THEOREM.** *A nonempty Peano continuum  $P$  on  $S$  satisfies either both or neither of the following conditions:*

(C<sub>1</sub>) *There exists a map  $f: \bar{D} \rightarrow S$ , meromorphic on  $D$ , for which  $P = f(\text{bdy } D)$ .*

(C<sub>2</sub>) *There exists a finite or countably infinite family of simply connected regions  $U_n$  ( $n = 1, 2, \dots$ ) such that:*

(a)  *$P$  is the closure of  $\bigcup_n \text{bdy}(U_n)$ ;*

(b) *for each  $n > 1$ , there exists  $m < n$  for which  $U_n \cap U_m \neq \emptyset$  and*

$$\text{bdy}(U_n) \cap \text{bdy}(U_m) \neq \emptyset;$$

*and, if the family is infinite,*

(c)  *$\limsup(U_n) \subset P$ .*

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The *limit superior* of  $\{U_n\}$  is the set of all points  $x$  in  $S$  such that every neighborhood of  $x$  contains points of infinitely many regions  $U_n$ . If (c) is replaced by “ $\text{diam}(U_n) \rightarrow 0$ ,” the resulting condition is equivalent (see 3.3 and 3.6). Condition  $C'_1$  is defined from  $C_1$  by replacing “meromorphic” by “analytic”; Condition  $C'_2$  is defined from  $C_2$  by adding the requirement that no region  $U_n$  contain infinity. *If  $P$  is a nonempty Peano continuum on  $S$ , other than a single point, then  $P$  satisfies  $C'_1$  if and only if it satisfies  $C'_2$ .*

Condition  $C_2$  has a degree of naturalness. Given any Peano continuum  $P$  on  $S$ , each of the countable number of components  $V_m$  of  $S - P$  is a simply connected region (indeed, its boundary is locally connected [25, p. 106, (2.2)]), and the diameters converge to zero [25, p. 113, (4.4)]. There is a countable family of disks  $D_m$  in  $\text{int} P$ , such that the bounding circles are dense in  $\text{int} P$  and  $\text{diam}(D_m) \rightarrow 0$ . The sets  $V_m$  and  $D_m$ , together, constitute regions  $U_n$  satisfying Condition  $C_2$  except for (b). In “most” cases, these sets can be modified to satisfy (b).

For example, suppose that  $P$  is a Peano continuum such that  $\dim P = 1$  and  $S - P$  has a finite number of components. Let  $V_m$  ( $m = 1, 2, \dots, k$ ) be the components of  $S - P$ , and let  $v_m$  be points on  $\text{bdy}(V_m)$ . Let  $\gamma_m$  ( $m = 1, 2, \dots, k - 1$ ) be arcs [24, p. 81, (3.11)] in  $P$  from  $v_m$  to  $v_{m+1}$  ( $\gamma_m = \{v_m\}$  if  $v_m = v_{m+1}$ ), let  $U_{2m-1} = V_m$  ( $m = 1, 2, \dots, k$ ) and  $U_{2m} = S - \gamma_m$  ( $m = 1, 2, \dots, k - 1$ ). Then  $P$  satisfies  $C_2$ , and thus  $C_1$ .

While the example of [3, p. 52, Remark] has dimension one, its complement has an infinite number of components, and it fails to satisfy  $C_1$  and  $C_2$ . If  $\{U_n\}$  is any sequence satisfying (a) and (b) of  $C_2$ , then an infinite number of regions  $U_n$  will contain the unbounded complementary component; thus the example will not satisfy  $C_2$  (c).

Also the universal plane curve and the triangular curve of Sierpiński [7, pp. 202–203] both satisfy  $C_2$  and thus  $C_1$ . In the latter case, using the notation of [7], the regions  $U_n$  in order are:  $T, T_0, T_1, T_2, T_{00}, T_{01}$ , etc.

In a different but related direction, Morse [9, p. 74], and Titus [21; 22] have given conditions for a function (i.e., a *parametrized curve*) to be a boundary function for a function  $f$  analytic on  $D$ , and continuous on  $\bar{D}$  (see §5).

1.2. Given a meromorphic function  $f: D \rightarrow S$ , the *global cluster set*  $C(f)$  is defined as follows:  $y \in C(f)$  if there exists a sequence of points  $z_n$  in  $D$  such that  $\lim |z_n| = 1$  and  $\lim f(z_n) = y$ . The set  $C(f)$  is a continuum, and, of course, if  $f$  is continuous on  $\bar{D}$ , then  $C(f) = f(\text{bdy} D)$ . In [5, p. 123], Collingwood and Cartwright asked whether every continuum  $C \subset S$  could arise as the global cluster set of some meromorphic function. Independently D. B. Potyagaïlo [13] and W. Rudin [15] gave a counter-example, which is not a Peano continuum. In [4] the author gave the Peano continuum example mentioned earlier, and a sufficient condition. (The condition is not equivalent, even for Peano continua, to a sufficient condition given by Potyagaïlo in [13].) For Peano continua that

condition [4, p. 53] is almost identical to, and is equivalent to (see 3.6)  $C_2$ . We now observe that for the class of Peano continua at least, the condition is also necessary.

**THEOREM.** *For Peano continua  $P \neq \emptyset$ , Condition  $C_2$  is equivalent to Condition  $C_3$ : There exists a function  $f$  meromorphic on  $D$  for which  $P$  is the global cluster set  $C(f)$ .*

Condition  $C'_3$  (infinity not in the range of  $f$ ) is equivalent to  $C'_2$ , except for single points.

The following condition,  $C_4$ , implies the denial of  $C_3$  for (arbitrary) continua  $P$ .

*There exist components  $K_i$  ( $i = 1, 2, \dots, I$ ) of  $S - P$  and a sequence  $\{R_n\}$  of regions such that, for each  $n$  ( $n = 1, 2, \dots$ ):*

- (a)  $R_n \neq S$ ,
- (b)  $P \cap R_n \neq \emptyset$ ,
- (c)  $\text{bdy}(R_n) \subset \bigcup_{i=1}^I \text{bdy}(K_i)$ , and
- (d) *each open set  $W$  which meets  $\text{bdy}(R_n)$ , contains a region  $R_m$  having  $m > n$ .*

The counter example of [4, §2] satisfies this condition where  $I = 1$ ,  $K_1 = S - P$ , and the regions  $R_n$  are the (open) disks. The proof that Condition  $C_4$  implies the denial of  $C_1$  is a generalization of that proof.

**Proof.** Suppose that there exists a continuum  $P$  satisfying both  $C_3$  and  $C_4$ . Then  $P$  is not a single point, so that  $f(D)$  is open. Let  $C$  be the set of points in  $D$  at which  $f$  is not one-to-one. Let  $B_i$  be  $K_i - f(C)$ , and let  $E_i$  be  $f^{-1}(B_i)$  ( $i = 1, 2, \dots, I$ ). As in [4, p. 52], if  $E_i \neq \emptyset$ , then  $E_i$  is a covering space of the base space  $B_i$  with projection map  $f$  and degree  $n_i$ ,  $1 \leq n_i < \infty$ . (If  $E_i = \emptyset$ , let  $n_i = 0$ .)

For each  $\varepsilon$ ,  $0 < \varepsilon < 1$ , let  $A_\varepsilon$  be the open annular region in  $D$  at distance less than  $\varepsilon$  from the circle  $\text{bdy } D$ . By (a) and (d) there exists a natural number  $n(1)$  such that  $\text{diam}(R_{n(1)}) < \text{diam } P$ . Since  $P$  satisfies Condition  $C_3$ ,  $\text{diam } P \leq \text{diam}(f(D))$ , and by (b)  $f(D)$  meets  $R_{n(1)}$ . Thus  $f(D)$  meets  $\text{bdy}(R_{n(1)})$ . As a result, there is an  $\varepsilon(1) > 0$  and a set  $U_1$  open in  $D - A_{\varepsilon(1)}$  such that  $\text{diam}(f(U_1)) < \text{diam } P$  and  $f(U_1)$  meets  $\text{bdy}(R_{n(1)})$ .

In general, suppose that open sets  $U_j \subset D$ , an integer  $n(k)$ , and  $\varepsilon(k) > 0$  have been given such that the sets  $U_j$  are mutually disjoint,  $f(U_j) \subset f(U_{j-1})$  ( $j = 1, 2, \dots, k$ ;  $U_0 = D$ ),  $\bigcup_{j=1}^k U_j$  does not meet  $A_{\varepsilon(k)}$ , and  $f(U_k)$  meets  $\text{bdy}(R_{n(k)})$ . By (d) there exists  $n(k+1)$  such that  $R_{n(k+1)} \subset f(U_k)$ . By (b)  $f(A_{\varepsilon(k)})$  meets  $R_{n(k+1)}$ , and thus meets  $\text{bdy}(R_{n(k+1)})$ . Hence there is an  $\varepsilon(k+1) > 0$  and an open set  $U_{k+1} \subset A_{\varepsilon(k)} - A_{\varepsilon(k+1)}$  such that  $f(U_{k+1}) \subset f(U_k)$  and  $f(U_{k+1})$  meets  $\text{bdy}(R_{n(k+1)})$ .

For each  $k$  ( $k = 1, 2, \dots$ ), there exists a component  $K_i$  ( $i = 1, 2, \dots, I$ ) such that  $f(U_k)$  meets  $K_i$  (by (c)). Thus  $n_i \geq k$ , and a contradiction results.

1.3. **REMARK.** Let  $g$  be a function analytic on  $D$  and continuous on  $\bar{D}$  such that  $g(\text{bdy } D) = \bar{D}$  (given, for example, by [12]). Suppose that  $g \notin (\bar{D})\bar{D}$ . Since infinity is not in  $g(\bar{D})$ ,  $\text{bdy}(g(\bar{D}))$  meets  $S - \bar{D}$ , say in  $q$ . Since  $q \notin \bar{D}$ ,  $q \notin g(\text{bdy } D)$ .

Thus  $q \in g(D)$ . Since  $g(D)$  is open, we have a contradiction. Thus  $g(\bar{D}) \subset \bar{D}$ , so that  $g(D) \subset D$ .

Let  $E$  be the upper half-plane, considered as a topological disk on  $S$ , and let  $h$  be a homeomorphism of  $\bar{D}$  on  $\bar{E}$  (closure in  $S$ ),  $h$  analytic on  $D$ . The function  $f$  defined by  $f(z) = (h(g(z)))^2$  is analytic on  $D$ , and  $f(\text{bdy } D) = S$ ; thus,  $S$  satisfies Condition  $C'_1$ . (This conclusion could have been proved from results in this paper, without the use of [12].)

Let  $\mathfrak{D}_j$  be the set of all open squares with sides parallel to the real and imaginary lines, sides of length  $2^{-j+1}$ , and centers  $(m + ik)2^{-j}$  ( $m, k = 1, 3, \dots, 2^j - 1$ ;  $j = 1, 2, \dots$ ). The squares of  $\mathfrak{D}_j$ , taken in order, constitute a sequence  $\{V_n\}$  of regions for the unit square  $\{x + iy : 0 \leq x, y \leq 1\}$  satisfying  $C'_2$ . Now  $S = D_1 \cup D_2$ , where  $D_1$  and  $D_2$  are closed topological disks. Let  $p \in D_1 \cap D_2$ , and let  $h_t$  be a homeomorphism of the unit square onto  $D_t$  ( $t = 1, 2$ ). The sequence  $\{U_n\}$  of regions defined by  $U_1 = S - \{p\}$ ,  $U_{2n} = h_1(V_n)$ , and  $U_{2n+1} = h_2(V_n)$  ( $n = 1, 2, \dots$ ) satisfies Condition  $C'_2$  for  $S$ .

Also, a single point satisfies both  $C_1$  and  $C_2$ . Thus, to prove 1.1, it is sufficient to assume hereafter that  $P$  is a fixed nonempty Peano continuum, neither  $S$  nor a single point.

The outline of the proof is given now. After preliminary lemmas (§2), it is shown (3.1) that each of Conditions  $C_2$  and  $C_3$  implies a certain condition. This condition implies (3.2, 3.5, 3.6) the existence of a chain (3.3), which is a family of regions  $U_n$  satisfying the properties of  $C_2$  and some others. As a result, Condition  $C_3$  implies  $C_2$ . In 4.2 it is proved that the existence of a chain implies  $C_1$ , except that  $f$  is only light open. The existence of the meromorphic function is given in 4.3, using the extension of Stoilow's Theorem given in [3], so that  $C_2$  implies  $C_1$ . Clearly,  $C_1$  implies  $C_3$ ; thus, the three conditions are equivalent, as are  $C'_1, C'_2$ , and  $C'_3$ . The special case in which the function of  $C_1$  is a homeomorphism on  $D$  is treated in 2.1, and extensions are discussed in §5.

The closure of  $X$  will be denoted either by  $\text{Cl}[X]$  or by  $\bar{X}$ ; the interior of  $X$ , by  $\text{int } X$ ; the distance between two points  $p$  and  $q$  on  $S$ , by  $d(p, q)$ ; the set of points of distance from  $p$  at most  $\varepsilon$ , by  $S(p, \varepsilon)$ ; and the null set by  $\emptyset$ . The term "map" will mean "continuous function."

**2. Preliminary results.** It is well known [1, p. 86] that if  $f$  is a conformal homeomorphism of the open unit disk  $D$  onto the interior  $X$  of a closed topological disk, then there is an extension of  $f$  to a homeomorphism of  $\bar{D}$  onto  $\bar{X}$ . The following result generalizes this theorem.

**2.1. THEOREM.** *If  $B$  is a simply connected region on  $S$  such that  $\text{bdy } B$  is a non-degenerate Peano continuum, then each prime end is a single point. Thus, if  $f$  is any conformal homeomorphism of  $D$  onto  $B$ , then  $f$  can be extended to  $\bar{D}$  to be continuous.*

**Proof.** We use the terminology of Piranian [11, pp. 45–46]; the term “chain” here should not be confused with that of 3.3. Suppose that there exists a prime end  $E$  whose impression  $I(E)$  contains (at least) two distinct points  $p$  and  $q$ ; we may suppose that  $p$  is a principal point. The equivalence class  $E$  contains a chain  $\{c_n\}$  such that each  $c_n \in S(p, 1/n)$  and the sets  $B_n$  are nested. (For each crosscut  $c_n$ ,  $B_n$  is that component of  $B - c_n$  which contains  $c_{n+1}$ ; the impression  $I(E)$  is  $\bigcap_n \bar{B}_n$ .) Let  $\{p_n\}$  and  $\{q_n\}$  be sequences of points such that each  $p_n \in c_n \cap B$ ,  $p_n \rightarrow p$ , each  $q_n \in B_{n-1} - B_n$ , and  $q_n \rightarrow q$ . Let  $\{\gamma_n\}$  be a sequence of arcs in  $B$ , each  $\gamma_n$  joining  $q_{n-1}$  to  $q_n$  and meeting  $c_n$  only in  $p_n$ . (Thus  $\gamma_n$  passes “through”  $c_n$ .) Choose  $\varepsilon > 0$  such that neither  $q$  nor any  $q_n$  is in  $S(p, \varepsilon)$ . Since  $\text{bdy } B$  is locally connected,  $S(p, \varepsilon) \cap \text{bdy } B$  contains a connected neighborhood  $N$  of  $p$  in  $\text{bdy } B$ ; there exists a natural number  $n$  such that

$$S(p, 1/n) \cap \text{bdy } B \subset N.$$

The arc  $\gamma_n$  separates  $S(p, \varepsilon)$  into at least two components, one containing  $p$ , another containing an endpoint  $r$  of  $c_n$ , with  $r$  in  $\text{bdy } B \cap S(p, 1/n)$ . The connectivity of  $N$  is thus contradicted, so that  $I(E)$  must be  $\{p\}$ .

The second statement follows from Carathéodory’s principal theorem on prime ends [2, p. 350].

2.2. DEFINITION. Let  $U$  be  $S$ , or  $D$  where  $\text{bdy } D \subset P$ . Let  $\varepsilon > 0$ , and let  $K_i$  ( $i = 1, 2, \dots, I$ ) be those components of  $U - P$  having  $\text{diam}(K_i) \geq \varepsilon/3$  (there are only a finite number of them [25, p. 113, (4.4)]). An  $\varepsilon$ -region (for  $P$  and  $U$ ) is a component of  $U - \bigcup_{i=1}^I K_i$ .

REMARKS. If  $V$  is a region in  $U$ ,  $\text{bdy } V \subset P$ , and  $\text{diam } V \leq \varepsilon/3$ , then  $V$  is contained in some  $\varepsilon$ -region  $R$ . Since  $P \cap \bar{D}$  is a Peano continuum [25, p. 112, (4.2) and p. 113, (4.4)], the  $\varepsilon$ -regions for  $P$  and  $D$  are the  $\varepsilon$ -regions for  $P \cap \bar{D}$  and  $S$ , except possibly for  $S - \bar{D}$ . Thus, it suffices to prove 2.3, 2.4, 2.5, and 2.6 for  $U = S$ .

2.3. LEMMA. For each  $\varepsilon > 0$ , an  $\varepsilon$ -region  $R$  has Property S [25, p. 20], and thus  $\bar{R}$  is a Peano continuum.

**Proof.** Let  $X_j$  ( $j = 1, 2, \dots, J$ ) be the components of  $\bigcup_{i=1}^I K_i$ . Since each  $\text{bdy}(K_i)$  is locally connected [25, p. 106, (2.2)], each  $K_i$  has Property S [25, p. 109, (3.2)], and  $\bar{K}_i$  does also [25, p. 20, (15.3)]; thus, each  $X_j$  has Property S and is locally connected. Let  $S_j$  be that component of  $S - X_j$  containing  $R$  ( $j = 1, 2, \dots, J$ ); then  $S_j$  has Property S [25, p. 106, (2.2) and p. 109, (3.2)].

Let  $\delta$  be any positive number less than one half the minimum distance from any of the sets  $\text{bdy}(S_j)$  to any other of them (since  $\text{bdy}(S_j) \subset X_j$ ,  $\delta > 0$ ;  $j = 1, 2, \dots, J$ ). Each  $S_j$  is the union of a finite family  $\mathfrak{Y}$  of connected subsets, each of diameter less than  $\delta$ ; let  $Y_k$  ( $k = 1, 2, \dots, N$ ) be those subsets (for all  $j$ ) contained in  $R$ . We will show that  $R \subset \bigcup_{k=1}^N Y_k$ . Let  $p \in R$ . If  $d(p, \text{bdy}(S_j)) \geq \delta$ , for all  $j$ , then any connected subset  $Y \in \mathfrak{Y}$  containing  $p$  is contained in  $R$ ; thus

it is a set  $Y_k$ . If  $d(p, \text{bdy}(S_j)) < \delta$ , for some  $j$ , then  $p \in Y$ , where  $Y \in \mathfrak{Y}$  is a connected subset for  $S_j$ . Since  $\text{diam } Y < \delta$ ,  $Y$  is disjoint from  $\text{bdy}(S_i)$ , for  $i \neq j$ ; as a result,  $Y$  is a set  $Y_k$ . Thus,  $R$  has Property S. Since  $\bar{R}$  also has Property S, it is locally connected [25, p. 20, (15.1) and (15.3)].

2.4. LEMMA. *If, for some  $\varepsilon > 0$ , there are an infinite number of distinct  $\varepsilon$ -regions  $R_n$  ( $n = 1, 2, \dots$ ), then  $\text{diam}(R_n) \rightarrow 0$ .*

**Proof.** We first prove that if  $R_1$  and  $R_2$  are two (ordinary) regions on  $S$  such that  $R_1$  is simply connected and

$$\text{bdy}(R_1) \cap \text{bdy}(R_2) = \emptyset,$$

then  $R_1 \cap R_2$  is a (possibly empty) region. We may suppose that  $R_1 \cap R_2$  is neither  $\emptyset$  nor  $R_2$ . If  $C$  is any component of  $R_1 \cap R_2$ , then there is an arc  $\gamma$  in  $R_2$  joining a point of  $C$  to a point of  $R_2 - (R_1 \cap R_2)$ . As a result,  $\text{bdy } C$  meets  $R_2 \cap \text{bdy}(R_1)$ , and since  $\text{bdy}(R_1)$  is connected,  $\text{bdy}(R_1) \subset R_2$ . Thus there is a neighborhood  $N$  of  $\text{bdy}(R_1)$  (in  $S$ ) such that  $N \cap R_1 \subset C$ ; since  $C$  is an arbitrary component of  $R_1 \cap R_2$ ,  $C = R_1 \cap R_2$ ; i.e.,  $R_1 \cap R_2$  is connected.

Let  $X_j$  ( $j = 1, 2, \dots, J$ ) be the components of  $\bigcup_{i=1}^I K_i$  (cf. 2.2); then, as in the proof of 2.3, each  $X_j$  is a Peano continuum. Let  $\delta > 0$  be given; it is desired to prove that only a finite number of  $\varepsilon$ -regions have diameter at least  $\delta$ .

Let  $R$  be any  $\varepsilon$ -region with  $\text{diam } R \geq \delta$ ; since  $R$  is a component of  $S - \bigcup_{j=1}^J X_j$ , it is a component of  $\bigcap_{j=1}^J R_{j,k(j)}$ , where each set  $R_{j,k(j)}$  is a component of  $S - X_j$ , and  $\text{diam}(R_{j,k(j)}) \geq \delta$ . From the conclusion of the first paragraph and induction, this intersection set is connected and thus is  $R$ . Since only a finite number of components of  $S - X_j$  have diameter at least  $\delta$  [25, p. 113, (4.4)], the conclusion follows.

2.5. LEMMA. *If  $R$  is an  $\varepsilon$ -region, and*

$$\text{diam } R < \varepsilon < \frac{1}{2} \text{ diam } S,$$

*then  $R$  is contained in a closed topological disk  $E$  where  $\text{int } E \subset U$ ,  $\text{diam } E = \text{diam } R$  and  $\text{bdy } E \subset P$ . In fact, if  $\text{diam } R < \varepsilon/3$ , then  $\bar{R} = E$ .*

**Proof.** First, suppose that  $U = S$ . Let  $p \in R$ ; then  $R \subset S(p, \varepsilon)$ . If  $W$  is that component of  $S - \bar{R}$  which contains  $S - S(p, \varepsilon)$ , then  $\text{bdy } W$  is locally connected (2.3 and [25, p. 106, (2.2)]) and thus contains a circle  $C$  separating  $W$  from  $R$  [24, p. 114, (6.7)]. Let  $E$  be the closure of the component of  $S - C$  containing  $R$ . Since  $C \subset \text{bdy } R$ ,  $\text{diam } C \leq \text{diam } R$ . But  $\text{diam } E = \text{diam } C$ . On the other hand, since  $R \subset E$ ,  $\text{diam } R \leq \text{diam } E$ .

Suppose  $\text{diam } R < \varepsilon/3$ . Then  $E$  does not contain any  $K_i$  ( $i = 1, 2, \dots, I$ ), and thus is disjoint from them. Since  $\text{int } E$  is a region it is contained in  $R$ ; therefore, it is  $R$ .

If  $U = D$ , then  $R$  is an  $\varepsilon$ -region for  $S$  and  $P \cap \bar{D}$  (by 2.2); thus, if  $E$  is the closed topological disk given by the above proof, then  $\text{bdy } E \subset P \cap \bar{D}$ . Since  $E \cap R \neq \emptyset$ ,  $\text{int } E \subset D$ .

2.6. LEMMA. *If  $R$  is an  $\varepsilon$ -region,  $\varepsilon < \text{diam } R$ , and  $\varepsilon < \frac{1}{2} \text{diam } S$ , then there are a finite number of closed topological disks  $E_m$  ( $m = 1, 2, \dots, M$ ) such that*

- (1)  $R \subset \bigcup_{m=1}^M \text{int}(E_m)$ ,
- (2) each  $\text{int}(E_m) \subset U$  and  $\text{bdy}(E_m) \subset P$ , and
- (3)  $\text{diam}(E_m) < \varepsilon$ .

**Proof.** We may suppose that  $U = S$ ; the proof in case  $U = D$  follows as in 2.5. By 2.3,  $R$  has Property S. Thus, it is the union of a finite number of open sets, each having diameter less than  $\varepsilon/3$  and having Property S [25, p. 21, (15.41)]. Let  $Y$  be one of these sets; it suffices to construct a closed disk  $E$  containing  $Y$  and also satisfying (2) and (3).

Let  $V_n$  ( $n = 1, 2, \dots$ ) be the components of  $S - P$  which meet  $Y$ , and let  $Z$  be the region  $Y \cup \bigcup V_n$ . Since each  $V_n \subset R$  and each  $\text{diam}(V_n) < \varepsilon/3$  (by the definition of  $\varepsilon$ -region),  $\text{diam } Z < \varepsilon$ . Let  $\delta > 0$ ;  $\bar{Y}$  is the union of a finite number of connected sets  $A_j$ , where  $\text{diam}(A_j) \leq \delta/3$  ( $j = 1, 2, \dots, J$ ). There is a natural number  $N$  such that, for all  $n > N$ ,  $\text{diam}(V_n) < \delta/3$  [25, p. 113, (4.4)]. Let  $B_j$  be the union of  $A_j$  with all the  $\bar{V}_n$  it meets ( $n > N$ ;  $j = 1, 2, \dots, J$ ). Each set  $B_j$  is connected, and  $\text{diam}(B_j) < \delta$ . Since

$$\text{Cl} \left[ \bigcup_{n=1}^{\infty} V_n \right] - \bigcup_{n=1}^{\infty} \bar{V}_n \subset \bar{Y},$$

$Z$  is the union of the sets  $B_j$  ( $j = 1, 2, \dots, J$ ) and  $\bar{V}_n$  ( $n = 1, 2, \dots, N$ ). Because  $\delta$  is an arbitrary positive number,  $Z$  has Property S and thus is locally connected.

Since  $\text{diam } Z < \varepsilon$ , for any  $p$  in  $Z$ ,  $Z \subset S(p, \varepsilon)$ . Let  $W$  be the component of  $S - Z$  containing  $S - S(p, \varepsilon)$ . Since  $\text{bdy } W$  is locally connected and separates  $W$  from  $Z$ , it contains a circle  $C$  which also separates  $W$  from  $Z$  [24, p. 114, (6.7)]. Let  $E$  be the closure of the open topological disk of  $S - C$  containing  $Z$ . Since  $\text{bdy } E \subset Z$ ,  $\text{diam } Z \geq \text{diam } E$ ;  $\varepsilon < \frac{1}{2} \text{diam } S$ , and (3) is satisfied. Suppose that there is a point  $z \in (\text{bdy } Z) - P$ . Since  $V_n \subset Z$  ( $n = 1, 2, \dots$ ),  $z$  is in a component of  $S - P$  which does not meet  $Y$  (and thus  $Z$ ), contradicting the fact that  $z \in \text{bdy } Z$ . Thus  $\text{bdy } Z \subset P$ . Since  $\text{bdy } E \subset \text{bdy } Z$ , (2) is also satisfied.

2.7. DEFINITION. Two simply-connected regions  $U$  and  $V$  on  $S$  meet properly if there is an arc  $\gamma$  ending in a point  $p$  on  $\text{bdy } U \cap \text{bdy } V$ , such that  $\gamma - \{p\} \subset U \cap V$ .

If  $\text{bdy } U$  is locally connected (but not a point), then there is a naturally associated prime end  $E$  of  $U$  whose impression  $I(E)$  [11, p. 45] is  $\{p\}$ . Let  $\varepsilon$  be less than  $\text{diam } U$ ,  $\frac{1}{2} \text{diam}(\text{bdy } U)$ , and  $\frac{1}{2} \text{diam } \gamma$ . There is a subarc  $\gamma_n$  of  $\gamma$  such that  $p \in \gamma_n$ , the other endpoint  $q_n$  of  $\gamma_n$  is on the circle  $\text{bdy}(S(p, \varepsilon/n))$ , and  $\gamma_n - (\{p\} \cup \{q_n\})$

is disjoint from that circle ( $n = 1, 2, \dots$ ). Let  $\lambda_n$  be the closure of the component of  $U \cap \text{bdy}(S(p, \varepsilon/n))$  containing  $q_n$ ; since  $\text{bdy}(S(p, \varepsilon/n))$  meets  $\text{bdy} U$ ,  $\lambda_n$  is a crosscut. The family  $\{\lambda_n\}$  defines a prime end  $E$ , and by (2.1) its impression  $I(E)$  is  $\{p\}$ .

There is a conformal homeomorphism of  $D$  onto  $U$ , and it has a continuous extension  $f$  to  $\bar{D}$  (by 2.1). The set

$$\text{Cl}[f^{-1}(\gamma - \{p\})] \cap \text{bdy} D$$

is connected (e.g., from [25, p. 14, (9.1)]). If it were not a point, then  $f$  would be constant along an arc, contradicting a slight modification of the Riesz-Nevanlinna Theorem [10, p. 19]. Thus  $\text{Cl}[f^{-1}(\gamma - \{p\})]$  is an arc  $\mu$  meeting  $\text{bdy} D$  in a single point  $O$ , which corresponds to the prime end  $E$  under the Carathéodory theorem [2, p. 350]; in fact,  $f(O) = p$ .

**2.8. LEMMA** *Let  $U$  and  $V$  be simply connected proper subregions of  $S$  such that  $\text{bdy} U$  and  $\text{bdy} V$  are locally connected,  $U \cap V \neq \emptyset$ , and  $\text{bdy} U \cap \text{bdy} V \neq \emptyset$ . Then  $U$  and  $V$  meet properly.*

**Proof.** If  $U \subset V$  or  $V \subset U$ , the conclusion follows from [25, p. 112, (4.2)]. Thus, we may suppose that there exist points  $q \in U \cap V$ ,  $r \in U - V$ , and  $s \in V - U$ . Since each of  $\text{bdy} U$  and  $\text{bdy} V$  has Property S, their union has it also; thus it is locally connected. Let  $W$  be the component of

$$S - (\text{bdy} U \cup \text{bdy} V)$$

containing  $q$ ;  $\text{bdy} W$  is locally connected [25, p. 106, (2.2)]. There is an arc  $\alpha$  joining  $q$  to  $r$  in  $U$ ; since  $\alpha \not\subset W$ ,  $\alpha$  meets  $\text{bdy} W$ , and thus  $\text{bdy} W$  meets  $\text{bdy} V$ . Similarly,  $\text{bdy} W$  meets  $\text{bdy} U$ . Since the union of the two closed sets  $\text{bdy} U$  and  $\text{bdy} V$  contains  $\text{bdy} W$ , which is connected, they have a common point  $p$ . There is an arc  $\gamma$  ending at  $p$ ,  $\gamma - \{p\} \subset W$ ; as a result,  $U$  and  $V$  meet properly.

### 3. Chains.

**3.1. LEMMA.** *Let  $P$  satisfy either Condition  $C_2$  or  $C_3$ , and let  $U$  be  $S$ , or  $D$  where  $\text{bdy} D \subset P$ ; suppose that  $U \not\subset P$ . Let  $K$  be a component of  $U - P$ ; let  $\varepsilon > 0$ ,  $\varepsilon$  less than the minimum of  $\text{diam} K$ ,  $\text{diam} P$ , and  $\frac{1}{2} \text{diam} S$ ; and let  $R_m$  ( $m = 1, 2, \dots$ ) be a collection of distinct  $\varepsilon$ -regions (for  $P$  and  $U$ ) such that  $\text{diam}(R_m) < \varepsilon/3$ ,  $P \cap R_m \neq \emptyset$ , and  $\text{bdy}(R_m) \cap \text{bdy} U$  contains no arc.*

*Then, there is a natural number  $M$  and, corresponding to the regions  $R_m$ , two families of nonempty open sets  $X_m$  and  $Y_m$  such that:*

- (a)  $\bar{Y}_m \subset X_m \subset U$ ;
- (b)  $Y_m \cap \text{bdy}(R_m) \neq \emptyset$ ;
- (c) if  $j < m$ , then either  $X_m \subset X_j$  or  $X_m \cap \bar{Y}_j = \emptyset$ ; and
- (d) if  $j(k)$  ( $k = 1, 2, \dots, m$ ) is a function such that  $j(k) < j(k+1)$  and  $X_{j(k+1)} \subset X_{j(k)}$  ( $k = 1, 2, \dots, m-1$ ), then  $m \leq M$ .

**Proof.** There are, actually, two separate proofs. First, suppose that  $P$  satisfies Condition  $C_3$ . Let  $K_i$  ( $i = 1, 2, \dots, I$ ) be as in 2.2 for  $\epsilon, P$ , and  $U$ ; then  $K$  is a region  $K_i$ . If  $Z$  is the set of zeros of the derivative of  $f$  in  $D$ , then  $f(Z) \cap K_i$  has no limit point in  $K_i$ , since  $K_i$  is disjoint from the global cluster set  $C(f) = P$ . The restriction of  $f$  to  $f^{-1}(K_i - f(Z))$  is a covering map (as in [4, p. 52]), and, for each  $q$  in  $K_i - f(Z)$ ,  $f^{-1}(q)$  has the same (finite) number  $M_i$  of points [18, p. 67]; let  $M$  be the maximum of the  $M_i$  ( $i = 1, 2, \dots, I$ ).

Let  $A_n$  be the annular region defined by  $1 - (1/n) < |z| < 1$  ( $n = 1, 2, \dots$ ). Since  $P \cap R_1 \neq \emptyset$ ,  $f(A_1) \cap R_1 \neq \emptyset$ ; but, since

$$\text{diam}(R_1) < \text{diam } P \leq \text{diam}(f(A_1)),$$

$(A_1)$  meets  $S - \bar{R}_1$ . Thus, there exists a set  $W_1$  open in  $A_1$ , such that  $\bar{W}_1 \subset A_1$  and the open set  $f(W_1)$ , call it  $X_1$ , meets  $\text{bdy}(R_1)$ . By 2.5 each set  $\bar{R}_m$  is a closed topological disk, and by hypothesis  $\text{bdy}(R_m) \cap \text{bdy } U$  contains no arc. Hence, we may suppose that  $X_1 \subset U$ . There exists an open set  $Y_1$  such that  $Y_1$  meets  $\text{bdy}(R_1)$ , and  $\bar{Y}_1 \subset X_1$ .

There also exists a natural number  $n(2)$  such that  $\bar{W}_1 \cap A_{n(2)} \neq \emptyset$ , and, in  $A_{n(2)}$ , an open set  $T$  such that  $\bar{T}$  is compact,  $f(T)$  meets  $\text{bdy}(R_2)$  and  $f(T) \subset U$ . There is an open set  $W_2 \subset T$  such that the open set  $f(W_2)$ , call it  $X_2$ , also meets  $\text{bdy}(R_2)$ , and either  $X_2 \subset X_1$  or  $X_2 \cap \bar{Y}_1 = \emptyset$ . Let  $Y_2$  be any open set such that  $Y_2$  meets  $\text{bdy}(R_2)$  and  $\bar{Y}_2 \subset X_2$ .

Using finite induction, suppose that we have defined:

- (1<sub>m</sub>) mutually disjoint open sets  $W_j$  such that  $\bar{W}_j \subset D$ ;
- (2<sub>m</sub>)  $X_j$  as  $f(W_j)$  with  $X_j \subset U$ ; and
- (3<sub>m</sub>) open sets  $Y_j$  such that  $\bar{Y}_j \subset X_j$ ,  $Y_j$  meets  $\text{bdy}(R_j)$ , and, if  $i < j$ , then either  $X_j \subset X_i$  or  $X_j \cap \bar{Y}_i = \emptyset$  ( $j = 1, 2, \dots, m$ ).

There exists a natural number  $n(m + 1)$  such that

$$A_{n(m+1)} \cap \bigcup_{j=1}^m \bar{W}_j = \emptyset,$$

and, in  $A_{n(m+1)}$ , an open set  $T$  such that  $\bar{T}$  is compact,  $f(T)$  meets  $\text{bdy}(R_{m+1})$ , and  $f(T) \subset U$ . There is an open set  $W_{m+1} \subset T$ , such that  $f(W_{m+1})$ , call it  $X_{m+1}$ , meets  $\text{bdy}(R_{m+1})$ , and, for each  $j$  ( $j = 1, 2, \dots, m$ ), either  $X_{m+1} \subset X_j$  or  $X_{m+1} \cap \bar{Y}_j = \emptyset$ . Let  $Y_{m+1}$  be any open set such that  $Y_{m+1}$  meets  $\text{bdy}(R_{m+1})$  and  $\bar{Y}_{m+1} \subset X_{m+1}$ . Then  $W_j, X_j$ , and  $Y_j$  ( $j = 1, 2, \dots, m + 1$ ) satisfy (1<sub>m+1</sub>), (2<sub>m+1</sub>), and (3<sub>m+1</sub>).

Thus, there exist families of sets  $X_m$  and  $Y_m$  satisfying conclusions (a), (b), and (c). For (d), suppose on the contrary, that there exist increasing natural numbers  $j(k)$  ( $k = 1, 2, \dots, M + 1$ ) such that each set  $X_{j(k+1)} \subset X_{j(k)}$ . By (a) and (b),  $X_{j(M+1)}$  meets  $\text{bdy}(R_{j(M+1)})$ , and thus meets one of the components  $K_i$  ( $i = 1, 2, \dots, I$ ). Since  $f^{-1}(X_{j(M+1)})$  has at least  $M + 1$  components, each mapped onto  $X_{j(M+1)}$  by  $f$ , the degree of the restriction of  $f$  to  $f^{-1}(K_i - f(Z))$  is at least  $M + 1$ , contradicting the definition of  $M$ .

Now, suppose that  $P$  satisfies  $C_2$ . If a simply connected region  $U_n$  meets a component  $K_i$  ( $n = 1, 2, \dots; i = 1, 2, \dots, I$ ), then (cf. Condition  $C_2(a)$  and (2.2))  $K_i \subset U_n$ . Since  $\limsup(U_n) \subset P$ , there exists  $M$  such that, if  $n > M$ , then

$$U_n \cap \bigcup_{i=1}^I K_i = \emptyset.$$

Since  $P \cap R_m \neq \emptyset$  ( $m = 1, 2, \dots$ ), there exists (by Condition  $C_2(a)$ ), a region  $U_r$  such that  $\text{bdy}(U_r)$  meets  $R_m$ . Because

$$\text{diam}(R_m) < \text{diam } P,$$

it follows from Condition  $C_2(a)$  that there is a region  $U_s$  whose boundary meets  $S - R_m$ . By  $C_2(b)$  there are regions  $U_{s(t)}$  ( $t = 1, 2, \dots, T$ ) such that  $s(1) = r$ ,  $s(T) = s$ ,  $U_{s(t)}$  meets  $U_{s(t+1)}$  and  $\text{bdy}(U_{s(t)})$  meets  $\text{bdy}(U_{s(t+1)})$  ( $t = 1, 2, \dots, T-1$ ). For some  $t$ ,  $U_{s(t)}$  meets  $\text{bdy}(R_m)$ , and  $\text{bdy}(U_{s(t)})$  meets  $\text{bdy}(R_m)$ . Let  $s(t)$  be denoted by  $n(m)$ . There exists an arc  $\gamma_m$  on  $\text{bdy}(R_m)$  (a topological circle by 2.5) such that  $\gamma_m$  ends in a point  $q_m$  on  $\text{bdy}(U_{n(m)})$  and  $\gamma_m - \{q_m\} \subset U_{n(m)}$  ( $m = 1, 2, \dots$ ).

Since  $R_1 \subset U$  and  $\text{bdy}(R_1) \cap \text{bdy } U$  contains no arc, there exist open sets  $X_1$  and  $Y_1$ , each meeting  $\gamma_1$ , such that

$$\bar{Y}_1 \subset X_1 \subset U \cap U_{n(1)}.$$

If  $q_2 \in \bar{Y}_1$ , choose open sets  $X_2$  and  $Y_2$ , each meeting  $\gamma_2$ , such that

$$\bar{Y}_2 \subset X_2 \subset X_1 \cap U_{n(2)}.$$

(Thus  $\bar{Y}_2 \subset U$ .) In this case, since

$$q_2 \in U_{n(1)} \cap \text{bdy}(U_{n(2)}),$$

$U_{n(1)} \neq U_{n(2)}$ . If  $q_2 \notin \bar{Y}_2$ , choose open sets  $X_2$  and  $Y_2$ , each meeting  $\gamma_2$ , such that

$$\bar{Y}_2 \subset X_2 \subset (U \cap U_{n(2)}) - \bar{Y}_1.$$

Using finite induction, suppose that open sets  $X_j$  and  $Y_j$  have been defined, each meeting  $\gamma_j$ , such that:

(1<sub>m</sub>)  $\bar{Y}_j \subset X_j \subset U$ ; and

(2<sub>m</sub>) for each  $i < j$ , either  $X_j \subset X_i \cap U_{n(j)}$  (in which case  $U_{n(i)} \neq U_{n(j)}$ ), or  $X_j \subset U_{n(j)} - Y_i$  ( $j = 1, 2, \dots, m$ ).

Let  $V$  be an open neighborhood of  $q_{m+1}$  such that: if  $q_{m+1} \notin \bar{Y}_j$ , then  $V \cap \bar{Y}_j = \emptyset$ , and if  $q_{m+1} \in \bar{Y}_j$ , then  $V \subset X_j$  ( $j = 1, 2, \dots, m$ ). Now  $V \cap U_{n(m+1)}$  meets  $\gamma_{m+1}$ ,  $\gamma_{m+1} \subset \bar{U}$ , and  $\gamma_{m+1} \cap \text{bdy } U$  has no arc, so that  $V \cap U_{n(m+1)} \cap U$  also meets  $\gamma_{m+1}$ . Let  $X_{m+1}$  be this open set, and let  $Y_{m+1}$  be an open set also meeting  $\gamma_{m+1}$ , such that  $\bar{Y}_{m+1} \subset X_{m+1}$ . The sets  $X_j$  and  $Y_j$  ( $j = 1, 2, \dots, m+1$ ) satisfy (1<sub>m+1</sub>) and (2<sub>m+1</sub>).

Thus, there exist nonempty open sets  $X_m$  and  $Y_m$  satisfying conclusions (a),

(b), and (c). For (d), suppose that, on the contrary, there exist increasing natural numbers  $j(k)$  ( $k = 1, 2, \dots, M + 1$ ) such that  $X_{j(k+1)} \subset X_{j(k)}$ ; thus, by the second inductive condition, the regions  $U_{n(j(k))}$  are distinct. Since  $X_{j(M+1)}$  meets  $\text{bdy}(R_{j(M+1)})$  and  $X_{j(M+1)} \subset U$ ,  $X_{j(M+1)}$  meets  $U \cap \text{bdy}(R_{j(M+1)})$ ; by the definition of  $\varepsilon$ -region (2.2),  $X_{j(M+1)}$  meets some component  $K_i$  ( $i = 1, 2, \dots, I$ ). As a result, each  $X_{j(k)}$  meets  $K_i$ . Since  $X_{j(k)} \subset U_{n(j(k))}$  (by the second inductive condition), the distinct sets  $U_{n(j(k))}$  each meet, and thus contain  $K_i$  ( $k = 1, 2, \dots, M + 1$ ), contradicting the choice of  $M$ .

3.2. LEMMA. *Let  $P, U, \varepsilon$ , and  $R_m$  ( $m = 1, 2, \dots$ ) satisfy the hypotheses of 3.1. Then there exist simply connected regions  $L_i^h$  ( $i = 1, 2, \dots, I$ , the integer  $I$  of 2.2;  $h = 1, 2, \dots, M$ , the integer  $M$  of 3.1) in  $U$  such that:*

- (a) *each  $\text{bdy}(L_i^h)$  is a Peano continuum;*
- (b) *there exists a function  $\Omega$  mapping the indices of the  $\varepsilon$ -regions  $R_m$  into pairs of natural numbers  $(i, h)$  such that  $R_m$  meets  $L_i^h$  properly in a point  $q_m$ ;*
- (c) *the prime ends of  $L_i^h$  associated with these points by 2.7, are distinct, and no one of the prime ends is a limit of others.*

The limit statement of conclusion (c) means that: if  $\bar{q}_m$  are the points on the circle  $\text{bdy}D$  given by the Carathéodory Theorem [2, p. 350] as corresponding to the prime ends, then no point  $\bar{q}_m$  is a limit point of others.

**Proof.** Let  $X_m$  and  $Y_m$  ( $m = 1, 2, \dots$ ) be the open sets given by 3.1. Consider a fixed set  $R_m$ ; by 2.5,  $\bar{R}_m$  is a closed topological disk. By 3.1(b), there exists an arc  $\alpha$  of the circle  $\text{bdy}(R_m)$  such that  $\alpha \subset Y_m$  and (since  $\text{bdy}(R_m) \cap \text{bdy}U$  contains no arc)  $\alpha \cap \text{bdy}U = \emptyset$ . Since (by 2.2)

$$\text{bdy}(R_m) \subset \bigcup_{i=1}^I \text{bdy}(K_i),$$

the union of the closed sets  $\alpha \cap \text{bdy}(K_i)$  ( $i = 1, 2, \dots, I$ ) is  $\alpha$ . Thus for some  $i$  (call it  $i(m)$ )

$$\dim(\alpha \cap \text{bdy}(K_{i(m)})) = 1$$

[6, p. 30], so that  $\alpha \cap \text{bdy}(K_{i(m)})$  contains an arc  $\beta_m$  [6, p. 44]; let  $p_m$  be an interior point of  $\beta_m$ .

Choose  $\eta$ ,  $0 < \eta < 1/2m$  such that

$$S(p_m, \eta) \cap \text{bdy}(R_m) \subset \beta_m$$

and  $S(p_m, \eta) \subset Y_m$ . Since  $\text{bdy}(K_{i(m)})$  is locally connected [25, p. 106, (2.2)], each point of  $\beta_m$  is regularly accessible from  $K_{i(m)}$  [25, p. 111 and p. 112, (4.2)]. Choose the regular accessibility  $\delta$  for  $\eta$  and  $p_m$ ,  $0 < \delta < \eta$ . Let  $q$  be a point of the component of  $\beta_m \cap S(p_m, \delta)$  containing  $p_m$ ,  $q \neq p_m$ , and similarly choose  $\zeta$  for  $\eta - \delta$  and  $q$ ,  $\zeta + d(p_m, q) < \delta$ ; choose  $r$  in  $S(q, \zeta) \cup K_{i(m)}$ . There exist arcs  $\rho$  joining  $r$  to  $p_m$  in

$$(K_{i(m)} \cap S(p_m, \eta)) \cup \{p_m\},$$

and  $\sigma$  joining  $r$  to  $q$  in

$$(K_{i(m)} \cap S(p_m, \eta)) \cup \{q\}.$$

Since  $\rho \cup \sigma$  has Property S, it is locally connected, and thus contains an arc  $\tau$  joining  $p$  and  $q$ ,

$$\tau - (\{p\} \cup \{q\}) \subset K_{i(m)} \cap S(p_m, \eta).$$

Since  $\tau \subset S(p_m, \eta) \subset Y_m$ ,  $\beta_m \subset \alpha \subset Y_m$ , and  $Y_m \subset U$ , (3.1(a)),  $\beta_m \cup \tau \subset U$ .

Let  $C_m$  be the component of  $U - (\beta_m \cup \tau)$  disjoint from  $R_m$ . Since  $\bar{C}_m$  and  $\bar{R}_m$  are closed topological disks (the latter by 2.5) whose intersection is an arc, their union is also a closed topological disk. Let  $E_m$  be the interior of  $\bar{C}_m \cup \bar{R}_m$ . Since  $\bar{C}_m \subset S(p_m, \eta)$ ,  $\bar{C}_m \subset Y_m$  and  $\text{diam}(\bar{C}_m) < 1/m$ . Let  $q_m$  be a point on  $\text{bdy}(R_m) - \beta_m$ ; thus

$$q_m \in \text{bdy}(E_m) \cup \text{bdy}(K_{i(m)} \cup E_m).$$

Let  $\gamma_m$  be an arc ending at  $q_m$ ,  $\gamma_m - \{q_m\}$  in the open topological disk (by 2.5)  $R_m$ .

Define a function  $\omega$  mapping the indices  $m$  of the sets  $R_m$  into the natural numbers by:  $\omega(m) = h$  if and only if  $h$  is the maximal positive integer such that there exists a function  $j(k)$  ( $k = 1, 2, \dots, h$ ) with  $j(k) < j(k + 1)$ ,  $j(h) = m$ , and  $X_{j(k+1)} \subset X_{j(k)}$  ( $h$  may be 1). Thus  $h \leq M$ , the natural number given by 3.1 (d). Define  $\Omega(m)$  as the ordered pair  $(i(m), \omega(m))$  and let  $L_i^h$  be the union of  $K_i$  with those open topological disks  $E_m$  for which  $\Omega(m) = (i, h)$  ( $i = 1, 2, \dots, I$ ;  $h = 1, 2, \dots, M$ ).

Let  $E_m$  and  $E_n$  be two open topological disks added to some region  $K_i$  in forming  $L_i^h$ , i.e.,  $\Omega(m) = \Omega(n) = (i, h)$ . Using the facts that  $E_m \subset \bar{C}_m \cup R_m$  and  $E_n \subset \bar{C}_n \cup R_n$  ( $R_m$  and  $R_n$  are open topological disks by 2.5), it will now be shown that  $E_m \cap E_n = \emptyset$ . Suppose that  $Y_m \cap Y_n \neq \emptyset$ ; by conclusions (a) and (c) of 3.1, if  $m < n$ , then  $X_n \subset X_m$ , so that  $\omega(m) < \omega(n)$ . Since  $\omega(m) = \omega(n) = h$ ,  $Y_m \cap Y_n = \emptyset$ . Thus, since  $\bar{C}_m \subset Y_m$  and  $\bar{C}_n \subset Y_n$ ,  $\bar{C}_m \cap \bar{C}_n = \emptyset$ . Moreover, the  $\varepsilon$ -regions  $R_m$  and  $R_n$  are disjoint, by Definition 2.2. Suppose  $R_n \cap \bar{C}_m \neq \emptyset$ . If  $R_n \subset \bar{C}_m$ , then, by conclusion (b) of (3.1),  $Y_n \cap \bar{C}_m \neq \emptyset$ , contradicting the fact that  $Y_n \cap Y_m = \emptyset$ . Thus, if  $R_n \cap \bar{C}_m \neq \emptyset$ , then  $R_n$  meets  $\text{bdy}(C_m)$ . But

$$\text{bdy}(C_m) \subset K_{i(m)} \cup \text{bdy}(R_m),$$

and  $R_n$  does not meet either, by the definition of  $\varepsilon$ -region (2.2); thus,  $R_n \cap \bar{C}_m = \emptyset$ . Similarly  $R_m \cap \bar{C}_n = \emptyset$ , and, hence,  $E_m$  and  $E_n$  are disjoint.

Each set  $L_i^h$  is a region. Let  $\Gamma$  be any closed path in  $L_i^h$ , i.e.,  $\Gamma$  is a map of the unit circle into  $L_i^h$ . The topological disks  $E_m$  for which  $\Omega(m) = (i, h)$  are mutually disjoint, and each  $\text{bdy}(E_m)$  meets  $K_i$  in an open arc. There is a function  $\Delta(t, x)$ , where  $t \in [0, 1]$  and  $x$  is on the unit circle, defined as follows: for  $x \notin \Gamma^{-1}(E_m)$ ,  $\Delta(t, z) = \Gamma(x)$ ; for  $x \in \Gamma^{-1}(E_m)$ ,  $\Delta(0, x) = \Gamma(x)$ ,

$$\Delta(t, x) \subset E_m \cup (K_i \cap \text{bdy}(E_m)),$$

$$\Delta(1, x) \subset K_i \cap \text{bdy}(E_m),$$

and the restriction map  $\Delta|_{\Gamma^{-1}(\bar{E}_m)}$  is continuous. We wish to prove that  $\Delta$  is continuous, i.e., if  $t_k \rightarrow t$  and  $x_k \rightarrow x$ , then  $\Delta(t_k, x_k) \rightarrow \Delta(t, x)$ . If  $x$  is in some set  $\Gamma^{-1}(E_m)$ , then all but a finite number of the  $x_k$  are in  $\Gamma^{-1}(E_m)$ , and the conclusion follows from the definition of  $\Delta$  on  $\Gamma^{-1}(E_m)$ . Thus we may suppose that  $x \notin \bigcup_m \Gamma^{-1}(E_m)$ , so that  $\Delta(t, x) = \Gamma(x)$ , and that either (1) all  $x_k \notin \bigcup_m \Gamma^{-1}(E_m)$  or (2) each  $x_k$  is in some set  $\Gamma^{-1}(E_m)$ , call it  $\Gamma^{-1}(E_{m(k)})$ . Since  $\Delta|_{\Gamma^{-1}(\bar{E}_m)}$  is continuous ( $m = 1, 2, \dots$ ), we may suppose in case (2) that the numbers  $m(k)$  are distinct ( $k = 1, 2, \dots$ ). In case (1)  $\Delta(t_k, x_k) = \Gamma(x_k)$ , so that  $\Delta(t_k, x_k) \rightarrow \Gamma(x_k)$ . In case (2),  $\Delta(t_k, x_k) \in \bar{E}_{m(k)}$ . Since  $\text{diam}(C_m) \rightarrow 0$ ,  $\text{diam}(R_m) \rightarrow 0$  (by 2.4), and  $\bar{E}_m = \bar{C}_m \cup \bar{R}_m$ ,  $\text{diam}(E_m) \rightarrow 0$ . Since  $\Gamma(x_k) \in \bar{E}_{m(k)}$ ,  $\bar{E}_{m(k)} \rightarrow \Gamma(x)$ , so that  $\Delta(t_k, x_k) \rightarrow \Gamma(x)$ . As a result  $\Delta$  is continuous, so that  $\Gamma$  is homotopic in  $L_i^h$  to a closed path in

$$L_i^h - \bigcup \{E_m : \Omega(m) = (i, h)\},$$

a subset of  $K_i$ . Since  $K_i$  is simply connected (it is a component of  $S - P$ ), this path is homotopic in  $L_i^h$  to a point. Thus,  $L_i^h$  is also a simply connected region, so that  $\text{bdy}(L_i^h)$  is a continuum.

Let  $\xi > 0$  be given. Since  $K_i$  has Property S [25, p. 106, (2.2) and p. 112, (4.2)],  $K_i$  is the union of a finite number of connected subsets, each of diameter less than  $\xi/3$ . Since  $\text{diam}(E_m) \rightarrow 0$ , only a finite number of the topological disks  $E_m$  have  $\Omega(m) = (i, h)$  and  $\text{diam}(E_m) \geq \xi/3$ , and each of these sets is also the finite union of connected subsets of diameter less than  $\xi/3$ . The sets  $E_m$  such that  $\Omega(m) = (i, h)$  and  $\text{diam}(R_m) < \xi/3$  each meet one of the connected subsets of  $K_i$ ; thus,  $L_i^h$  is the union of a finite number of connected subsets, each of diameter less than  $\xi$ . Hence,  $L_i^h$  has Property S, and by [25, p. 112, (4.2)],  $\text{bdy}(L_i^h)$  is locally connected. Thus, conclusion (a) is satisfied.

We have seen that  $E_m \cap E_n = \emptyset$  for  $\Omega(m) = \Omega(n) = (i, h)$ , and that

$$q_m \in \text{bdy}(E_m) \cup \text{bdy}(K_i \cup E_m);$$

thus  $q_m \in \text{bdy}(L_i^h)$ , so that conclusion (b) is satisfied (using the arc  $\gamma_m$ ). Also, because of the disjointness of the sets  $E_n$ , the open arc  $\text{bdy}(R_m) - \beta_m$  of  $\text{bdy}(K_i \cup E_m)$  is contained in  $\text{bdy}(L_i^h)$ . Moreover, a neighborhood of one side of it is contained in  $R_m \cap L_i^h$ ; thus, corresponding to each point of the open arc  $\text{bdy}(R_m) - \beta_m$  is a unique prime end "from  $R_m$ ." Now  $q_m$  is on this open arc, and  $\gamma_m - \{q_m\}$  is contained in the open topological disk  $R_m$ ; because of the disjointness of the sets  $E_n$ , conclusion (c) follows.

3.3. Although the chains of prime end theory were used in the proof of 2.1, it is convenient to give a new concept the same name.

DEFINITION. A *chain* (for  $P$ ) is a finite or countably infinite family of simply connected regions  $U_n$  in  $S$  ( $n = 1, 2, \dots$ ) such that:

- (a) The space  $P = \text{Cl}[\bigcup_n \text{bdy}(U_n)]$ .
- (b) Each  $\text{bdy}(U_n)$  is a nondegenerate Peano continuum.
- (c) There is a function  $\psi$  sending the natural numbers greater than one into the natural numbers such that  $\psi(n) < n$ , and  $U_n$  meets  $U_{\psi(n)}$  properly (2.7) in a point  $p_n$ . The accessibility arc defines a natural prime end  $P_n$  of  $U_{\psi(n)}$  (see 2.7) whose impression is the point  $p_n$ .
- (d) For each natural number  $k$  and the set of all natural numbers  $n(m)$  ( $m = 1, 2, \dots$ ) such that  $\psi_{(n(m))} = k$ , the prime ends  $P_{n(m)}$  (of  $U_k$ ) are distinct and distinct from the point  $p_k$ , and no prime end  $P_{n(m)}$  is a limit of others (see 3.2).
- (e) Let

$$\Psi_n = \bigcup \{U_k : \psi^h(k) = n, h = 0, 1, 2, \dots\},$$

where  $\psi^h$  is the  $h$ th iteration of  $\psi$ , and  $\psi^0(k) = k$ . If the family  $\{U_n\}$  is infinite, then  $\text{diam}(\Psi_n) \rightarrow 0$ . In particular,  $\text{diam}(U_n) \rightarrow 0$ .

The *range* of the chain  $\{U_n\}$  is  $\bigcup_n U_n$ . If  $P$  has a chain, then  $P$  satisfies condition  $C_2$ .

3.4. DEFINITIONS. Let  $f$  be a function of  $A$  into  $B$ , where  $A$  and  $B$  are topological spaces. If, whenever  $U$  is open in  $A$ ,  $f(U)$  is open in  $B$ , then  $f$  is *interior* (*open*). If, for each  $p$  in  $f(A)$ , the components of  $f^{-1}(p)$  are single points, then  $f$  is *light*. It is well known that any nonconstant meromorphic function is light interior.

Let  $f: D \rightarrow S$  be light interior with the Peano continuum  $P$  its global cluster set  $C(f)$ . Let  $f = gh$  be the factorization given by [19, p. 121], where  $h$  is a homeomorphism and  $g$  is meromorphic. We may suppose that  $h(D)$  is either  $D$  or  $S - \{\infty\}$ . Since  $P$  is neither  $S$  nor a single point,  $h(D) = D$ ; hence  $P$  is the global cluster set  $C(g)$ , so that  $P$  satisfies Condition  $C_3$ . Thus Condition  $C_3$  is a topological condition in the sense that  $P$  may be replaced by the set  $h(P)$ , where  $h$  is any homeomorphism of  $S$  onto itself.

As a result, the properties of the hypothesis and conclusion of the following lemma are preserved under a homeomorphism of  $S$  onto itself. In particular,  $D$  may be replaced by the interior  $E$  of any topological closed disk.

3.5. LEMMA. Suppose that  $P$  satisfies either Condition  $C_2$  or  $C_3$ , that  $\varepsilon > 0$ , that  $U$  is  $S$  or  $D$ , and, if  $U$  is  $D$ , then  $u \in \text{bdy } D$  and  $\text{bdy } D \subset P$ . Then there exists a family  $\{V_n\}$  such that:

- (1)  $\{V_n\}$  is a chain, except that, instead of Condition (a), each  $\text{bdy}(V_n) \subset P$ .
- (2) If  $U = D$ , then  $V_1 = U$ ; if  $\psi(n) = 1$ , then  $p_n \neq u$  ( $n = 1, 2, \dots$ ).
- (3) Each  $V_n \subset U$ .
- (4) There exists a natural number  $N$  such that, for all  $n > N$ :
  - (a)  $\bar{V}_n$  is a closed topological disk; and

- (b)  $\text{diam}(V_n) < \varepsilon$ .
- (5) For all  $n$ ,  $\psi(n) \leq N$ .
- (6) For each point  $p$  in  $P \cap \bar{U}$ , either  $p \in \text{bdy}(V_n)$  ( $n = 1, 2, \dots, N$ ), or  $p \in \bar{V}_n$  ( $n > N$ ).

If  $P$  satisfies either Condition  $C'_2$  or  $C'_3$ ,  $\infty \notin P$ , and  $U = S$ , then  $\{V_n\}$  may be chosen so that  $\infty \notin V_n$  ( $n = 1, 2, \dots$ ).

**Proof.** If  $U \subset P$ , then, since  $P \neq S$ ,  $U = D$ . Let  $h$  be a homeomorphism of the unit square  $I^2$  onto  $\bar{D}$ , let  $\delta$  be the uniform continuity  $\delta$  for  $h$  and  $\varepsilon$ , and let  $m$  be a natural number such that  $2^{-m} < \delta$ . Let  $S_n$  ( $n = 1, 2, \dots$ ) be the squares of the families  $\mathfrak{D}_j$  (defined in 1.3;  $j = 1, 2, \dots$ ) taken in order, and let  $N$  be the index of the last square of side at most  $2^{-m-1}$ . The sets  $V_n$  are  $h(S_n)$ .

Thus, it may be assumed that  $U - P \neq \emptyset$ , and that  $\varepsilon$  is less than  $\text{diam } P$ ,  $\frac{1}{2} \text{diam } S$  and  $\text{diam } K$ , where  $K$  is some component of  $U - P$ . Let  $K_i$  ( $i = 1, 2, \dots, I$ ) be the components of  $U - P$  in 2.2, and let  $E_j$  ( $j = 1, 2, \dots, J$ ) be the interiors of the closed topological disks given by 2.5 and 2.6 for the  $\varepsilon$ -regions (for  $P$  and  $U$ ) of diameter at least  $\varepsilon/3$ .

Let  $\mathfrak{B}$  be the family of those  $\varepsilon$ -regions  $R$  (for  $P$  and  $U$ ) such that  $\text{diam } R < \varepsilon/3$  and  $R \cap P \neq \emptyset$ . Each region  $R$  in  $\mathfrak{B}$  is the interior of a closed topological disk (by 2.5). If  $U = D$ , let  $\mathfrak{Q}$  be the family of those  $\varepsilon$ -regions  $R$  in  $\mathfrak{B}$  such that  $\text{bdy } R \cap \text{bdy } D$  contains an open arc, call it  $\mu(R)$ , and let  $q(R) \in \mu(R)$ . Then the open arcs  $\mu(R)$  for  $R$  in  $\mathfrak{Q}$  are disjoint, and, if  $L$  is the closure of  $\bigcup \{q(R) : R \in \mathfrak{Q}\}$ , then  $L$  contains no arc. If  $U = D$ , let  $\mathfrak{R} = \mathfrak{B} - \mathfrak{Q}$ . If  $U = S$ , let  $\mathfrak{R} = \mathfrak{B}$ . If  $\mathfrak{R} \neq \emptyset$ , let  $R_m$  be an enumeration of the family  $\mathfrak{R}$ , and let the regions  $L_i^h$  ( $i = 1, 2, \dots, I$ ;  $h = 1, 2, \dots, M$ ), the function  $\Omega$ , and the points  $q_m$  ( $m = 1, 2, \dots$ ) be as given in 3.2.

If  $U = D$ , let  $V_1 = D$ ; if  $U = S$ , let  $V_1 = K_1$  (since  $U - P \neq \emptyset$ ,  $I \neq 0$ ). Let  $H$  be the number of regions  $K_i$  (other than  $K_1$ , if  $V_1 = K_1$ ),  $L_i^h$  (there are none if  $\mathfrak{R} = \{R_m\} = \emptyset$ ), and  $E_j$  ( $i = 1, 2, \dots, I$ ;  $h = 1, 2, \dots, M$ ;  $j = 1, 2, \dots, J$ ). Let  $V_n$  ( $n = H + 2, H + 3, \dots, 2H + 1$ ) be any enumeration of these sets with the sets  $E_j$  at the end, and let  $N$  be the index of the last region  $V_n$  not a topological disk  $E_j$  ( $N = H + 1$  if all the sets are topological disks  $E_j$ ). We will now define simply connected regions  $V_n$  ( $n = 2, 3, \dots, H + 1$ ) such that  $V_n$  meets both  $V_1$  and  $V_{n+H}$  properly,  $V_n \subset U$ ,  $\text{bdy}(V_n) \subset P$ , and the points  $p_n$  of accessibility (3.3(c)) are distinct, different from  $u$  and from  $q_m$  ( $m = 1, 2, \dots$ ), and in case  $U = D$ , not in  $L$ .

If  $U = D$ , then  $V_1 = D$  and  $V_{n+H} \subset D$ ; let  $\lambda_1 = \text{bdy } D$ . If  $U = S$ , let  $\lambda_1$  be an arc on  $\text{bdy}(V_1)$ . If  $\text{bdy}(V_{n+H}) \subset \lambda_1$ , let  $\lambda_2 = \lambda_3 = \emptyset$ . Otherwise, there exists an arc  $\lambda_2$  disjoint from  $\lambda_1$  and contained in  $\text{bdy}(V_{n+H})$ . There is an arc  $\lambda_3$  in  $P$  joining a point of  $\lambda_2$  to a point of  $\lambda_1$ ,  $\lambda_3$  meeting  $\lambda_1 \cup \lambda_2$  only in its endpoints. Let  $V_n = U - (\lambda_1 \cup \lambda_2 \cup \lambda_3)$ ; then  $V_n$  is a simply connected subregion of  $U$  whose boundary  $\lambda_1 \cup \lambda_2 \cup \lambda_3$  is a nonempty and nondegenerate Peano continuum contained in  $P$ . Since  $\lambda_1 \subset \text{bdy}(V_1) \cap \text{bdy}(V_n)$  and  $\lambda_2 \subset \text{bdy}(V_n) \cap \text{bdy}(V_{n+H})$ ,

and since each point of  $\text{bdy}(V_1)$  (resp.,  $\text{bdy}(V_{n+H})$ ) is regularly accessible from  $V_1$  (resp.,  $V_{n+H}$ ) [25, p. 112, (4.2)],  $V_1$  (resp.,  $V_{n+H}$ ) meets  $V_n$  properly (2.7) in an arc of points of accessibility. Define  $\psi(n)$  as 1,  $\psi(n+H)$  as  $n$ , and  $p_n$  and  $p_{n+H}$  ( $n = 2, 3, \dots, H+1$ ) as accessibility points, chosen so that they are distinct, none is  $u$  or  $q_m$  ( $m = 1, 2, \dots$ ), and, in case  $U = D$ , none is in the 0-dimensional set  $L$ .

The regions  $V_n$  ( $n = 2H+2, 2H+3, \dots$ ) are defined to be the open topological disks of  $\mathfrak{B}$ , i.e., of  $\mathfrak{Q}$  and  $\mathfrak{R}$ , added (say) in order of decreasing diameter. If  $V_n = R$  in  $\mathfrak{Q}$ , then  $U = D = V_1$ ; let  $\psi(n) = 1$ , and let  $p_n$  be  $q(R)$ . Then  $V_n$  meets  $V_1 = D$  properly in  $p_n$ . Suppose that  $V_n = R_m$  in  $\mathfrak{R}$ ; if  $\Omega(m) = (i, h)$  and  $V_k = L_i^h$ , then let  $\psi(n) = k$  and  $p_n = q_m$ .

Conclusions (1) (a), (b), (c) have already been noted or are immediate. For (1) (d) and for (2), observe that the points  $p_n$  ( $n = 2, 3, \dots, 2H+1$ ) and  $u$  were defined to be distinct. Thus, to prove that (1) (d) and (2) are satisfied for all natural numbers  $k$ , it is sufficient to consider only those  $k$  for which there exists  $n > 2H+1$  with  $\psi(n) = k$ . For each  $n > 2H+1$ ,  $V_n \in \mathfrak{Q} \cup \mathfrak{R}$ . If  $V_n \in \mathfrak{Q}$ , then  $\psi(n) = 1$  and  $U = D = V_1$ ; conversely,  $\psi(n) = 1$  only for  $V_n$  in  $\mathfrak{Q}$  and for  $n = 2, 3, \dots, H+1$ . Since  $p_n = q(R)$  for  $V_n = R$  in  $\mathfrak{Q}$ , conclusions (1)(d) and (2) follow for  $k=1$  from the definition of  $p_n$  ( $n = 2, 3, \dots, 2H+1$ ). The set  $V_n$  is in  $\mathfrak{R}$  (i.e.,  $V_n$  is an open topological disk  $R_m$ ) if and only if  $\psi(n) = k$ , where  $V_k = L_i^h$  and  $\Omega(m) = (i, h)$  ( $k = 2, 3, \dots$ ); the point  $p_n$  is the point  $q_m$ , so that (1) (d) is satisfied for these  $k$  by 3.2(c) and by the choice of  $p_k$ .

Conclusions (3) and (5) follow immediately from the construction of  $\{V_n\}$ , and conclusion (4) from the fact that, for  $n > N$ ,  $\bar{V}_n$  is a closed topological disk of  $\{E_j\}$  ( $j = 1, 2, \dots, J$ ) or of  $\mathfrak{B} = \mathfrak{Q} \cup \mathfrak{R}$ . By conclusion (5)  $\Psi_n = V_n$  (for  $n > N$ ), and since all but a finite number of these sets are  $\varepsilon$ -regions, (1) (e) follows from 2.4. For (6), each point  $P$  in  $P \cap \bar{U}$  is either in  $\text{bdy} D$  (in case  $U = D$ , so that  $V_1 = D$ ), in  $\text{bdy}(K_i)$  ( $i = 1, 2, \dots, I$ ), or in some  $\varepsilon$ -region  $R$  (by 2.2). Thus it suffices to prove that for each  $\varepsilon$ -region  $R$ ,

$$P \cap R \subset \bigcup_{n=N+1}^{\infty} V_n.$$

If  $\text{diam} R \geq \varepsilon/3$ , then  $R \subset \bigcup_{j=1}^J E_j$ ; if  $\text{diam} R < \varepsilon/3$ , and  $R$  meets  $P$ , then  $R \in \mathfrak{B} = \mathfrak{R} \cup \mathfrak{Q}$ . Conclusion (6) follows.

Lastly, suppose that  $P$  satisfies either Condition  $C'_2$  or  $C'_3$ ,  $\infty \notin P$ , and  $U = S$ . Let  $R$  be  $\bigcup_n U_n$ , or the range of  $f$ , as the case may be; then  $R$  is a region (in case  $C'_2$  use  $C'_2$ (b)) not containing  $\infty$ . Let  $K$  be the component of  $S - P$  containing  $\infty$ ; suppose that  $K \cap R \neq \emptyset$ . In case  $C'_2$  some region  $U_n$  ( $n = 1, 2, \dots$ ) meets  $K$ . Since  $\text{bdy}(U_n) \subset P$  and  $\infty \notin U_n$ ,  $\infty \notin \bar{U}_n$ . Since  $\infty \in K$ ,  $K$  meets  $\text{bdy}(U_n)$ , and a contradiction results. In case  $C'_3$ , while  $f(D)$  meets  $K$ ,  $\infty \notin f(D)$  and  $\infty \notin P$ , which is the global cluster set  $C(f)$ . Thus  $\infty \notin \text{Cl}[f(D)]$ , and there exists a point

$$q \in K \cap \text{bdy}(\text{Cl}[f(D)]).$$

Since  $\text{Cl}[f(D)] = f(D) \cup C(f)$ ,  $C(f) = P$ , and  $P \cap K = \emptyset$ ,  $q \in f(D)$ . But  $f(D)$  is open, contradicting the definition of  $q$ . Thus  $K \cap R = \emptyset$  in both cases  $C'_2$  and  $C'_3$ .

Since  $\text{bdy } K$  is a Peano space [25, p. 106, (2.2)], and  $P \subset \bar{R}$ ,  $\text{bdy } K$  is a simple closed curve  $C$  separating  $K$  from  $R$  [24, p. 114, (6.7)]. Let  $E$  be that open topological disk of  $S - C$  containing  $R$ . Then  $P \subset \bar{E}$  and  $C \subset \text{bdy } K \subset P$ . Application of this lemma under the previous hypothesis ( $C_2$  or  $C_3$ ) to the case  $U = E$  (see 3.4) yields a possibly finite sequence of regions  $V_n \subset E$  by conclusion (3). Thus  $\infty \notin V_n$  ( $n = 1, 2, \dots$ ).

A family  $\{V_n\}$  satisfying conclusions (1) through (6) of the lemma is almost a chain, intuitively speaking, differing only by sets  $V_n$  of diameter less than  $\varepsilon$ . (Compare 3.3(a) with (1) and (6).) Each of these small sets  $V_n$  is a topological disk, and there exists a "finer almost-chain" for  $P$  and it. If one continues in this way, a chain for  $P$  is constructed; this is the idea of the next lemma.

**3.6. LEMMA.** *If  $P$  satisfies either Condition  $C_2$  or  $C_3$ , then it has a chain  $\{U_n\}$ . If it satisfies either  $C'_2$  or  $C'_3$ , then  $\{U_n\}$  may be chosen so that its range does not contain  $\infty$ .*

**Proof.** Repeated applications of 3.5 will be used to define a chain  $\{U_n\}$ , where the function  $\psi^*$  and the accessibility points  $q_n$  are induced by that lemma. Define  $U_n$  as  $V_n$  for  $U = S$  and  $\varepsilon = 1$  ( $n = 1, 2, \dots, N_1$ ), where  $N_1$  is the integer  $N$  of conclusion (4) of 3.5; define  $\psi^*$  (on  $n = 2, 3, \dots, N_1$ ) as the function  $\psi$  and  $q_n$  as the point  $p_n$  of 3.5 (1) (c). Let  $\mathfrak{E}_1$  be the family of those regions (interiors of closed topological disks, by 3.5(4)(a))  $V_n$  such that  $n > N_1$ ; for each region  $E$  in  $\mathfrak{E}_1$ ,  $\text{diam } E < 1$  (by (4)(b)). Let  $\omega_1$  be the function of  $\mathfrak{E}_1$  into the natural numbers  $1, 2, \dots, N_1$  such that: if  $E = V_n$ , then  $\omega_1(E) = \psi(n)$  (see 3.5(1)(c) and (5)(a)); let  $q_E$  be  $p_n$ . In the final chain  $\{U_n\}$  any region  $U_n$  such that  $\psi^*(n) = k$  ( $k = 1, 2, \dots, N_1$ ) will be a topological disk  $E$  in  $\mathfrak{E}_1$  with  $\omega_1(E) = k$ ; and the points  $q_n$  will be the points  $q_E$ . As a result, for these numbers  $k$ , condition 3.3(d) will result from the corresponding condition, (1)(d), of 3.5. Eventually, each topological disk of  $\mathfrak{E}_1$  will be included in  $\{U_n\}$ . (If  $\mathfrak{E}_1 = \emptyset$ , then  $\{U_n\}$  ( $n = 1, 2, \dots, N_1$ ) is the entire chain.)

Enlarge the family  $U_n$  ( $n = 1, 2, \dots, N_1$ ) with those topological disks  $E$  of  $\mathfrak{E}_1$  such that  $\text{diam } E \geq \frac{1}{2}$ ; by 3.5(1)(e) there are only a finite number (and there may be none). Let  $I(1)$  be the subscript of the last region  $U_n$ , and let  $\psi^*$  and  $q_n$  be defined for these additional  $n$  (from  $\omega_1$  and the points  $q_E$ ) as above.

There is no  $n$  ( $n = 1, 2, \dots, I(1)$ ) such that  $\psi^*(n) = k$  ( $k = N_1 + 1, N_1 + 2, \dots, I(1)$ ); indeed, there is no topological disk  $E$  in  $\mathfrak{E}_1$  such that  $\omega_1(E) = k$ , for such a  $k$  (from 3.5(5)). Each region  $U_k$  is the interior of a closed topological disk (3.5(4)(a)), and  $\text{bdy}(U_k) \subset P$  (3.5(1)). For each such  $k$ , form the family  $\{V_n\}$ , call it  $\{V_{n,k}\}$ , given by 3.5 for  $U = U_k$ ,  $\varepsilon = 1/2$ , and  $u = p_k$ . Add each region

$V_{n,k}$  ( $n=2,3, \dots, N(k)$ , the  $N$  of 3.5(4)) as a set  $U_j$ , where  $\psi^*(j)$  and  $q_j$  are induced by 3.5 in the natural way, i.e., if  $U_j = V_{n,k}, \psi(n)$  (for  $U = U_k$ ) is  $m$ , and  $V_{m,k} = U_s (V_{1,k} = U_k)$ , then  $\psi^*(j) = s$ . Let  $N_2$  be the index of the last region  $U_n$  thus far defined.

Let  $\mathfrak{E}_2$  consist of the regions  $V_{n,k}$  for  $n > N(k)$  (3.5(4)), together with the regions  $E$  of  $\mathfrak{E}_1$  other than  $U_n$  ( $n = 1, 2, \dots, I_1$ ). Let  $\omega_2$  be the function of  $\mathfrak{E}_2$  into the natural numbers such that: on  $\mathfrak{E}_1 \cap \mathfrak{E}_2$ ,  $\omega_2$  agrees with  $\omega_1$ , and on  $\mathfrak{E}_2 - \mathfrak{E}_1$ ,  $\omega_2$  (with the points  $q_E$ ) is induced by the function  $\psi$  of 3.5(1)(c), as was  $\omega_1$ . In the final chain  $\{U_n\}$  any region  $U_n$  such that  $\psi^*(n) = k$  ( $n > N_2; k = I(1) + 1, I(1) + 2, \dots, N_2$ ) will be a topological disk  $E$  in  $\mathfrak{E}_2 - \mathfrak{E}_1$  with  $\omega_2(E) = k$ ; for each such  $k$ , all such sets  $U_n$  result from a single application of 3.5 for  $U$  the interior  $U_m$  of a closed topological disk ( $m = N_1 + 1, N_1 + 2, \dots, I(1)$ ),  $U_n \cup U_k \subset U_m$ , and  $q_n$  will be  $q_E$ . Thus, condition 3.3(d) for such  $k$  results from the corresponding condition, (1)(d), of 3.5. Add to  $\{U_n\}$  each disk  $E$  of  $\mathfrak{E}_2$  with  $\text{diam } E \geq 1/3$ . Let  $I(2)$  be the subscript of the last region  $U_n$  thus far defined, and, if  $E = U_n$ , let  $\psi^*(n) = \omega_2(E)$  and  $q_n = q_E$ .

The  $m$ th stage in the construction is similar to the second, where  $\mathfrak{E}_1$  and  $\mathfrak{E}_2$  are replaced by  $\mathfrak{E}_{m-1}$  and  $\mathfrak{E}_m$ , respectively,  $\varepsilon = \frac{1}{2}$ , by  $\varepsilon = 1/m$ ;  $I(1)$  by  $I(m-1)$ ; and  $\omega_1$  and  $\omega_2$ , by  $\omega_{m-1}$  and  $\omega_m$ , respectively.

A (possibly finite) sequence  $\{U_n\}$  is thus defined; it will now be proved that  $\{U_n\}$  is a chain. (A formal proof by induction is not given because it is so involved.) Since each region  $U_n$  ( $n = 1, 2, \dots$ ) arose from the use of 3.5,  $\text{bdy}(U_n)$  is a non-degenerate Peano continuum contained in  $P$  (3.5(1)); thus, 3.3(b) is satisfied. For each region  $E$  in  $\mathfrak{E}_m$ ,  $\text{diam } E < 1/m$ , and (from the use of 3.5(6))

$$P \subset \left[ \bigcup_{n=1}^{I(m)} \text{bdy}(U_n) \right] \cup \bigcup \{ \bar{E} : E \in \mathfrak{E}_m \}$$

( $m = 1, 2, \dots$ ). Since each region  $E$  in  $\mathfrak{E}_m$  meets some region  $U_n$  properly ( $n = 1, 2, \dots, I_m$ ),  $\bar{E}$  meets  $\text{bdy}(U_n)$ ; as a result  $\bigcup_{n=1}^{I(m)} \text{bdy}(U_n)$  is  $(1/m)$ -dense in  $P$ , so that

$$P \subset \text{Cl} \left[ \bigcup \text{bdy}(U_n) \right],$$

and condition 3.3(a) is satisfied. Condition (c) of 3.3 follows from the construction (i.e., from the use of the corresponding condition, (1)(c), of 3.5).

Condition 3.3(d) has already been discussed (e.g., see the end of the first paragraph). Consider 3.3(e); let  $(\psi^*)^h$  be the  $h$ th iteration of  $\psi^*$  ( $h = 1, 2, \dots$ ), and let  $(\psi^*)^0$  be the identity function. For each  $n > I(m)$ , there exists a topological disk  $U_k$  in  $\mathfrak{E}_m$ ,  $k \leq n$ , such that for some  $h$ ,  $(\psi^*)^h(n) = k$  ( $h = 0, 1, 2, \dots$ ). But  $\Psi_k \subset U_k$ , because these sets arose through successive applications of 3.5, first for  $U$  as  $U_k$ , then as subsets of  $U_k$  (from 3.5(3)). Since  $U_k \in \mathfrak{E}_m$ ,  $\text{diam}(U_k) < 1/m$ ,  $\text{diam}(\Psi_k) < 1/m$  and 3.3(e) satisfied.

Suppose that  $P$  satisfies condition  $C'_2$  or  $C'_3$ . If  $\infty \notin P$ , then the  $C'_2 - C'_3$

version of 3.5 may be applied, so that none of the regions  $V_n$  defined in the first paragraph of this proof contains  $\infty$ ; from the subsequent use of 3.5 and from 3.5(3), no  $U_n$  contains  $\infty$ .

Now suppose that  $\infty \in P$ . For each  $n$  with  $\infty \notin U_n$ , let  $\lambda_n = \emptyset$ . If  $\infty \in U_1$ , let  $\lambda_1$  be any arc in  $P$  joining  $\infty$  to a point  $r_1$  of  $\text{bdy}(U_1)$ ,  $\lambda_1 - \{r_1\} \subset U_1$ . If  $\infty \in U_n$  ( $n = 2, 3, \dots$ ), and  $\infty \notin U_{\psi^*(n)}$ , then there is an arc  $\alpha$  joining  $\infty$  to a point  $x$  of  $\text{bdy}(U_n) \cup \text{bdy}(U_{\psi^*(n)})$ ,  $\alpha - \{x\} \subset U_n - \bar{U}_{\psi^*(n)}$ . If  $x \in \text{bdy}(U_n)$ , let  $\beta = \emptyset$ ; otherwise, there is an arc  $\beta \subset \text{bdy}(U_{\psi^*(n)})$  joining  $x$  to a point  $y$  on  $\text{bdy}(U_n)$  (by 3.3(c)),  $\beta - \{y\} \subset U_n$ . Let  $\lambda_n = \alpha \cup \beta$ , and let  $r_n$  be its endpoint on  $\text{bdy}(U_n)$ .

If  $\infty \in U_n \cap U_{\psi^*(n)}$ , we may suppose (by induction) that  $\lambda_{\psi^*(n)}$  has been defined,  $\lambda_{\psi^*(n)} - r_{\psi^*(n)} \subset U_{\psi^*(n)}$ . If  $\lambda_{\psi^*(n)}$  meets  $\text{bdy}(U_n)$ , let  $\lambda_n$  be that subarc of it which contains  $\infty$  and meets  $\text{bdy}(U_n)$  only in its other endpoint, call it  $r_n$ . Otherwise, there is an arc  $\alpha$  in  $\text{bdy}(U_{\psi^*(n)})$  joining  $r_{\psi^*(n)}$  to a point  $r_n$  on  $\text{bdy} U_n$  (by 3.3(c)),  $\alpha - \{r_n\} \subset U_n$ ; let  $\lambda_n = \alpha \cup \lambda_{\psi^*(n)}$ . In any case,  $U_{\psi^*(n)} \cap \lambda_n \subset \lambda_{\psi^*(n)}$ .

Let  $X_n = U_n - \lambda_n$ , and let  $W_{2n+1} = W_{2n} = X_n$ ; let  $\psi(2n) = 2\psi^*(n) + 1$ , and  $\psi(2n + 1) = 2n$  ( $n = 1, 2, \dots$ ). That  $\{W_m\}$  satisfies 3.3 except possibly for conditions (c) and (d), and that its range does not contain  $\infty$  is immediate.

For each  $k$  and  $n$  ( $k = 1, 2, \dots; \psi^*(n) = k$ ),  $X_n$  and  $X_k$  meet properly in  $q_n$  (in fact, in a prime end of  $X_k$  corresponding to  $Q_n$  in  $U_k$ ) unless  $\infty \in U_k$  (i.e.,  $\lambda_k \neq \emptyset$ ),  $q_n = r_k$ ,  $\lambda_k - \{r_k\} \not\subset U_n$  and  $(\lambda_k - \{r_k\}) \cap U_n \neq \emptyset$ . There is at most one such  $n$ , call it  $M(k)$ , and  $X_k$  meets  $X_{M(k)}$  properly in a point  $x_{M(k)}$  in  $\lambda_k - \{r_k\}$ . Let  $p_{2n} = q_n$  for  $n \neq M(k)$ , and  $p_{2M(k)} = x_{M(k)}$ ; let  $p_{2k+1}$  be any point on  $\text{bdy}(W_{2k+1})$  other than  $p_{2k}$  and the points  $p_m$  for  $\psi^*(m) = 2k$ . Since  $W_{2k+1} = W_{2k}$ ,  $p_{2k+1}$  is a point of accessibility [25, p. 112, (4.2)]. Thus  $\{W_m\}$  satisfies 3.3(c) and (d), and is a chain whose range does not contain  $\infty$ .

**3.7. COROLLARY.** *Condition  $C_1$  implies  $C_3$ , and  $C_3$  implies  $C_2$ . Also,  $C'_1$  implies  $C'_3$  and  $C'_3$  implies  $C'_2$ .*

**Proof.** The first and third implications are immediate from the definitions. The second and fourth follow from 3.6 and the definition of chain (3.3).

Thus, to complete the proof of 1.1, it suffices to prove that  $C_2$  implies  $C_1$ , and  $C'_2$  implies  $C'_1$ .

**4. Construction of the function.**

**4.1. LEMMA.** *Let  $\sigma$  be a closed topological disk, and let the map  $g : \sigma \rightarrow S$  be an orientation-preserving homeomorphism on  $\text{int} \sigma$ . Let  $\lambda$  be an arc on  $\text{bdy} \sigma$ , and let  $\mu$  be an arc in  $\text{int} \sigma$ , except for its endpoints, one on  $(\text{bdy} \sigma) - \lambda$ , the other,  $O$ , on  $\text{int} \lambda$ . Lastly, let  $U$  be a simply connected region in  $S$  such that  $\text{bdy} U$  is a nonempty Peano continuum other than a single point, and  $g(\mu)$  is an arc inside  $U$  except for an endpoint  $g(O)$  on  $\text{bdy} U$ . Then there exists a topological closed disk  $\tau$  in  $\sigma$  and a map  $h$  of  $\sigma$  onto  $g(\sigma) \cup U$  such that:*

- (1)  $\tau \cap \text{bdy } \sigma$  is a subarc  $\nu$  of  $\text{int } \lambda$ , and  $O \in \text{int } \nu$ ;
- (2)  $h$  agrees with  $g$  on a neighborhood (in  $\sigma$ ) of  $\text{bdy } \sigma - \text{int } \lambda$ ;
- (3)  $h$  is 2-to-1 interior on  $\text{int } \sigma$ ;
- (4)  $h$  on  $\tau$  is topologically equivalent to a map on  $\bar{D}$  which on  $D$  is a conformal homeomorphism onto  $U - g(\omega)$ , where  $\omega$  is a subarc of  $\mu$ ;
- (5)  $h(\text{Cl}[\lambda - \tau]) = g(\lambda)$ ; and
- (6)  $h(\lambda \cap \text{bdy } \tau) = \text{bdy } U$ .

**Proof.** We may as well assume that  $\sigma$  is the triangle with vertices in rectangular coordinates  $(0, 2)$ ,  $(-3, 0)$ , and  $(3, 0)$ , that  $\lambda$  is its (horizontal) base, and that  $\mu$  is its altitude, where  $O$  is the origin. Let  $P$  be  $(0, 1)$ ,  $Q_1$  be  $(-2, 0)$ ,  $Q_2$  be  $(2, 0)$ ,  $R_1$  be  $(-1, 0)$ , and  $R_2$  be  $(1, 0)$ ; let  $\tau$  be the triangle whose vertices are  $P$ ,  $R_1$ , and  $R_2$ . Outside of the open triangle whose vertices are  $P$ ,  $Q_1$ , and  $Q_2$ , let  $h$  agree with  $g$ ; thus (1) and (2) are satisfied.

Let  $\omega$  be that subarc of  $\mu$  having endpoints  $P$  and  $O$ . The set  $U - g(\omega)$  is a simply connected region, and its boundary is a Peano continuum (since it is the union of two sets having Property S [25, p. 20]). There exists a conformal homeomorphism of  $D$  onto  $U - g(\omega)$ , and its extension  $j$  to  $\bar{D}$  is continuous (by (2.1)). The circle  $\text{bdy } D$  is the union of three arcs  $a$ ,  $b$ , and  $c$  which meet one another only in their endpoints,  $a$  and  $b$  each mapped homeomorphically onto  $g(\omega)$ ,  $a$  on one side (in the sense of prime ends of  $U - g(\omega)$ ),  $b$  on the other side. There exists an orientation-preserving homeomorphism of  $\text{bdy } \tau$  onto  $\text{bdy } D$ , sending the side  $PR_1$  onto one of these two arcs, and the side  $PR_2$  onto the other. This map may be extended to a homeomorphism  $k$  of  $\tau$  onto  $\bar{D}$ , by sending concentric triangles onto concentric circles in the corresponding way. On  $\tau$ ,  $h$  is defined as  $jk$ ; thus (4) and (6) are satisfied. If we imagine that  $g(\omega)$  is vertical, with  $g(O)$  at the bottom, then  $h$  sends the right side of  $PR_1$  onto the right side of  $g(\omega)$ , and the left side of  $PR_2$  onto the left side of  $g(\omega)$ .

Let  $m$  be any orientation-preserving homeomorphism of the boundary of the triangle  $PQ_1R_1$  onto the boundary of  $PQ_1O$  which is the identity map on  $PQ_1$  and  $g^{-1}h$  on  $PR_1$ . Let  $n$  be the homeomorphism of the triangles themselves, defined by sending concentric triangles onto concentric triangles in the corresponding way. On the triangle  $PQ_1R_1$  the map  $h$  is defined as  $gn$ , and on  $PQ_2R_2$  it is defined analogously. Conclusion (5) and the fact that  $h$  is continuous follow. Since  $g$  and  $n$  are orientation-preserving,  $h$  maps the left side of  $PR_1$  onto the left side of  $g(\omega)$ , and the right side of  $PR_2$  onto the right side of  $g(\omega)$ . Thus there is no folding along the straight line segments  $PQ_1$ ,  $PQ_2$ ,  $PR_1$  and  $PR_2$ , and hence,  $h$  is a local homeomorphism except at  $P$ . As a result,  $h$  is interior [26, p. 82, (2.3)], yielding conclusion (3). (In fact,  $h$  locally at  $P$  is topologically equivalent to the analytic function  $f(z) = z^2$ .)

4.2. LEMMA. *If  $P$  has a chain  $\{U_n\}$ , then there exists a map  $f: \bar{D} \rightarrow S$*

such that  $f$  is light interior on  $D$ ,  $f(D)$  is contained in the range of  $\{U_n\}$ , and  $f(\text{bdy}D) = P$ .

**Proof.** There exists a conformal homeomorphism of  $D$  onto  $U_1$ , and its extension  $f_1$  to  $\bar{D}$  is continuous (by 2.1). If  $U_1$  is the only region of the chain, let  $f$  be  $f_1$ .

Otherwise, let  $\gamma_n$  and  $p_n$  ( $n = 2, 3, \dots$ ) be the accessibility arc and point for  $U_n$  and  $U_{\psi(n)}$  (given by 3.3 (c)). For each  $n$  such that  $\psi(n) = 1$ ,  $\text{Cl}[f_1^{-1}(\gamma_n - \{p_n\})]$  is an arc  $\eta_n$  in  $\bar{D}$  meeting  $\text{bdy}D$  in a single point  $O_n$  (by 2.7). The points  $O_n$  correspond to the prime ends  $P_n$  of 3.3(c) under the Carathéodory correspondence, i.e.,  $f(O_n) = p_n$ , so that no one of the points  $O_n$  is a limit point of others (3.3(d)). Thus there are subarcs  $\mu_n$  of  $\eta_n$  ending in  $O_n$  which are mutually disjoint (for  $n$  with  $\psi(n) = 1$ ); we may also suppose that  $\text{diam}(\mu_n) < 1/n$  and that  $\text{diam}(f_1(\mu_n)) < 1/n$ . Thus for each  $n$  and  $k$  such that  $\psi(n) = \psi(k)$ , either  $p_n \neq p_k$  and  $f(\mu_n) \cap f(\mu_k) = \emptyset$  or  $p_n = p_k$  and  $f(\mu_n) \cap f(\mu_k)$  is the point  $p_n$ .

For each  $n$  such that  $\psi(n) = 1$ , there exists a (closed) topological disk  $\sigma_n$  in  $\bar{D}$  such that:

- (1) each set  $\sigma_n \subset D$ , except for an arc  $\lambda_n$  on  $\text{bdy}D$  containing  $O_n$  in its interior;
- (2) each arc  $\mu_n \subset \text{int}(\sigma_n)$ , except for its ( $\mu_n$ 's) endpoints,  $O_n$  and one in  $D$ ;
- (3)  $\text{diam}(\sigma_n) < 1/n$ , and (by the continuity of  $f_1$ )

$$f_1(\sigma_n) \subset S(p_n, 1/n);$$

- (4) the topological disks  $\sigma_n$  are mutually disjoint; and
- (5) no one of them contains a limit point of others.

Let  $f_2$  be the map given by:  $f_2$  agrees with  $f_1$  off  $\text{int}(\sigma_2)$ , and on  $\sigma_2$  is the map  $h$  given by 4.1, where  $g$  is the restriction of  $f_1$  to  $\sigma_2$ ,  $\lambda = \lambda_2$ ,  $\mu = \mu_2$ , and  $U = U_2$ . Then (by 4.1 (2) and (3))  $f_2$  is finite-to-one interior on  $D$ ,  $f_2(D)$  is  $U_1 \cup U_2$ , and (by 4.1 (2), (5) and (6))

$$f_2(\text{bdy}D) = \text{bdy}(U_1) \cup \text{bdy}(U_2).$$

(This construction of  $f_2$  is for clarity only, and it will not be used hereafter.)

Now, suppose that maps  $f_i: \bar{D} \rightarrow S$  ( $i = 1, 2, \dots, m$ ) have been defined such that:

(1) For each  $n$  such that  $\psi(n) = 1, 2, \dots, m$  there is a closed topological disk  $\sigma_n$  contained in  $D$  except for an arc  $\lambda_n$  on  $\text{bdy}D$ .

(2) (a) There is an arc  $\mu_n$  such that the set  $f_{\psi(n)}(\mu_n)$  is a subarc of  $\gamma_n$ .

(b) Moreover,  $\mu_n \subset \text{int}(\sigma_n)$ , except for its two endpoints, one in  $D$ , the other,  $O_n$ , in  $\text{int}(\lambda_n)$ ; and  $f_{\psi(n)}(O_n) = p_n$ .

(3) Let  $\mathfrak{S}_i$  ( $i = 1, 2, \dots, m$ ) be the set of topological disks  $\sigma_n$ , where  $n > i$  and  $\psi(n) \leq i$ .

(a) The topological disks of  $\mathfrak{S}_i$  are disjoint, and no one of them has a limit point of others.

(b) If  $j < i$  then for each topological disk  $\sigma_n$  of  $\mathfrak{S}_i$ , either  $\sigma_n \in \mathfrak{S}_j$ , or there exists a topological disk  $\sigma_k$  in  $\mathfrak{S}_j$  such that

$$\sigma_n \subset \text{int}(\sigma_k) \cup \text{int}(\lambda_k).$$

(c) Given two distinct topological disks  $\sigma_n$  and  $\sigma_k$  in  $\bigcup_{i=1}^m \mathfrak{S}_i$ , then  $\sigma_n \subset \sigma_k$  if and only if  $\psi^h(n) = k$  ( $h = 1, 2, \dots$ ).

(d) Furthermore,  $f_i$  agrees with  $f_{i-1}$  on  $\text{Cl}[D - \sigma_i]$  ( $i = 2, 3, \dots, m$ ).

(4) For each topological disk  $\sigma_n$  in  $\mathfrak{S}_i$ ,  $f_i$  is a homeomorphism on  $\text{int}(\sigma_n)$  ( $i = 1, 2, \dots, m$ ).

(5) For each topological disk  $\sigma_n$  in  $\bigcup_{i=1}^m \mathfrak{S}_i$  and  $j$  ( $j = 1, 2, \dots, m$ ):

(a)  $\text{diam}(\sigma_n) < 1/n$ , and

(b)  $f_j(\sigma_n) \subset S(p_n, 1/n) \cup \Psi_n$  (see 3.3(e)).

(6) (a) On  $D$ ,  $f_i$  is finite-to-one interior,

(b)  $f_i(D) \subset \bigcup_{j=1}^i U_j$ , and

(c)  $f_i(\text{bdy } D) = \bigcup_{j=1}^i \text{bdy}(U_j)$  ( $i = 1, 2, \dots, m$ ).

Call these six conditions Property  $\mathfrak{Q}_m$  of the set of functions  $f_1, f_2, \dots, f_m$ . Observe that  $f_1$  satisfies  $\mathfrak{Q}_1$  (we will not give a separate proof that the set  $f_1, f_2$  satisfies  $\mathfrak{Q}_2$ ). If the chain  $\{U_n\}$  consists of more than  $m$  regions, we will now construct a function  $f_{m+1}$  such that the set  $f_1, f_2, \dots, f_{m+1}$  has Property  $\mathfrak{Q}_{m+1}$ .

Each topological disk  $\sigma_k \in \mathfrak{S}_{k-1}$  ( $k = \psi(m+1) + 1, \psi(m+1) + 2, \dots, m$ ); since  $\sigma_{m+1} \in \mathfrak{S}_{k-1}$ ,  $\sigma_{m+1} \cap \sigma_k = \emptyset$  (by  $\mathfrak{Q}_m, (3)(a)$ ). Thus  $f_m$  agrees with  $f_{\psi(m+1)}$  on  $\sigma_{m+1}$  (by  $\mathfrak{Q}_m, (3)(d)$ ), so that  $f_m(\mu_{m+1})$  is a subarc of  $\gamma_{m+1}$  and  $f_m(O_{m+1}) = p_{m+1}$  (by  $\mathfrak{Q}_m, (2)$ ).

The map  $f_{m+1}$  will agree with  $f_m$  off the closed topological disk  $\sigma_{m+1}$ . Let  $g$  be the restriction of  $f_m$  to this topological disk;  $f_{m+1}$  on  $\sigma_{m+1}$  is the map  $h$  given by 4.1 for  $\lambda = \lambda_{m+1}$ ,  $\mu = \mu_{m+1}$ , and  $U = U_{m+1}$ ; let  $\tau$  be the topological closed disk also given by 4.1. It follows that  $f_{m+1}$  is continuous, and that conditions  $\mathfrak{Q}_{m+1}, (3)(d), (6)(a)$  and  $(6)(b)$  are satisfied (by inductive hypothesis, and 4.1(2) and (3)).

By inductive hypothesis, condition  $\mathfrak{Q}_{m+1}, (5)(b)$  is satisfied except for the cases  $j = m+1$  or  $\psi(n) = m+1$ . Suppose that  $j = m+1$  and  $\psi(n) = 1, 2, \dots, m$ . If  $\sigma_n \cap \sigma_{m+1} = \emptyset$ , then, since  $f_m$  agrees with  $f_{m+1}$  off  $\sigma_{m+1}$ ,  $\mathfrak{Q}_{m+1}, (5)(b)$  again follows from the inductive hypothesis. Thus we may suppose that  $\sigma_n \cap \sigma_{m+1} \neq \emptyset$ . Since  $\sigma_n \in \mathfrak{S}_{\psi(n)}, \sigma_{m+1} \in \mathfrak{S}_{\psi(m+1)}, \psi(n) \leq m$ , and  $\psi(m+1) \leq m$ , it follows from conditions  $\mathfrak{Q}_m, (3)(a), (b)$ , and  $(c)$  that  $\psi^h(m+1) = n$  ( $h = 1, 2, \dots$ ) and  $\sigma_{m+1} \subset \sigma_n$ . Now since  $f_m = f_{m+1}$  off  $\sigma_{m+1}$ ,

$$f_{m+1}(\sigma_n) \subset f_m(\sigma_n) \cup f_{m+1}(\sigma_{m+1}).$$

By 4.1,

$$f_{m+1}(\sigma_{m+1}) \subset f_m(\sigma_{m+1}) \cup U_{m+1} \subset f_m(\sigma_n) \cup \bar{\Psi}_n.$$

As a result, by the inductive hypothesis,

$$f_{m+1}(\sigma_n) \subset S(p_n, 1/n) \cup \bar{\Psi}_n.$$

Thus, to complete the proof that condition  $\mathfrak{Q}_{m+1}$ , (5)(b) is satisfied, we need only define the sets  $\sigma_n$  such that  $\psi(n) = m + 1$ , and prove that the condition is satisfied for  $\psi(n) = m + 1$ .

By 4.1(4),  $f_{m+1}$  on  $\tau$  is topologically equivalent to a map on  $\bar{D}$  which on  $D$  is a conformal homeomorphism onto  $U_{m+1} - \alpha$ , where  $\alpha$  is a subarc of  $\gamma_{m+1}$  containing  $p_{m+1}$ . For each  $n$  such that  $\psi(n) = m + 1$ ,  $p_n \neq p_{m+1}$  (3.3(d)), so that we may suppose that  $\gamma_n \cap \alpha = \emptyset$ . The set

$$\text{Cl}[(\text{int } \tau) \cap f_{m+1}^{-1}(\gamma_n)]$$

is an arc  $\eta_n$  which meets  $\text{bdy } D \cap \text{bdy } \tau$  in a single point  $O_n$  (by 2.7), and  $f_{m+1}(O_n) = p_n$ . Since the points  $O_n$  with  $\psi(n) = m + 1$  are disjoint and no one is a limit point of others (3.3(d)), there is a subarc  $\mu_n$  of  $\eta_n$  containing  $O_n$  such that the arcs  $\mu_n$  are mutually disjoint,  $\text{diam}(\mu_n) < 1/n$ , and  $\text{diam}(f_j(\mu_n)) < 1/n$  ( $j = 1, 2, \dots, m + 1$ ). Thus  $\mathfrak{Q}_{m+1}$ , (2)(a) is satisfied.

For each  $n$  such that  $\psi(n) = m + 1$ , there exists a closed topological disk  $\sigma_n$  in  $\tau$  such that:

(1) Each  $\sigma_n \subset \text{int } \tau$  except for an arc  $\lambda_n \subset \text{int}(\lambda_{m+1})$ , and  $O_n \in \text{int}(\lambda_n)$ . (Thus  $\mathfrak{Q}_{m+1}$ , (1) is satisfied.)

(2) The arc  $\mu_n \subset \text{int}(\sigma_n)$  except for its endpoints, one in  $\text{int } \tau$ , the other  $O_n$  (thus  $\mathfrak{Q}_{m+1}$ , (3)(b) is satisfied).

(3) Each  $\text{diam}(\sigma_n) < 1/n$  (so that  $\mathfrak{Q}_{m+1}$ , (5)(a) is satisfied), and each  $f_j(\sigma_n) \subset S(p_n, 1/n)$  ( $j = 1, 2, \dots, m + 1$ ;  $\psi(n) = m + 1$ ; yielding the remainder of  $\mathfrak{Q}_{m+1}$ , (5)(b)).

(4) The disks  $\sigma_n$  are mutually disjoint.

(5) No one of them contains a limit point of others.

Let  $\mathfrak{S}_{m+1}$  consist of the topological closed disks just defined, together with disks of  $\mathfrak{S}_m - \{\sigma_{m+1}\}$ . Condition  $\mathfrak{Q}_{m+1}$ , (3)(a) follows from (4) and (5) above (and, of course, the inductive hypothesis); and  $\mathfrak{Q}_{m+1}$ , (3)(b) from (1) since  $\tau \subset \sigma_{m+1}$ . Consider condition  $\mathfrak{Q}_{m+1}$ , (3)(c). For  $\psi(n) \leq m$  and  $\psi(k) \leq m$ , it follows by inductive hypothesis; for  $\psi(n) = m + 1 = \psi(k)$ , it follows from (4) above. Suppose  $\psi(n) = m + 1$  and  $\psi(k) \leq m$ . Then  $\sigma_n \subset \sigma_{m+1}$ , and (by  $\mathfrak{Q}_m$ , (3)(a), (b), and (c)) either  $\sigma_k \cap \sigma_{m+1} = \emptyset$ , or  $\sigma_{m+1} \subset \sigma_k$  and  $\psi^h(m + 1) = k$  ( $h = 1, 2, \dots$ ). Thus condition  $\mathfrak{Q}_{m+1}$ , (3)(c) is satisfied.

Each topological disk  $\sigma_n$  of  $\mathfrak{S}_{n+1}$  either has  $\psi(n) \leq m$  so that  $\sigma_n \in \mathfrak{S}_m$  and  $\sigma_n \cap \sigma_{m+1} = \emptyset$  (by  $\mathfrak{Q}_m$ , (3)(a)), or  $\psi(n) = m + 1$  and  $\text{int}(\sigma_n) \subset \text{int } \tau$ . Since  $f_{m+1}$  agrees with  $f_m$  off  $\sigma_{m+1}$ , and since  $f_{m+1}$  on  $\text{int } \tau$  is a homeomorphism (by 4.1(4)), condition  $\mathfrak{Q}_{m+1}$ , (4) follows.

The map  $f_{m+1}$  agrees with  $f_m$  off  $\sigma_{m+1}$ , so that the restriction maps  $f_{m+1}|_{\text{bdy } D}$  and  $f_m|_{\text{bdy } D}$  agree off  $\lambda_{m+1}$  (see (1)). By the construction of  $f_{m+1}$  and by 4.1(5) and (6),

$$f_{m+1}(\text{Cl}[\lambda_{m+1} - \tau]) = f_m(\lambda_m)$$

and

$$f_{m+1}(\lambda_m \cap \text{bdy } \tau) = \text{bdy}(U_{m+1}).$$

Condition  $\mathfrak{Q}_{m+1}$ , (6)(c) then follows from the induction hypothesis. Thus, the set of functions  $f_1, f_2, \dots, f_{m+1}$  satisfies condition  $\mathfrak{Q}_{m+1}$ .

If the chain  $\{U_n\}$  is finite, consisting of  $m$  regions, let  $f$  be  $f_m$ ; it has the desired properties by  $\mathfrak{Q}_m$ , (6). If the chain is infinite, then, by the induction, there exists a sequence of functions  $f_m$ ; the function  $f$  will now be defined. Let  $p$  be any point of  $D$ , and let  $M$  be such that  $1/M < d(p, \text{bdy } D)$ . Then for all  $m \geq M$ , the maps  $f_m$  agree (by  $\mathfrak{Q}_m$ , (3)(d) and (5)(a)) and are finite-to-one interior (by  $\mathfrak{Q}_m$ , (6)(a)) on  $\{z : |z| < 1 - 1/M\}$ , which contains  $p$ . The map  $f$  is defined on this set to agree with these maps; thus the restriction map  $f|D$  is defined and is light interior. It follows from  $\mathfrak{Q}_m$ , (6)(b) that  $f(D)$  is contained in the range of  $U_n$ .

We will prove that the maps  $f_m|D$  converge uniformly to  $f|D$ . Given  $\varepsilon > 0$ , choose  $M$  ( $M = 1, 2, \dots$ ) such that  $1/M < \varepsilon/4$  and  $\text{diam}(\Psi_m) < \varepsilon/4$  (by 3.3(e);  $m = M, M + 1, \dots$ ). Since  $f|D$  agrees with  $f_m|D$  except on  $\bigcup_{n=m+1}^\infty \text{int}(\sigma_n)$  (by  $\mathfrak{Q}_m$ , (3)(d)), it suffices to prove that, for each  $z$  in  $\text{int}(\sigma_n)$ ,  $d(f_m(z), f(z)) < \varepsilon$  ( $m = M, M + 1, \dots; n = M + 1, M + 2, \dots$ ). Now

$$f_m(\sigma_n) \subset S(p_n, 1/n) \cup \bar{\Psi}_n$$

(by  $\mathfrak{Q}_m$ , (5)(b)), and  $p_n \in \bar{U}_n \subset \bar{\Psi}_n$  (by 3.3(e)), so that  $\text{diam}(f_m(\sigma_n)) < \varepsilon/2$ . Let  $y \in D \cap \text{bdy}(\sigma_n)$ ; then  $f_m(y) = f(y)$  ( $m = M, M + 1, \dots$ ). There exists  $k \geq M$  such that  $f_k(z) = f(z)$ . Since  $d(f_k(z), f_k(y)) < \varepsilon/2$ ,  $d(f_m(z), f_m(y)) < \varepsilon/2$ , and  $f_k(y) = f_m(y)$ , it results that  $d(f_m(z), f(z)) < \varepsilon$ . Thus the maps  $f_m|D$  converge uniformly to  $f|D$ .

Since  $\bar{D}$  is compact, each map  $f_m$  is uniformly continuous; it follows that  $f|D$  is uniformly continuous. Thus  $f|D$  has a unique continuous extension to  $\bar{D}$ , call it  $f$ , and the maps  $f_m$  converge uniformly to  $f$ .

For each  $n$  and  $m$  ( $n = 1, 2, \dots; m = n, n + 1, \dots$ ) and point  $w \in \text{bdy}(U_n)$  there is a point  $z_m \in \text{bdy } D$  such that  $f_m(z_m) = w$  (by  $\mathfrak{Q}_m$ , (6)(c)). The sequence  $\{z_m\}$  has a subsequence  $\{z_{m(k)}\}$  ( $k = 1, 2, \dots$ ) convergent to a point  $z$  on  $\text{bdy } D$ . Since the functions  $f_{m(k)}$  converge uniformly to  $f$ , and the points  $f(z_{m(k)})$  converge to  $f(z)$ , it follows that the points  $f_{m(k)}(z_{m(k)})$  converge to  $f(z)$ ; thus  $f(z) = w$ . As a result,  $\text{bdy}(U_n) \subset f(\text{bdy } D)$ , so that

$$\text{Cl} \left[ \bigcup_{n=1}^\infty \text{bdy}(U_n) \right] \subset f(\text{bdy } D).$$

By  $\mathfrak{Q}_m$ , (3)(a) and (b), and (5)(a),

$$\text{bdy } D - \bigcap_{m=1}^\infty \bigcup_{n=m}^\infty \sigma_n$$

is dense in  $\text{bdy } D$ . Since, by  $\mathfrak{Q}_m$ , (3)(d) and (6)(c),

$$f\left(\text{bdy } D - \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \sigma_n\right) \subset \bigcup_{n=1}^{\infty} \text{bdy}(U_n),$$

it follows that

$$f(\text{bdy } D) \subset \text{Cl}\left[\bigcup_{n=1}^{\infty} \text{bdy}(U_n)\right].$$

4.3. LEMMA. *Condition  $C_2$  implies  $C_1$ , and  $C'_2$  implies  $C'_1$ .*

**Proof.** If  $P$  satisfies condition  $C_2$  (respectively,  $C'_2$ ), then, from 3.6 and 4.2, there exists a continuous function  $f: \bar{D} \rightarrow S$ ,  $f$  light interior on  $D$ , such that  $f(\text{bdy } D) = P$  (respectively, and  $\infty \notin f(D)$ ). Let  $gh$  be the factorization given by [3], where  $h$  is continuous on  $\bar{D}$  and a homeomorphism on  $D$ , and  $g$  is continuous on  $h(\bar{D})$  and meromorphic (respectively, analytic) on  $h(D)$ . The functions  $h$  and  $g$  are extensions of the functions  $H$  and  $G$ , respectively, defined in the first paragraph of that proof, and thus  $h(D)$  may be supposed to be either  $D$  or the complement in  $S$  of a single point. In the latter case, since  $h(\text{bdy } D)$  and  $g(\text{bdy } D)$  would be single points,  $f(\text{bdy } D)$  would be also; but it was assumed in 1.3 that  $P$ , which is  $f(\text{bdy } D)$ , is neither  $S$  nor a single point. Thus,  $h(D) = D$ , and  $g$  is continuous on  $\bar{D}$  and meromorphic (respectively, analytic) on  $D$ ; as a result,  $P$  satisfies condition  $C_1$  (respectively,  $C'_1$ ).

Theorems 1.1 and 1.2 follow from 3.7 and 4.3.

## 5. Concluding remarks.

5.1. Condition  $C_2$  is topological in the sense that: if  $P$  is a Peano continuum in  $S$ , and  $h$  is a homeomorphism of  $S$  onto itself, then  $P$  satisfies  $C_2$  if and only if  $h(P)$  does. It is not sufficient that  $h$  be a homeomorphism on  $P$  alone; i.e., the embedding is significant. Let  $P$  be the example of [4], modified by replacing each closed disk by its bounding circle and a radius segment;  $P$  does not satisfy condition  $C_1$  (see the first remark of §2 of [4]). Let  $P'$  be  $P$ , except that each radius segment is turned outside the disk; then  $P'$  is the boundary of a simply connected region, but  $P$  and  $P'$  are homeomorphic.

One might ask: can  $D$  in 1.1 be replaced by an arbitrary simply connected region  $U$  whose boundary is a nonempty and nondegenerate Peano continuum? In this case  $C_1$  implies  $C_2$ . For the proof, let  $f$  be the function meromorphic on  $U$  such that  $f(\text{bdy } U) = P$ . There exists a continuous function  $g: \bar{D} \rightarrow \bar{U}$ , such that  $g$  is a meromorphic homeomorphism on  $D$  (by the Riemann Mapping Theorem and 2.1), and  $P = (fg)(\text{bdy } D)$ . Thus, 1.1 can be applied. On the other hand, condition  $C'_2$  does not imply  $C'_1$  in this case. Let  $\lambda$  be a radius segment in  $D$ , let  $U = D - \lambda$ , and suppose that there exists a continuous function  $f: \bar{U} \rightarrow \bar{D}$  with  $f$  analytic on  $U$  and  $f(\text{bdy } U) = \text{bdy } D$ . From the openness of  $f$  and the fact that  $f(U) \subset D$  it follows that  $f$  on  $\text{bdy } U$  as a function of the prime ends (whose im-

pressions are single points) is either sense-preserving or -reversing. As a result,  $f$  could not be continuous on  $\text{bdy } U$ ; thus  $U$  does not satisfy condition  $C'_1$ .

Next, consider the following question: for what functions  $\alpha: \text{bdy } D \rightarrow S - \{\infty\}$  is there a continuous function  $f: \bar{D} \rightarrow S$  with  $f|_D$  analytic and  $f|_{\text{bdy } D} = \alpha$ ? Morse [9] and Titus [21 and 22] considered this question for the special case of functions  $\alpha$  with continuous and nonzero first derivative. Morse gave a sufficient condition for a function  $\alpha$  to be a boundary function in this sense, and Titus considered a subclass called normal representations and gave a necessary and sufficient condition for such functions to be boundary functions. Related results are given in Titus [20] and Titus and Young [23].

For the general question (no assumption on the differentiability of  $\alpha$ ), a necessary condition in terms of the range of  $\alpha$  is, of course, given by 1.1. Also, by Riesz-Nevalinna Theorem [10, p. 19] either  $\alpha$  is a constant function, or  $\alpha$  is constant on no arc. More generally, let  $f: \bar{D} \rightarrow S$  be continuous on  $\bar{D}$ , light open on  $D$ , with  $f(\text{bdy } D)$  neither  $S$  nor a single point; let  $gh$  be the factorization of 4.3, and let  $\alpha$ ,  $\beta$ , and  $\gamma$  be the restrictions of  $f$ ,  $h$  and  $g$  to  $\text{bdy } D$ , respectively. Then  $\beta$  is a sense-preserving (or -reversing) map of the circle onto itself, and thus differs from a homeomorphism only in that some point inverses may be arcs. Since no point inverse of  $\alpha$  is an arc,  $\beta$  (and thus  $h$ ) is a homeomorphism. To within topological equivalence, then, the question is reduced to a topological question (which still seems imposing), namely: What is the class of boundary functions  $\alpha$  for light open maps, where  $\alpha$  maps no arc onto a single point?

In [21] and [22] Titus used a special case of this approach. He proved that for each normal representation  $\alpha$  in his class, there is a map  $f: \bar{D} \rightarrow S - \{\infty\}$ ,  $f$  light open on  $D$  and a local homeomorphism on a neighborhood of  $\text{bdy } D$ . As he observed, in this case the above factorization follows directly from the theorems of Stoilow [19, p. 121] and Carathéodory [1, p. 86], where  $h$  is a homeomorphism of  $\bar{D}$  onto  $\bar{D}$ .

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