

ON POINTS OF JACOBIAN RANK k

BY
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1. **Introduction.** Let $f: M^n \rightarrow N^p$ be C^n , where M^n and N^p are C^n manifolds of dimensions n and p , respectively. Let $R_k(f)$ be the set of points in M^n at which the Jacobian matrix of f has rank at most k . In [3, (1.3)] it was shown that $\dim(f(R_k)) \leq k$. Here the following result is proved:

1.1. **THEOREM.** *If $i_*: \pi_m(N^p - f(R_k)) \rightarrow \pi_m(N^p)$ is the homomorphism on the m th homotopy groups induced by the inclusion map i , then i_* is an isomorphism (onto) for $m + k \leq p - 2$, and is onto for $m + k \leq p - 1$.*

For example, if $f: E^2 \rightarrow E^5$ is a C^2 map, then $\pi_1(E^5 - f(E^2)) = 0$. It is easy to see that some differentiability condition is required.

Also the embedding of the set $f(E)$ of [2, Theorem] (proved in [3]) is discussed. For $n \geq 2$, $F_{n,d}$ is the canonical open map defined by:

$$F_{n,d}(x_1, x_2, \dots, x_n) = (u_1, u_2, \dots, u_n),$$

where $u_1 + iu_2 = (x_1 + ix_2)^d$ and $u_j = x_j$ ($i = \sqrt{-1}$; $d = 1, 2, \dots$; $j = 3, 4, \dots, n$).

1.2. **THEOREM.** *Let $f: M^n \rightarrow N^n$ be C^n and open ($n \geq 2$), where M^n is compact or f is light. Then there exists a closed set E_f , $\dim(E_f) \leq n - 3$, such that for each x in $M^n - E_f$ there exists a neighborhood of x on which f is topologically equivalent to one of the canonical maps $F_{n,d}$ ($d = 1, 2, \dots$). Moreover, $\pi_1(N^n - f(E_f))$ is isomorphic to $\pi_1(N^n)$ under the homomorphism induced by inclusion.*

The set E_f is (defined to be) the minimal set satisfying the second sentence.

NOTATION. The set $R_q(h)$ may be denoted simply by R_q , when this causes no ambiguity. The restriction of a function f to a space X will be denoted by $f|X$, the interior of X by $\text{int } X$, and the closure of X by \bar{X} or $\text{Cl}[X]$. For $f: X \rightarrow E^p$ the coordinate functions will be denoted by f_i ($i = 1, 2, \dots, p$).

The distance between two points x and y will be denoted by $d(x, y)$, and the function d may refer (even in the same sentence) to distance on two different metric spaces; from the context, however, the meaning will be clear. An open sphere about x of radius ε will be denoted by $S(x, \varepsilon)$.

Throughout the paper M^n and N^p will be manifolds (not necessarily connected) of the indicated dimensions.

Although the basic ideas of Lemmas 2.1 and 2.4 are not new, the author could not locate the results in the literature, and thus he includes the proofs for completeness.

2. Preliminary lemmas.

2.1. LEMMA. *Let Q be a finite polyhedron, let $\alpha : Q \rightarrow N^p$ be continuous, let $\nu > 0$, and let V be an open p -cell in N^p . Let $P \subset V$ be a finite polyhedron (in a triangulation of V) with $\dim P + \dim Q \leq p$, let $E \subset P$ be a compact set with $\dim E + \dim Q \leq p - 1$, and let W be an open neighborhood in V of $P \cap \alpha(Q)$. Then there exists a map $\beta : Q \rightarrow N^p$ such that*

- (1) $d(\alpha, \beta) < \nu$;
- (2) β agrees with α on $\alpha^{-1}(N^p - W)$;
- (3) $\beta(\alpha^{-1}(W)) \subset W$; and
- (4) $\beta(Q) \cap E = \emptyset$, and $\beta(Q) \cap P$ is a finite set in the simplices of P with largest dimension.

Proof. Let X and Y be open sets with compact closures (in V) such that $P \cap \alpha(Q) \subset Y$, $\bar{Y} \subset X$, and $\bar{X} \subset W$. Let R be a finite subpolyhedron of Q such that $\alpha^{-1}(\bar{X}) \subset \text{int } R$ and $R \subset \alpha^{-1}(W)$. Choose $\mu > 0$ such that $\mu < \nu$,

- (a) $\mu < d(\alpha(R), V - W)$,
- (b) $\mu < d(\bar{Y}, V - X)$, and
- (c) $\mu < d(\alpha(Q), P \cap (V - Y))$.

Let γ be a simplicial approximation to $\alpha|_R$ such that $d(\alpha|_R, \gamma) < (1/3)\mu$. By a general position argument we may move the polyhedron $\gamma(R)$ slightly, defining a simplicial map $\delta : R \rightarrow V$ such that $d(\alpha|_R, \delta) < (2/3)\mu$ and either (i) $\delta(R) \cap P = \emptyset$, in case $\dim P + \dim Q < p$, or (ii) $\delta(R) \cap P$ is a finite set of points disjoint from the $((\dim P) - 1)$ -skeleton of P , in case $\dim P + \dim Q = p$. In case (i), let the map $\lambda : R \rightarrow V$ be δ . In case (ii) E is nowhere dense in P [5, p.46]; thus by moving $\delta(R)$ slightly we may define λ such that $d(\alpha|_R, \lambda) < \mu$ and $\lambda(R) \cap E = \emptyset$.

Define β by: $\beta = \alpha$ on $Q - \text{int } R$, $\beta = \lambda$ on $\alpha^{-1}(\bar{X})$, and $\beta = t \cdot \lambda + (1 - t)\alpha$ elsewhere, where

$$t(x) = \frac{d(x, Q - \text{int } R)}{d(x, Q - \text{int } R) + d(x, \alpha^{-1}(\bar{X}))}.$$

That β is continuous and satisfies conclusions (1) and (2) is immediate. For (3), it suffices to prove that $\beta(R) \subset W$, which follows from (a) and the fact that $d(\alpha, \beta) < \mu$. For (4) $\beta(R - \alpha^{-1}(X)) \subset V - Y$ by (b), and $\beta(R - \alpha^{-1}(X)) \cap P = \emptyset$ by (c). Thus $\beta(Q - \alpha^{-1}(X)) \cap P = \emptyset$, and (4) follows.

2.2. REMARK. Let $f : M^n \rightarrow E^p$ be a C^n map, where M^n is a C^n manifold. There exists a countable number of open Euclidean coordinate neighborhoods N_i which cover M^n , and C^n diffeomorphisms $h^i : N_i \rightarrow E^n$ ($i = 1, 2, \dots$). We may suppose that the sets $h^i(N_i)$ are mutually disjoint; let $V = \bigcup_{i=1}^{\infty} h^i(N_i)$, and define $g : V \rightarrow E^1$ by the restriction $g|_{h^i(N_i)} = f((h^i)^{-1})$. Then $g(R_0(g)) = f(R_0(f))$.

If X_j is the set of critical points of g_j (the points at which all first partials are zero), then the Lebesgue measure of $g_j(X_j)$ is zero [8, p. 68, (4.3)] ($j = 1, 2, \dots, p$). Since $R_0(g) = \bigcap_{j=1}^p X_j$, for each $\varepsilon > 0$ there exist real numbers a_j^i ($i = 0, \pm 1, \dots; j = 1, 2, \dots, p$) such that $|a_j^i - a_j^{i+1}| < \varepsilon$ and the $(p - 1)$ -planes $g_j = a_j^i$ are disjoint from $g(R_0(g)) (= f(R_0(f)))$.

2.3. REMARK. In [3, (1.1)], if the dependent variables are ordered so that $\det [D_j h_i](\bar{x}) \neq 0$ ($i = 1, 2, \dots, q$; some q values of j), then k^2 may be chosen to be the identity; thus $g^{-1}(\alpha)$ is actually $h^{-1}(\alpha)$. To see this, replace the version of the rank theorem used in [3] by [7, pp. 7-8].

2.4. LEMMA. If W_i is an open cover of the locally compact metric space R , then there exists a cover X_i closed in R such that $X_i \subset W_i$ ($i = 1, 2, \dots, L$).

Proof. If $L = 1$, then $X_1 = W_1$. Otherwise we proceed by induction. Let $Y = R - \bigcup_{i \neq 1} W_i$. For each $y \in Y$ let $S(y, \varepsilon^y)$ be a sphere in R with compact closure, $d(y, R - W_1) > 2\varepsilon^y$. Let S be the union of the closed spheres, and let $X_1 = \bar{S}$ (closure in R). Suppose that $x^j \rightarrow x$, where $x^j \in S$ and $x \in R$, and that $x^j \in \text{Cl}[S(y^j, \varepsilon^j)]$. If $\varepsilon^j \rightarrow 0$, then $y^j \rightarrow x$. Thus x is in the closed set Y , and thus is in W_1 . If $\varepsilon^j \not\rightarrow 0$, then there is a subsequence (which we still call ε^j) such that $\varepsilon^j \geq \varepsilon > 0$. For some J $d(x^J, x) < \varepsilon$, and thus $d(y^J, x) < 2\varepsilon^J$. Hence $x \notin R - W_1$. As a result, $Y \subset \text{int}(X_1)$ and $X_1 \subset W_1$.

Since $R - \text{int}(X_1)$ is locally compact with open cover $W_i \cap (R - \text{int}(X_1))$ ($i = 2, 3, \dots, L$), there exists by inductive hypothesis a closed (in $R - \text{int}(X_1)$) cover X_i . The sets X_i ($i = 1, 2, \dots, L$) are the desired closed covers of R .

DEFINITION. An open (resp., closed) coordinate p -rectangle in E^p is the set of points (x_1, x_2, \dots, x_p) such that $a_i < x_i < b_i$ (resp., $a_i \leq x_i \leq b_i$) for some real a_i and b_i ($i = 1, 2, \dots, p$).

2.5. LEMMA. Let L be a compact subset of E^p , and let $\{T_i\}$ ($i = 1, 2, \dots, r$) be a cover of L by open coordinate p -rectangles. Then there exist a bounded open (in E^p) cover $\{H_j\}$ ($j = 1, 2, \dots, s$) of L and sets $\mathcal{N}_j \subset \{1, 2, \dots, r\}$ such that:

- (1) if $i \in \mathcal{N}_j$, then $H_j \subset T_i$;
- (2) if $H_j \cap H_k \neq \emptyset$ and $j > k$, then $\mathcal{N}_k \subset \mathcal{N}_j$;
- (3) there exists $\delta > 0$ such that each sphere $S(y, \delta)$ meets at most 2^p sets of the cover H_j ;
- (4) there is a cover $\{G_j\}$ of L by closed coordinate p -rectangles with $G_j \subset H_j$ ($j = 1, 2, \dots, s$).

Proof. There is a cover $\{U_i\}$ of L by closed coordinate p -rectangles with $U_i \subset T_i$; let $\mathcal{U}(\{i\}) = \{U_i\}$ ($i = 1, 2, \dots, r$). Since $U_h \cap U_i$ is compact, it has a finite cover by open coordinate p -rectangles each of whose closures is contained in $T_h \cap T_i$ (if $U_h \cap U_i = \emptyset$, then the cover is empty). Let $\mathcal{U}(\{h, i\})$ be the set of these closed p -rectangles, and let $U(\{h, i\})$ be their union ($h \neq i; h, i = 1, 2, \dots, r$).

We continue by induction. Suppose that, for all $\mathcal{I} \subset \{1, 2, \dots, r\}$ with at most A elements, $\mathcal{U}(\mathcal{I})$ and $U(\mathcal{I})$ have been defined such that:

(A, 1) $U(\mathcal{I}) \subset \bigcap T_i \ (i \in \mathcal{I})$.

(A, 2) Suppose that $x \in U(\mathcal{I}) \cap U(\mathcal{K})$, where each of \mathcal{I} and \mathcal{K} has less than A elements, $\mathcal{I} \not\subset \mathcal{K}$ and $\mathcal{K} \not\subset \mathcal{I}$. Then x is in the interior of a p -rectangle of $\mathcal{U}(\mathcal{I})$, for some \mathcal{I} with more elements than either \mathcal{I} or \mathcal{K} . (In general, this \mathcal{I} is not unique.)

Let $\mathcal{I} \subset \{1, 2, \dots, r\}$ with $A + 1$ elements be given. For each pair of subsets \mathcal{J} and \mathcal{K} of \mathcal{I} such that $\mathcal{J} \not\subset \mathcal{K}$, $\mathcal{K} \not\subset \mathcal{J}$, and $\mathcal{J} \cup \mathcal{K} = \mathcal{I}$, the compact set $U(\mathcal{J}) \cap U(\mathcal{K})$ has a finite cover (possibly empty) by open coordinate p -rectangles, each of whose closures is contained in $\bigcap T_i \ (i \in \mathcal{I})$, by (A, 1). Let $\mathcal{U}(\mathcal{I})$ be the set of all these closed p -rectangles for all such pairs \mathcal{J} and \mathcal{K} , and let $U(\mathcal{I})$ be the union of the p -rectangles of $\mathcal{U}(\mathcal{I})$. Then the sets $\mathcal{U}(\mathcal{I})$ and $U(\mathcal{I})$ for \mathcal{I} with $A + 1$ elements satisfy (A + 1, 1) and (A + 1, 2). Thus $\mathcal{U}(\mathcal{I})$ and $U(\mathcal{I})$ are defined for all $\mathcal{I} \subset \{1, 2, \dots, r\}$.

Consider those $(p - 1)$ -planes each of which contains a $(p - 1)$ -face of a p -rectangle of a set $\mathcal{U}(\mathcal{I}) \ (\mathcal{I} \subset \{1, 2, \dots, r\})$. The planes bound a finite number of closed coordinate p -rectangles; let $G_j \ (j = 1, 2, \dots, s)$ be those each of which is contained in a set $U(\mathcal{I})$. Then $\{G_j\}$ is a cover of L of order at most 2^p .

Let \mathcal{N}_j be a maximal subset \mathcal{I} of $\{1, 2, \dots, r\}$ such that $G_j \subset U(\mathcal{I})$; it follows from (r, 2) that there is only one maximal set. We may suppose that the set $\{1, 2, \dots, s\}$ of indices of G_j has been ordered so that $k < j$ implies that \mathcal{N}_k does not have more elements than \mathcal{N}_j . That the G_j satisfy conclusion (1) follows from (r, 1). Suppose that there exists $x \in G_j \cap G_k$, where $j > k$ and $\mathcal{N}_k \not\subset \mathcal{N}_j$; then $x \in U(\mathcal{N}_j) \cup U(\mathcal{N}_k)$, contradicting (r, 2) and the definition (maximality) of \mathcal{N}_j . Thus the sets G_j satisfy conclusion (2). Choose $\delta > 0$ such that each edge of each p -rectangle G_j has length greater than 4δ , and let

$$H_j = \{x \in E^p : d(x, G_j) < \delta \text{ and } x \in T_i \ (i \in \mathcal{N}_j)\}.$$

That the sets $H_j \ (j = 1, 2, \dots, s)$ satisfy the conclusions is immediate.

3. The proof of (1.1).

3.1. LEMMA. Let $h : U \rightarrow E^p$, where U is open in E^n , h is C^n , and E^p is a Euclidean coordinate neighborhood of N^p . Suppose that some $q \times q$ subdeterminant D of the Jacobian determinant of h is nonzero on $U \ (q = 1, 2, \dots; \text{ no assumption is made if } q = 0)$, and that Y is a compact subset of U . Let Q be a finite polyhedron such that $q + \dim Q \leq p - 1$, let $\alpha : Q \rightarrow N^p$ be continuous, and let $\Delta \subset Q$ be compact with

$$\alpha(\Delta) \cap h(Y \cap R_q(h)) = \emptyset.$$

Then, given $\eta > 0$, there exists a map $\gamma : Q \rightarrow N^p$ such that

- (1) $d(\alpha, \gamma) < \eta$,
- (2) γ agrees with α on $\Delta \cup \alpha^{-1}(N^p - E^p)$, and
- (3) $\gamma(Q) \cap h(Y \cap R_q(h)) = \emptyset$.

Proof. By (2.2) there exist in E^p a finite number of disjoint closed coordinate p -rectangles of diameter less than $\eta/2$, such that the union K is disjoint from $\alpha(\Delta)$ and $h(R_0(h) \cap Y) \subset \text{int} K$. Let β be the map given by (2.1) for α , $V = E^p$, $E = P$ the union of the center points of the p -rectangles of K , $W = \text{int} K$, and $v = \eta/2$. Let $g : N^p - P \rightarrow N^p - \text{int} K$ be the identity on $N^p - \text{int} K$, and, on each rectangle of K , projection from the center point to the boundary. Then $\gamma = g\beta$ is the desired map for $q = 0$.

For $q = 1, 2, \dots$ we may as well suppose that the dependent variables of D are indexed by $1, 2, \dots, q$, and write E^p as $E^q \times E^{p-q}$. Let $\pi : E^p \rightarrow E^q$ be the projection map. By [3, (1.1)] and (2.3), for each $w \in E^q$, $h^{-1}(\{w\} \times E^{p-q})$ is a C^n $(n - q)$ -manifold A_w (not necessarily connected). Also $R_q(h) \cap A_w = R_0(h|_{A_w})$, where $h|_{A_w}$ is the restriction map; hereafter, R_q denotes $R_q(h)$.

By (2.2) and the above remark, for each $w \in E^q$, there exists in E^{p-q} a finite union $K(w)$ of mutually disjoint closed coordinate $(p - q)$ -rectangles of diameter less than $\eta \cdot 2^{-q-1}$, such that

$$\begin{aligned} h(R_q \cap A_w \cap Y) &= h(R_q \cap Y) \cap (\{w\} \times E^{p-q}) \\ &\subset (\{w\} \times \text{int}(K(w))). \end{aligned}$$

We may suppose that

$$\eta \cdot 2^{-q-1} < d(h(R_q \cap Y), \alpha(\Delta)).$$

Since $h(R_q \cap Y)$ is compact, for each $w \in E^q$ and for v sufficiently near w ,

$$(*) \quad h(R_q \cap Y) \cap (\{v\} \times E^{p-q}) \subset (\{v\} \times \text{int}(K(w))).$$

Thus there exists an open coordinate q -rectangle neighborhood $T(w)$ such that $(*)$ holds for each $v \in \bar{T}(w)$, and $\alpha(\Delta) \cap (\bar{T}(w) \times K(w)) = \emptyset$. The compact subset $\pi(h(R_q \cap Y))$ in E^q has a finite subcover $\{T(w^j)\}$ ($j = 1, 2, \dots, r$). Let G_i, H_i, \mathcal{A}_i ($i = 1, 2, \dots, s$) and $\delta > 0$ be the sets and number given by (2.5) for $L = \pi(h(R_q \cap Y))$; we may suppose that $\delta < \eta/2$. Let K_i be the intersection of $K(w^j)$ for $j \in \mathcal{A}_i$. Then $\{w\} \times K_i$ is also the finite union (possibly empty) of mutually disjoint closed coordinate $(p - q)$ -rectangles, containing in its interior

$$h(R_q \cap Y) \cap (\{w\} \times E^{p-q})$$

for each w in H_i (by relation $(*)$ above, and by conclusion (1) of (2.5)). Thus

$$h(Y \cap R_q) \subset \bigcup_{i=1}^s (G_i \times \text{int}(K_i)).$$

Also $\alpha(\Delta) \cap (\bar{H}_i \times K_i) = \emptyset$ ($i = 1, 2, \dots, s$).

Let S_i be the union of the center points of the $(p - q)$ -rectangles in K_i . Then $P_i = G_i \times S_i$ is a q -dimensional finite subpolyhedron in (the triangulated set) E^p ($i = 1, 2, \dots, s$). Let β^1 be the map given by (2.1) for $\alpha, \nu = \delta \cdot 2^{-q}$, $E = P = P_1$ and $W = H_1 \times \text{int}(K_1)$. Thus $\beta^1(Q) \cap P_1 = \emptyset$ by conclusion (4) of (2.1).

Given any map $\beta^i: Q \rightarrow N^p$ such that $\beta^i(Q) \cap P_i = \emptyset$, we now define a map $\gamma^i: Q \rightarrow N^p$ ($i = 1, 2, \dots, s$). Choose $\mu_i > 0$, μ_i less than the minimum of $d(\beta^i(Q), P_i)$, and $d(G_i, E^q - H_i)$; and let

$$F_i = \{w \in E^q : d(w, G_i) < \mu_i\}.$$

Thus $P_i \subset F_i \times S_i$, and $\beta^i(Q) \cap (F_i \times S_i) = \emptyset$. We will define a map g^i such that its domain and range are $N^p - (F_i \times S_i)$, and on $N^p - (F_i \times K_i)$ g^i is the identity map. For each $w \in F_i$,

$$g^i(\{w\} \times (E^{p-q} - S_i)) \subset (\{w\} \times (E^{p-q} - S_i)),$$

i.e., $\pi g^i = \pi$. For each $w \in G_i$, g^i is the natural retraction map, i.e., for $x \in E^{p-q} - K_i$, g^i is the identity map, and for x in a $(p - q)$ -rectangle of K_i , g^i projects x from the center of the rectangle to the boundary. For $w \in F_i - G_i$ replace each (coordinate) $(p - q)$ -rectangle of K_i , say of side t , with a coordinate $(p - q)$ -rectangle of the same center and side $(\mu_i - d(w, G_i))(\mu_i)^{-1}t$. Define g^i on $\{w\} \times (E^{p-q} - S_i)$ analogously to its definition for $w \in G_i$. Then g^i is continuous.

We shall show that the map γ^i defined by $\gamma^i = g^i \beta^i$ ($i = 1, 2, \dots, s$) satisfies the following properties:

- (a) $\pi \gamma^i = \pi \beta^i$;
- (b) $d(\gamma^i, \beta^i) < \eta \cdot 2^{-q-1}$;
- (c) $\gamma^i = \beta^i$ on $Q - (\beta^i)^{-1}(H_i \times K_i)$ and $\gamma^i((\beta^i)^{-1}(H_i \times K_i)) \subset H_i \times K_i$;
- (d) if $\beta^i(Q) \cap (G_j \times \text{int}(K_j)) = \emptyset$ ($j = 1, 2, \dots, i - 1$), then

$$\gamma^i(Q) \cap (G_m \times \text{int}(K_m)) = \emptyset \quad (m = 1, 2, \dots, i).$$

Property (a) holds because $\pi g^i = \pi$, (b) because the diameters of the $(p - q)$ -rectangles of K_i are less than $\eta \cdot 2^{-q-1}$, and (c) is immediate.

Property (d) is clear for $m = i$. For those j such that $H_i \cap H_j = \emptyset$, (d) follows from (c). If $H_i \cap H_j \neq \emptyset$ ($j = 1, 2, \dots, i - 1$), then $\mathcal{N}(j) \subset \mathcal{N}(i)$ (by conclusion (2) of (2.5)), so that $K_i \subset K_j$. Suppose that $\gamma^i(Q) \cap (G_j \times \text{int}(K_j)) \neq \emptyset$. Then, by (c) and the hypothesis of (d),

$$\gamma^i((\beta^i)^{-1}(H_i \times \text{int}(K_i))) \cap (G_j \times \text{int}(K_j)) \neq \emptyset.$$

Since $\pi g^i = \pi$ and $g^i = \gamma^i(\beta^i)^{-1}$,

$$\gamma^i((\beta^i)^{-1}((H_i \cap G_j) \times \text{int}(K_i))) \cap (G_j \times \text{int}(K_j)) \neq \emptyset.$$

Now $K_i \subset K_j$, so that

$$\gamma^i((\beta^i)^{-1}(G_j \times \text{int}(K_j))) \cap (G_j \times \text{int}(K_j)) \neq \emptyset;$$

by the hypothesis of (d), we have a contradiction. Thus (d) is satisfied.

A map γ^1 has been defined. We continue by induction. A set of maps $\{\gamma_i\}$ ($i = 1, 2, \dots, I$) is said to satisfy property \mathcal{P}_I if

- (a) $d(\pi\gamma^i, \pi\gamma^{i-1}) < \delta \cdot 2^{-q}$ ($\gamma^0 = \alpha$),
- (b) $d(\gamma^i, \gamma^{i-1}) < \eta \cdot 2^{-q}$,
- (c) $\gamma^i = \gamma^{i-1}$ on $\alpha^{-1}(N^p - E^p) \cup \Delta \cup (Q - (\gamma^{i-1})^{-1}(H_i \times K_i))$,
- (d) $\gamma^i(Q) \cap (G_j \times \text{int}(K_j)) = \emptyset$ ($j = 1, 2, \dots, i; i = 1, 2, \dots, I$). The set $\{\gamma^i\}$

satisfies \mathcal{P}_I (for (c) remember that $\alpha(\Delta) \cap (\bar{H}_i \times K_i) = \emptyset$). Given $\{\gamma^i\}$ ($i = 1, 2, \dots, I$) satisfying \mathcal{P}_I , we will now define γ^{I+1} such that $\{\gamma^i\}$ ($i = 1, 2, \dots, I + 1$) satisfies \mathcal{P}_{I+1} .

Let $X = P_{I+1} \cap \gamma^I(Q)$. Since $P_{I+1} = G_{I+1} \times S_{I+1}$, if $H_i \cap H_{I+1} = \emptyset$, then $X \cap (G_i \times K_i) = \emptyset$ ($i = 1, 2, \dots, I$). If $H_i \cap H_{I+1} \neq \emptyset$, then $\mathcal{N}_i \subset \mathcal{N}_{I+1}$ by conclusion (2) of (2.5), so that $K_{I+1} \subset K_i$. Since $P_{I+1} \subset (E^q \times \text{int}(K_{I+1}))$, $X \cap (G_i \times \text{bdy}(K_i)) = \emptyset$. By \mathcal{P}_I , (d), X is disjoint from $G_i \times \text{int}(K_i)$ and thus from $G_i \times K_i$. Thus $X \subset W$, where

$$W = (H_{I+1} \times \text{int}(K_{I+1})) - \bigcup_{i=1}^I (G_i \times K_i).$$

Apply (2.1) for $\alpha = \gamma^I$, $E = P = P_{I+1}$, W , and $\nu < \delta \cdot 2^{-q}$, calling the resulting map β^{I+1} . Then $\beta^{I+1}(Q) \cap P_{I+1} = \emptyset$. Let γ^{I+1} be the map given for β^{I+1} .

That $\{\gamma^i\}$ ($i = 1, 2, \dots, I + 1$) satisfies \mathcal{P}_{I+1} follows from the corresponding properties for $\{\beta^i\}$ ($i = 1, 2, \dots, I$), the definitions of β^{I+1} and of γ^{I+1} from β^{I+1} , and the fact that $\delta < \eta/2$. For (d), observe that $\beta^{I+1} = \gamma^I$ on $(\gamma^I)^{-1}(N^p - W)$ and $\beta^{I+1}(W) \subset W$. Thus $\beta^{I+1}(Q) \cap (G_i \times \text{int}(K_i)) = \emptyset$ ($i = 1, 2, \dots, I$) by \mathcal{P}_I , (d).

The desired map γ is γ^s . Given any point $\alpha(x)$ in E^p , $S(\pi\alpha(x), \delta)$ (in E^q) meets H_i for at most 2^q different i , by the definition of δ . As a result, by \mathcal{P}_s , (a) and (c), $\pi\gamma^i(x) \in S(\pi\alpha(x), \delta)$ ($i = 1, 2, \dots, s$). By \mathcal{P}_s , (b) and the definition of δ , $d(\gamma, \alpha) < (\eta \cdot 2^{-q}) \cdot 2^q = \eta$. The remaining properties of γ are immediate from \mathcal{P}_s , and the fact that

$$h(Y \cap R_q) \subset \bigcup_{i=1}^s (G_i \times \text{int}(K_i)).$$

3.2. LEMMA. *Let $f: M^n \rightarrow N^p$ be a C^n map, and let k be a non-negative integer and Q a finite polyhedron such that $k + \dim Q \leq p - 1$. Let ε be positive, let $\Delta \subset Q$ be compact, and let $\alpha: Q \rightarrow N^p$ be a map such that $\alpha(\Delta) \cap f(R_k(f)) = \emptyset$. Then there exists a map $\gamma: Q \rightarrow N^p$ such that $\gamma(Q) \cap f(R_k(f)) = \emptyset$, $\gamma|_\Delta = \alpha|_\Delta$, and $d(\alpha, \gamma) < \varepsilon$.*

Proof. Choose compact sets $A_i \subset M^n$, and Euclidean coordinate neighbor-

hoods $U_i \subset M^n$ and $V_i \subset N^p$ ($i = 1, 2, \dots$) such that $R_k \subset \bigcup_i A_i$, $A_i \subset U_i$, and $f(U_i) \subset V_i$. For each i and q ($q = 1, 2, \dots, \min(n, k)$) let $D_{i,q,j}$ be an enumeration of the $q \times q$ subdeterminants of the Jacobian determinant of $f|U_i$; let $W_{i,q,j}$ be the set of points in U_i at which $D_{i,q,j} \neq 0$. Then each set $R_q \cap W_{i,q,j}$ is open in R_q , and $R_q \cap \bigcup_j W_{i,q,j} = U_i \cap (R_q - R_{q-1})$. Let $X_{i,q,j}$ be the sets closed in $A_i \cap (R_q - R_{q-1})$, given by (2.4), such that

$$X_{i,q,j} \subset A_i \cap (R_q - R_{q-1}) \cap W_{i,q,j}$$

and $\bigcup_j X_{i,q,j} = A_i \cap (R_q - R_{q-1})$; let $X_{i,0,1} = A_i \cap R_0$.

We may order the sets $X_{i,q,j}$ lexicographically (for each i , there are only a finite number of q and j), and rename them C_s ($s = 1, 2, \dots$). Some of the sets may be empty. For each S there exist L, Q , and T ($S, T = 1, 2, \dots; T \leq S; L, Q = 0, 1, \dots$) such that

$$\bigcup_{s=1}^T C_s = \left[\left(\bigcup_{i=1}^L A_i \right) \cap R_k \right] \cup (A_{L+1} \cap R_Q)$$

and $\bigcup C_s$ ($s = T + 1, T + 2, \dots, S$) is the union of sets $X_{L+1, Q+1, j}$ for none, some or all of the possible j . Since each of the limit points (in M^n) of the second set is either in that set or in $A_{L+1} \cap R_Q$, $\bigcup C_s$ ($s = 1, 2, \dots, S$) is compact. Let G_S denote this set.

Let γ^1 be the map given by (3.1) for $h = f|U_1$, $E^p = V_1$, $q = 0$, $Y = A_1$, and $\eta = \varepsilon/2$. Let $\varepsilon_1 = \varepsilon/2$, and let $\delta_1 \leq d(\gamma^1(Q), f(C_1))$, $\delta_1 > 0$.

Suppose that a set of maps $\gamma^s: Q \rightarrow N^p$ and numbers $\varepsilon_s > 0$ and $\delta_s > 0$ ($s = 1, 2, \dots, S$) have been defined such that

(S, 1) $\gamma^s | \Delta = \gamma^{s-1} | \Delta$ ($\gamma^0 = \alpha$),

(S, 2) $d(\gamma^s(Q), f(G_s)) \geq \delta_s$, and

(S, 3) $d(\gamma^s, \gamma^{s-1}) < \varepsilon_s$, where ε_s is the minimum of $\varepsilon \cdot 2^{-s}$ and $\delta_r \cdot 2^{-s+r-1}$ ($r = 1, 2, \dots, s - 1$).

Let ε_{S+1} be defined as in (S, 3), let

$$Z = \{x \in N^p : d(x, f(G_S)) < \delta_S/2\},$$

and let $Y = (N^p - Z) \cap f(C_{S+1})$. Since $\bar{C}_{S+1} - C_{S+1} \subset G_S$, Y is compact. Let γ^{S+1} be the map given by (3.1) for $h = f|W_{i,q,j}$ (where $C_{S+1} = X_{i,q,j}$), $E^p = V_i$, the above set Y , $\alpha = \gamma^S$, and $\eta = \varepsilon_{S+1}$. Then $\gamma^{S+1} | \Delta = \gamma^S | \Delta$, and $\gamma^{S+1}(Q) \cap f(Y) = \emptyset$. Since $d(\gamma^{S+1}, \gamma^S) < \varepsilon_{S+1} \leq \delta_S/4$, $\gamma^{S+1}(Q) \cap Z = \emptyset$. Thus there exists $\delta_{S+1} > 0$ such that

$$d(\gamma^{S+1}(Q), f(G_{S+1})) \geq \delta_{S+1}.$$

Let γ be the limit of the maps γ^s ($s = 1, 2, \dots$); then $d(\alpha, \gamma) < \varepsilon$ by (S, 3) and $\alpha | \Delta = \gamma | \Delta$ by (S, 1). Now (for $r = 1, 2, \dots, s - 1; s = 1, 2, \dots$),

$$d(\gamma^s(Q), f(G_r)) > \delta_r - \sum_{j=r+1}^s \varepsilon_j \geq \delta_r \left(1 - \sum_{j=r+1}^s 2^{-s+j-1} \right) \geq \delta_r/2.$$

Thus $\gamma(Q) \cap f(G_r) = \emptyset$ ($r = 1, 2, \dots$), so that $\gamma(Q) \cap f(R_k) = \emptyset$.

The proof of (1.1). That i_* is one-to-one follows from (3.2) with $Q = S^m \times [0, 1]$ and $\Delta = (S^m \times \{0, 1\}) \cup (\{x_0\} \times [0, 1])$ ($m + k \leq p - 2$). It is well known that, given $\alpha : S^m \rightarrow N^p$, $\alpha(x_0) = y_0$, there exists $\varepsilon > 0$ such that, if $\gamma : S^m \rightarrow N^p$, $\gamma(x_0) = y_0$, then γ is homotopic to α in N^p . That i_* is onto follows from (3.2) for $Q = S^m$ and $\Delta = \{x_0\}$ ($m + k \leq p - 1$).

3.3. REMARKS. Actually, we have proved more than (1.1) states. Given any $\varepsilon > 0$ and a map or homotopy α in N^p , the map or homotopy γ found in $N^p - f(R_k)$ has $d(\alpha, \gamma) < \varepsilon$.

For all n and p and $f : M^n \rightarrow N^p$ only C^{n-1} ($n \geq 2$), (1.1) is not true. Let $g_j : E^j \rightarrow E^1$ be the maps given by [9] with critical arcs A_j ; let $H_j = g_j^{-1}((0, 1))$ ($[0, 1] = g_j(A_j)$), let $r : (0, 1) \rightarrow E^1$ be a C^∞ homeomorphism, and let $h_j : H_j \rightarrow E^1$ be the restriction map $rg_j|_{H_j}$ ($j = 1, 2, \dots, n$). For $p = 1$, h_n is the desired example f since $E^1 = h_n(R_0(h_n))$. Given p and j ($j = 2, 3, \dots, \min(n, p)$; $p = 2, 3, \dots$), define

$$f : H_j \times E^{n-j} \rightarrow E^p$$

by $f_i(x_1, x_2, \dots, x_n) = h_j(x_1, x_2, \dots, x_j)$ ($i = 1, 2, \dots, j$); $f_i(x_1, x_2, \dots, x_n) = x_i$ ($i = j + 1, j + 2, \dots, \min(n, p)$); and $f_i(x_1, x_2, \dots, x_n) = 0$ ($i = \min(n, p) + 1, \min(n, p) + 2, \dots, p$). Then $f(R_{\min(n,p)-j}(f))$ is a tame open $(\min(n, p) - j + 1)$ -cell in E^p , and

$$\pi_{p-\min(n,p)+j-2}(E^p - f(R_{\min(n,p)-j}(f))) \neq \emptyset.$$

Also, it is easy to see that (1.1) does not hold for $f(R_k(f))$ replaced by $Cl[f(R_k(f))]$.

4. **The proof of (1.2).** The set E_f is the minimal set E satisfying [3, (2.1)], and $E_f \subset B_f$ (definition in [4, p. 528]).

4.1. LEMMA. Suppose that in (3.1) $n = p \geq 3$, h is C^n light open, $q = n - 2$, and $\dim Q \leq 2$ (rather than $q + \dim Q \leq p - 1$). If $R_q(h)$ is replaced by E_h , then the corresponding conclusion holds.

Proof. We will describe only the changes in the proof of (3.1) which are required here. Of course the first paragraph of that proof is irrelevant in this case. In the second paragraph, and thereafter, $R_q(h)$ is replaced by E_h (and the statement relating R_0 and R_q is omitted).

The restriction map $h|_{A_w}$ is light open [10, p. 147, (7.2)], and the union of the branch sets $B(h|_{A_w})$ is B_h [3, proof of (2.1)]. Thus

(1) $h(B_h \cap Y)$, and in particular $h(E_h \cap Y)$, meets each plane $\{w\} \times E^2$ in a finite set [10, p. 198, (5.1)].

(2) Let $\Lambda = \pi(h(E_h \cap Y))$; then Λ is a compact nowhere dense subset of E^{n-2} [3, proof of (2.1)]. Thus $\dim \Lambda \leq n - 3$ [5, p. 44]. (Actually in [3, proof of (2.1)] the analogous—and equivalent—statement for the projection of E_h itself was shown.)

Because of (1), the sets $G_i, H_i,$ and K_i may be defined as before, with $L = \Lambda$. Let β^1 be the map given by (2.1) for $\alpha, v = \delta \cdot 2^{-q}, V = E^n, P = P_1, E = (G_1 \cap \Lambda) \times S_1$ ($\dim E \leq n - 3$ by (2)), and $W = H_1 \times \text{int}(K_1)$. Thus $\beta^1(Q) \cap P_1$ is a finite set in $\text{int}(P_1)$, and

$$\beta^1(Q) \cap ((G_1 \cap \Lambda) \times S_1) = \emptyset.$$

Given any map $\beta^i : Q \rightarrow N^p$ such that $\beta^i(Q) \cap P_i$ is a finite set $\{a_{i,j}\}$ ($j = 1, 2, \dots, k_i$) and

$$\beta^i(Q) \cap ((G_i \cap \Lambda) \times S_i) = \emptyset,$$

we now define a map $\gamma^i : Q \rightarrow N^p$. Define μ_i so that, in addition to its previous properties,

$$\mu_i < d(\pi(a_{i,j}), \Lambda \cup \text{bdy}(G_i))$$

($j = 1, 2, \dots, k_i$). Define g^i as before except on $\{w\} \times K_i$ for $(w, \pi(a_{i,j})) \leq \mu_i$. For each such w replace each (coordinate) 2-rectangle of K_i , say of side t , with a coordinate 2-rectangle of the same center and side $d(w, \pi(a_{i,j}))(\mu_i)^{-1}t$. Define g^i on $\{w\} \times (E^2 - S_i)$ analogously to its definition for w elsewhere in G_i .

Beginning with the properties (a), (b), (c) and (d) of the map γ^i obtained from β^i , the sets G_j should be replaced by $G_j \cap \Lambda$ for the rest of the proof (P_j remains $G_j \times S_j$). The only other change is that in the definition of β^{I+1} (2.1) is applied for $\alpha = \gamma^I, P = P_{I+1}, E = (G_{I+1} \cap \Lambda) \times S_I$,

$$W = (H_{I+1} \times \text{int}(K_{I+1})) - \bigcup_{i=1}^I ((G_i \cap \Lambda) \times K_i),$$

and $v < \delta \cdot 2^{-q}$. Then $\beta^{I+1}(Q) \cap P_{I+1}$ is a finite set, and $\beta^{I+1}(Q) \cap E = \emptyset$.

4.2. LEMMA. *Let $f : M^n \rightarrow N^n$ be a C^n light open map, $n \geq 3$, and let Q be a finite polyhedron with $\dim Q \leq 2$. Let ε be positive, let $\Delta \subset Q$ be compact, and let $\alpha : Q \rightarrow N^n$ be a map such that $\alpha(\Delta) \cap f(E_f) = \emptyset$. Then there exists a map $\gamma : Q \rightarrow N^n$ such that $\gamma(Q) \cap f(E_f) = \emptyset, \gamma|_{\Delta} = \alpha|_{\Delta}$, and $d(\alpha, \gamma) < \varepsilon$.*

The proof is a slight modification of that of (3.2), and we only give the changes and additions. The compact sets A_i are defined so that $E_f = \bigcup_i A_i$.

Define the sets $X_{i,q,j}$ ($q = 0, 1, \dots, n - 3$) as before. Given the locally compact space $A_i - R_{n-3}$, the sets $A_i \cap W_{i,n-2,j}$ form an open cover since $E_f \subset B_f \subset R_{n-2}$ [3, (1.6)]; let $X_{i,n-2,j}$ be the closed cover of $A_i - R_{n-3}$ given by (2.4). As in (3.2) G_s is compact ($s = 1, 2, \dots$).

The only other change occurs in the case that C_{S+1} is a set $X_{i,n-2,j}$ (rather than $X_{i,q,j}; q = 0, 1, \dots, n - 3$): let γ^{S+1} be the map given by (4.1), rather than by (3.1).

The proof of (1.2). We need only prove the last sentence [3, (2.1)]. For $n = 2$

the conclusion is immediate since $E_f = \emptyset$. For $n \geq 3$ the proof is identical to that of (1.1), except that (4.2) is used instead of (3.2) and that $m = 1$.

The last statement (the extension) of (1.1) answers a question S. S. Cairns asked the author. It might be useful because locally such a map is a pseudo-covering map [4, p. 529, Definition 5 and (1.4)].

4.3. REMARK. Let $n = 3$, let V be a coordinate 3-cell in N^3 , and let Y be a compact set in M^3 such that $f(Y) \subset V$. Then *the (at most) 0-dimensional set $f(Y \cap E_f)$ is tame in V* , i.e., there exists a homeomorphism $h: V \rightarrow E^3$ such that $h(f(Y \cap E_f))$ is contained in a straight line.

By arguments analogous to those preceding one may show that, for each $\varepsilon > 0$, there exist mutually disjoint polyhedral 3-cells C_i ($i = 1, 2, \dots, m$) such that $f(E_f \cap Y) \subset \bigcup \text{int}(C_i)$. The conclusion follows from [1, pp. 435–436]. That (1.2) does not imply the conclusion follows from [6]. (The author is indebted to L. V. Keldys for supplying this reference.)

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