REPRODUCING KERNELS AND BEURLING'S THEOREM(1)

BY

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Introduction. Let us denote by l_2 complex square-summable sequences $\{a_n\}$, $r = 0, 1, 2, \cdots$, made into a Hilbert space in the usual way, and by A the "shift operator" on l_2: A maps $(a_0, a_1, \cdots)$ onto $(0, a_0, a_1, \cdots)$. Beurling [3] posed and solved the problem of identifying all closed subspaces of l_2 invariant with respect to A. Passing to the "Fourier transform" $f(z) = \sum_{n=0}^{\infty} a_n z^n$ the problem becomes one of pure function theory: Identify all closed subspaces of H_2 with the property: $f \in S$ implies $zf \in S$. The solution is then given in terms of a product representation (see below).

For purposes of our generalization we prefer to state the condition of invariance: $f \in S$ implies $\phi f \in S$, for every bounded analytic ϕ. Since polynomials span the bounded analytic functions in the topology of bounded pointwise convergence almost everywhere, this formulation is equivalent. P. Lax [8] similarly investigated closed invariant subspaces relative to the semigroup $\{A_\tau\}$ of right translations in the Hilbert space $L^2(0, \infty)$ where $A_\tau f(t)$ is the function equal to 0 for $t < \tau$ and to $f(t - \tau)$ for $t \geqq \tau$ $(\tau > 0)$. Passing to the Fourier transform $F(z) = \int_0^\infty f(t) e^{itz} dt$ (Im $z > 0$) the problem similarly reduces to identifying closed subspaces of the "Paley-Wiener space" of the upper half-plane which are mapped into themselves upon multiplication by bounded analytic functions(2).

In this paper we obtain a product representation for the closed invariant (in the above sense) subspaces of a class of Hilbert spaces of analytic functions which includes both H_2 and the Paley-Wiener space of the half-plane. The class of Hilbert spaces in question (which we define axiomatically) is not very general; however, a Beurling-type theorem does not seem to be true in much greater generality, as may be seen by the consideration of certain simple Hilbert spaces of analytic functions which do not satisfy our axioms. Our treatment of the subject will be based on a systematic exploitation of the notion of reproducing kernel (r.k.). That the Beurling factorization stands in close relationship with the notion of

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(2) Lax and several other authors consider also extensions to operator-valued functions. Other proofs of Beurling's theorem have been published by Halmos [6], Helson and Lowdenslager [7] and Rovnyak [10].

r.k. is suggested by features in Lax's proof, which operates however with Fourier analysis and does not use properties of the r.k.(3).

In §3 we show how, analogously to the discussion in Bergman [2, Chapter VII] and Garabedian [4], important extremal properties of bounded analytic functions, generalized Blaschke products, etc., can be based on the r.k. and its properties. We obtain these results by use of the theorem of §2. Thus the existence of a Hilbert space of analytic functions which satisfies certain axioms can be made to yield properties of bounded analytic functions which are usually deduced from the existence of Green's function, or the Riemann mapping theorem. We believe our approach to have some methodological and computational interest, insofar as in certain domains (e.g., circle, half-plane, strip) Hilbert spaces satisfying our axioms and the corresponding r.k. are easily constructed directly.

In §1 we give definitions and preliminary remarks. In §2 we state and prove our main theorem. §3 contains applications to bounded functions. §4 contains some concluding remarks concerning a possible extension.

1. Preliminaries.

1.1. Throughout this paper Ω denotes an open set in the complex z-plane, with(4) boundary Γ. We consistently use the letters z, ζ, λ to denote points of Ω, and t to denote points of Γ. We denote by H a Hilbert space whose elements are functions $f(z)$ analytic and single-valued in Ω. The inner product of f and g is written $\langle f, g\rangle$. We assume moreover that H satisfies the following axioms:

A1. For every $\lambda \in \Omega$ the linear functional $L_\lambda f = f(\lambda)$ is bounded.

A2. (Boundary behavior). We assume that there is given a family of subsets of Γ, called *negligible sets*, with the properties

(i) An enumerable union of negligible sets is negligible.

(ii) Γ is not negligible.

A property enjoyed by all points of Γ with the exception of a negligible set will be said to hold *nearly everywhere* (n.e.). We now assume further,

(iii) To every $f \in H$ there is associated a *boundary function* (which we denote by $f(t)$) defined and nonvanishing nearly everywhere.

(iv) To nearly every point $t \in \Gamma$ there is associated a nonempty class of sequences $\{z_n\}$, $z_n \in \Omega$ (called: *admissible sequences belonging to the point* t) such that for every $f \in H$, the relation $\lim_{n\to\infty} f(z_n) = f(t)$ holds nearly everywhere.

A3. The norm of f is uniquely determined by the values of $|f(t)|$ nearly everywhere.

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(3) On p. 168 of [8] P. Lax remarks "observe the curious skew-symmetry in the dependence of B on μ and z displayed by formula (2.7)." We wish to point out that this is a consequence of the identity $B_\mu(z) = K_{-\bar{\mu}}(z)$ where K is the reproducing kernel of the Paley-Wiener Hilbert space; Lax's $B_\mu(z)$ is not defined in terms of a reproducing property but in terms of orthogonal projection of the exponential function, which is readily seen to imply this relationship with the r.k.

(4) *Added in proof.* More generally, Γ can be the set of "boundary elements" of Ω, or "prime ends" in the sense of Carathéodory; the theory we shall develop applies also in this generality.

A4. (Multipliers). There is given a set M of functions $\{\phi(z)\}$ (called *multipliers*) analytic and single-valued in Ω with the following properties:

(i) If $f \in H$, then $\phi f \in H$.

(ii) For every four functions $f_1, f_2, g_1, g_2 \in H$ which satisfy for all $\phi \in M$ the two relations

(a) $\langle f_1, \phi g_1 \rangle = \langle f_2, \phi g_2 \rangle$,

(b) $\langle \phi f_1, g_1 \rangle = \langle \phi f_2, g_2 \rangle$

we have

(c) $f_1(t)\overline{g_1(t)} - f_2(t)\overline{g_2(t)} = 0$ nearly everywhere on Γ.

1.2. Remarks on the axioms. From A1 we infer that H possesses a uniquely determined *reproducing kernel* (r.k.) which we write $K_\zeta(z)$ or occasionally $K(z,\zeta)$. We assume known to the reader the following properties of reproducing kernels (whose proofs are immediate; the properties in question have nothing to do with analytic functions and are valid in any Hilbert space H of functions on a space Ω for which A1 holds; see [1]):

(1) $K_\zeta(z) \in H$, for every $\zeta \in \Omega$,

(2) $\langle f, K_\zeta \rangle = f(\zeta)$, for every $f \in H$, $\zeta \in \Omega$,

(3) $K_\zeta(z) = \overline{K_z(\zeta)}$,

(4) for every $f \in H$ we have the inequality

$$|f(\zeta)|^2 \leqq K_\zeta(\zeta) \|f\|^2.$$

From A3 it follows that multiplication by a $\phi \in M$ for which $|\phi(t)| = 1$ nearly everywhere(5) preserves norms, hence also inner products. It also follows that the inner product of two functions is uniquely determined by the boundary functions and that $f(z)$ is uniquely determined by its boundary function.

The axiom A4(i) is necessary for the very formulation of the problem of "invariant subspaces" to be meaningful. A4(i) implies, since the transformation $f \to \phi f$ has a closed graph, that it is bounded, and this is readily seen to imply that the functions in M are necessarily bounded. A4(ii) is perhaps the only axiom which does not seem natural; it ensures a sufficient richness of the class of multipliers. This purpose could perhaps be achieved with a simpler axiom, but some axiom of this type is indispensable for Beurling's theorem.

The simplest example of a space H satisfying our axioms is H_2, or more generally the space $E_2(\Omega)$ (see Privalov [9]) where Ω is a Jordan domain with rectifiable boundary. Sets of measure zero on Γ are to be understood as negligible sets. Boundary functions are the usual nontangential limits. To each boundary point t where Γ admits a tangent we may associate any sequence $\{z_n\}$ which converges nontangentially to t as an admissible sequence belonging to t, and then A2(iv) holds. The multiplier set M may be taken as the bounded analytic functions in Ω,

(5) *Added in proof.* For the interpretation of this property when ϕ is not in H, see the remark following the statement of Theorem 1.

or equivalently, polynomials. Then the key axiom A4(ii) holds since the real parts of bounded analytic functions are dense in the bounded measurable functions on Γ, in the topology of bounded pointwise convergence almost everywhere. Similar remarks apply to the Paley-Wiener space of the half-plane. If on the contrary Ω is not simply connected, the axiom A2(iv) (and only this one) fails, and indeed, Theorem 1 below is false in this case (for further remarks see the concluding section). However, in the multiply connected case it is possible to construct Hilbert spaces satisfying all the axioms by restricting consideration to only those functions of $E_2(\Omega)$, and those multipliers, which are invariant under a suitably chosen group of mappings of Ω on itself.

We remark finally that analyticity is only used rather weakly in the proof of Theorem 1, and could be dispensed with at the expense of a further complication in formulation. The ultimate Hilbert spaces to which our method applies are vaguely suggestive of Dirichlet algebras.

2. Generalized Beurling theorem.

THEOREM 1. *Let H be a Hilbert space of analytic functions as described in* §1, *satisfying axioms* A1-A4. *Let S denote a closed subspace of H with the property that $\phi f \in S$ whenever $\phi \in M$, $f \in S$. Then $S = H\omega$, where $\omega(z)$ is analytic and bounded in Ω and $|\omega(t)| = 1$ nearly everywhere on Γ. $\omega(z)$ is uniquely determined apart from a constant factor of modulus* 1.

Since multiplication by such an ω is an isometry of H, it is clear that, conversely, $H\omega$ is always a closed subspace of H which is mapped into itself upon multiplication by every $\phi \in M$. Note that the statement $|\omega(t)| = 1$ nearly everywhere on Γ is here understood to mean that, for nearly every $t \in \Gamma$, $\lim \omega(z_n)$ exists and has modulus one for every admissible sequence $\{z_n\}$ belonging to t.

Proof of theorem.

1. Let $K_\zeta(z)$ denote the reproducing kernel of S. We prove first that for every $\lambda, \zeta \in \Omega$

$$k_\lambda(\zeta)K_\lambda(t)\overline{K_\zeta(t)} = K_\lambda(\zeta)k_\lambda(t)\overline{k_\zeta(t)} \tag{1}$$

for nearly all $t \in \Gamma$.

Indeed, applying A4(ii) with

$$f_1(z) = k_\lambda(\zeta)K_\lambda(z), \quad g_1(z) = K_\zeta(z),$$

$$f_2(z) = K_\lambda(\zeta)k_\lambda(z), \quad g_2(z) = k_\zeta(z),$$

we will have proved (1) if we verify the two relations

$$k_\lambda(\zeta)\langle K_\lambda, \phi K_\zeta\rangle = K_\lambda(\zeta)\langle k_\lambda, \phi k_\zeta\rangle, \tag{2}$$

$$k_\lambda(\zeta)\langle \phi K_\lambda, K_\zeta\rangle = K_\lambda(\zeta)\langle \phi k_\lambda, k_\zeta\rangle, \tag{3}$$

for every $\phi \in M$. Using the facts that $\phi k_\zeta \in S, \phi k_\lambda \in S$, and the reproducing properties of the kernels k and K, these relations are equivalent to

$$(2') \qquad k_\lambda(\zeta)\overline{\phi(\lambda)K_\zeta(\lambda)} = K_\lambda(\zeta)\overline{\phi(\lambda)k_\zeta(\lambda)},$$

$$(3') \qquad k_\lambda(\zeta)\phi(\zeta)K_\lambda(\zeta) = K_\lambda(\zeta)\phi(\zeta)k_\lambda(\zeta)$$

and these are identities, in the case of (2′) because of the skew-symmetry of reproducing kernels.

Let us now set(6)

$$(4) \qquad \psi_\lambda(z) = \frac{k_\lambda(z)}{K_\lambda(z)}$$

so that (1) may be written

$$(5) \qquad \psi_\lambda(\zeta) = \psi_\lambda(t)\overline{\psi_\zeta(t)}, \text{ nearly all } t \in \Gamma.$$

In particular, setting $\lambda = \zeta$ in (5) gives

$$(6) \qquad |\psi_\lambda(t)|^2 = \psi_\lambda(\lambda) = \frac{k_\lambda(\lambda)}{K_\lambda(\lambda)}, \text{ nearly all } t \in \Gamma.$$

We deduce readily from (1.24) with $f(z) = k_\lambda(z)$ that

$$(7) \qquad k_\lambda(\lambda) \leqq K_\lambda(\lambda)$$

and so in particular

$$(8) \qquad |\psi_\lambda(t)| \leqq 1, \text{ nearly all } t \in \Gamma.$$

Let E denote an enumerable dense subset of Ω. Then (8) remains true except for t in some fixed negligible set, for all $\lambda \in E$. Thus, given any fixed $\lambda \in \Omega$, for all t in a set whose complement is negligible the relations $|\psi_\lambda(t)| \leqq 1$ as well as

$$(9) \qquad |\psi_\zeta(t)| \leqq 1, \quad \zeta \in E,$$

$$(10) \qquad \psi_\lambda(\zeta) = \psi_\lambda(t)\overline{\psi_\zeta(t)}, \quad \zeta \in E,$$

are valid. Choosing such a t-value we then see from (10),

$$(11) \qquad |\psi_\lambda(\zeta)| \leqq 1, \quad \zeta \in E.$$

Since $\psi_\lambda(z)$ is meromorphic in Ω and bounded on a dense set, if follows that it is analytic and bounded by one in Ω.

2. Next, let us restrict both λ and ζ to the set E. Then, outside a fixed negligible t-set all functions $\psi_\lambda(z)$ satisfy

$$(12) \qquad \psi_\lambda(\zeta) = \psi_\lambda(t)\overline{\psi_\zeta(t)}, \quad \lambda \in E, \quad \zeta \in E,$$

and by A2(iv) we can choose a point $t_0 \in \Gamma$, and a sequence $\{z_n\}$, such that

(6) *Added in proof.* Since the set of λ for which $K_\lambda(z) \equiv 0$ is at most countable, $\psi_\lambda(z)$ is meromorphic in Ω except for a countable set of λ.

(13) $$\lim_{n\to\infty} \psi_\lambda(z_n) = \psi_\lambda(t_0) \text{ exists and is finite, } \lambda \in E,$$

(14) $$\psi_\lambda(\zeta) = \lim_{n\to\infty} \psi_\lambda(z_n)\,\overline{\psi_\zeta(z_n)}, \quad \lambda \in E, \quad \zeta \in E.$$

The point t_0 and the sequence $\{z_n\}$ will remain fixed in the remainder of the discussion. Let now (note that $K_{z_n}(z) \not\equiv 0$ for n large enough):

(15) $$\omega_n(z) = \psi_{z_n}(z) = \frac{k_{z_n}(z)}{K_{z_n}(z)}, \text{ for } n > n_0.$$

Then we have,

(16) $$|\,\omega_n(z)\,| \leqq 1, \text{ for } n > n_0$$

and, from (13) and (14), by the skew-symmetry of $\psi_\lambda(z)$ we have

(17) $$\lim_{n\to\infty} \omega_n(\lambda) = \overline{\psi_\lambda(t_0)}, \quad \lambda \in E,$$

(18) $$\lim_{n\to\infty} \overline{\omega_n(\lambda)}\,\omega_n(\zeta) = \psi_\lambda(\zeta), \quad \lambda \in E, \quad \zeta \in E.$$

Thus the sequence of functions $\omega_n(z)$ is uniformly bounded by one and converges pointwise on a dense set. Hence it converges uniformly on compact subsets of Ω to a function $\omega(z)$ satisfying

(19) $$\psi_\lambda(\zeta) = \overline{\omega(\lambda)}\ \omega(\zeta)$$

initially only for λ and ζ in E, and hence for all λ and ζ in Ω, by continuity. From (19) we get, setting $\lambda = \zeta$:

(20) $$\psi_\lambda(\lambda) = |\,\omega(\lambda)\,|^2$$

and also

(21) $$|\,\psi_\lambda(\zeta)\,|^2 = |\,\omega(\lambda)\,|^2 |\,\omega(\zeta)\,|^2;$$

holding λ fixed and letting ζ run through an admissible sequence belonging to the boundary point t, we see from (6), (20), (21) that $|\,\omega(\zeta)\,| \to 1$ for nearly all t. We may express this relationship as follows:

(22) $$|\,\omega(t)\,| = 1 \text{ nearly everywhere.}$$

3. We can now easily complete the proof of Theorem 1. Let S_1 denote the closed subspace(7) $H\omega$ of H, and note from (19) that

(23) $$k_\zeta(z) = K_\zeta(z)\,\overline{\omega(\zeta)}\,\omega(z)$$

as a function of z, is in S_1. Moreover, for every $g = \omega f$ in S_1 we have

(7) *Added in proof.* It follows from (19) that $\omega(z)$ multiplies $K_\zeta(z)$ into H. Moreover, it follows easily from (23) that $\omega(z)$ multiplies all of H into H. Since this map is an isometry, $H\omega$ is closed.

$$\begin{aligned}\langle g, k_\zeta\rangle &= \langle \omega(z)f(z), K_\zeta(z)\,\overline{\omega(\zeta)}\omega(z)\rangle\\ &= \omega(\zeta)\langle \omega f, \omega K_\zeta\rangle = \omega(\zeta)\langle f, K_\zeta\rangle\\ &= g(\zeta).\end{aligned}$$

Hence $k_\zeta(z)$ is a reproducing kernel for S_1. Since a subspace is determined by its r.k., $S_1 = S$. Finally, the uniqueness of $\omega(z)$ is immediate from (20), which shows that $|\omega(z)|$ is uniquely determined for all $z \in \Omega$. Theorem 1 is completely proved.

COROLLARY. *Under the conditions of Theorem* 1, $S^\perp$ *is precisely the closure of the linear manifold spanned by the functions*

$$g_\lambda(z) = \left[\omega(z) - \frac{1}{\overline{\omega(\lambda)}}\right]K_\lambda(z), \text{ for } \lambda \in \Omega.$$

This follows from (23) and the fact that the r.k. of $S^\perp$ is $K_\lambda(z) - k_\lambda(z)$.

3. **Applications to bounded functions.**

3.1. In what follows we assume that M is the class $B(\Omega)$ of all bounded analytic functions on Ω, and also impose two restrictions on the Hilbert space H:

A5. For every $\phi \in B$, the map $f \to \phi f$ is a bounded linear transformation with norm $\leqq \sup |\phi(z)|$.

A6. No $z \in \Omega$ is a common zero of all $f \in H$.

REMARK. It is easily deduced that the norm of the transformation $f \to \phi f$ is $\geqq \sup |\phi(z)|$; hence it is precisely $\sup |\phi(z)|$. Indeed, denoting this norm by M, we have, since $\phi(z)K_\zeta(z)$ has norm $\leqq MK_\zeta(\zeta)^{1/2}$, by (142.)

$$|\phi(\zeta)|^2[K_\zeta(\zeta)]^2 \leqq M^2[K_\zeta(\zeta)]^2.$$

3.2 Let us consider the following two extremal problems in $B(\Omega)$, whereby we use $\|\phi\|_\infty$ to denote $\sup|\phi(z)|$:

PROBLEM A. $\phi \in B(\Omega)$, $\phi(a) = 0$, $\|\phi\|_\infty \leqq 1$. Maximize $|\phi(\zeta)|$. (Here a, ζ, are fixed points of Ω.)

PROBLEM B. $\phi \in B(\Omega)$, $\phi(a) = 0$, $\|\phi\|_\infty \leqq 1$. Maximize $|\phi'(a)|$.

The solutions of these problems are of course well known(8) (see for example Bergman [2, Chapter VII], Garabedian [4]). The extremal is in each case (recall that we are still assuming Ω simply connected) the Riemann mapping function of Ω onto $|w| < 1$ taking a into $w = 0$. Moreover the solution to problem B is the same without the assumption $\phi(a) = 0$ since the extremal function in the wider class must in fact vanish at a. Here however we do not presuppose this information, as we wish to discuss problems A and B from the standpoint of Theorem 1.

(8) The reader may also consult the work *Hilbertsche Räume mit Kernfunktion* of Herbert Meschkowski, Springer, Berlin, 1962.

Denote by $S = S(a)$ the closed subspace of H consisting of all $f \in H$ which vanish at a. This is an invariant subspace and so, by Theorem 1, and (2.23)

$$k_\zeta(z) = K_\zeta(z)\,\overline{\omega(\zeta)}\,\omega(z) \tag{1}$$

where k_ζ is the r.k. of S, and K_ζ is the r.k. of H. Here $\omega(z) \in B(\Omega)$ and has modulus one n.e. on Γ. Moreover $\omega(a) = 0$ because of A6. *We show that this function* $\omega(z) = \omega_a(z)$ *is the unique extremal function for problems* A *and* B (we use the word "unique" to mean: unique apart from a constant factor of modulus one). Consider first problem A, and let ϕ be a "competing function" of norm one. Then ϕK_ζ belongs to S and so

$$\begin{aligned} |\phi(\zeta)|^2 K_\zeta(\zeta)^2 &= |\langle \phi(z)K_\zeta(z),\ k_\zeta(z)\rangle|^2 \\ &= |\omega(\zeta)|^2\,|\langle \phi(z)K_\zeta(z), \omega(z)K_\zeta(z)\rangle|^2 \\ &\leqq |\omega(\zeta)|^2 K_\zeta(\zeta)^2 \end{aligned}$$

by A5 and Schwartz' inequality. This proves the assertion regarding problem A, the uniqueness following by the condition for equality in Schwartz' inequality. As for problem B, we have, for $f \in S$:

$$f(\zeta) = \langle f(z), k_\zeta(z)\rangle;$$

hence

$$f'(\zeta) = \left\langle f(z), \frac{\partial}{\partial \bar{\zeta}}\, k_\zeta(z)\right\rangle \tag{2}$$

and from (1) we have

$$\frac{\partial}{\partial \bar{\zeta}}\, k_\zeta(z) = K_\zeta(z)\,\overline{\omega'(\zeta)}\,\omega(z) + \left[\frac{\partial}{\partial \bar{\zeta}} K_\zeta(z)\right]\overline{\omega(\zeta)}\omega(z). \tag{3}$$

Let us now substitute in (2) $\phi(z)K_a(z)$ in place of $f(z)$, and a for ζ, and apply the Schwartz inequality:

$$\begin{aligned} |\phi'(a)|^2 K_a(a)^2 &= |\omega'(a)|^2\,|\langle \phi(z)K_a(z), \omega(z)K_a(z)\rangle|^2 \\ &\leqq |\omega'(a)|^2 K_a(a)^2, \end{aligned}$$

and the assertion follows as before.

Now, $k_\zeta(z)$ can be expressed in terms of $K_\zeta(z)$ by the formula (here it is convenient to write $k(z,\zeta)$, $K(z,\zeta)$ for $k_\zeta(z)$, $K_\zeta(z)$ respectively):

$$k(z,\zeta) = \frac{\begin{vmatrix} K(z,\zeta) & K(z,a) \\ K(a,\zeta) & K(a,a) \end{vmatrix}}{K(a,a)}. \tag{4}$$

(4) is evident by the uniqueness of the r.k. since the right-hand side belongs to S for each ζ, and is *a* r.k. for S.

Comparing this with (1) gives

(5)
$$\overline{\omega(\zeta)}\,\omega(z) = \frac{\begin{vmatrix} K(z,\zeta) & K(z,a) \\ K(a,\zeta) & K(a,a) \end{vmatrix}}{K(z,\zeta)K(a,a)}.$$

Setting $z = \zeta$ gives

(6)
$$|\,\omega(\zeta)\,|^2 = \frac{\begin{vmatrix} K(\zeta,\zeta) & K(\zeta,a) \\ K(a,\zeta) & K(a,a) \end{vmatrix}}{K(\zeta,\zeta)K(a,a)} = 1 - \frac{|\,K(a,\zeta)\,|^2}{K(a,a)K(\zeta,\zeta)}.$$

Another formula gotten from (5) is

(7)
$$\frac{\omega(z_1)}{\omega(z_2)} = \frac{K(z_2,\zeta)}{K(z_1,\zeta)}\,\frac{\begin{vmatrix} K(z_1,\zeta) & K(z_1,a) \\ K(a,\zeta) & K(a,a) \end{vmatrix}}{\begin{vmatrix} K(z_2,\zeta) & K(z_2,a) \\ K(a,\zeta) & K(a,a) \end{vmatrix}}$$

in which the right-hand side is actually independent of ζ. We also obtain, clearing off fractions in (5) and differentiating:

(8)
$$|\,\omega'(a)\,|^2 = \frac{\begin{vmatrix} \dfrac{\partial^2 K(z,\zeta)}{\partial z \partial \bar{\zeta}} & \dfrac{\partial K(z,a)}{\partial z} \\ \dfrac{\partial K(a,\zeta)}{\partial \bar{\zeta}} & K(a,a) \end{vmatrix}_{z=\zeta=a}}{K(a,a)^2}.$$

These results may be summarized as

THEOREM 2. *The common solution* (*unique apart from a constant factor*) *of the extremal problems* A *and* B *is* $\omega(z) = \omega_a(z)$, *which is uniquely determined from the formulas* (5), (6) *once* arg $\omega(z)$ *is given at some point. The maximum in problem* A *is the right-hand side of* (6); *the maximum in problem* B *is the right-hand side*(9) *of* (8).

3.3. Let us now consider a generalization of problem A.

PROBLEM A′. $\phi \in B$, $\|\,\phi\,\|_\infty \leqq 1$, $\phi(a_1) = \cdots = \phi(a_n) = 0$. Maximize $|\,\phi(\zeta)\,|$. Here $a_1, \cdots, a_n$ are distinct points of Ω.

Denote by S_n the closed (invariant) subspace of H consisting of all functions vanishing at $a_1, \cdots, a_n$. Let $k_n(z,\zeta)$ denote the r.k. of S_n. Then as before,

(1)
$$k_n(z,\zeta) = K(z,\zeta)\,\overline{\omega_n(\zeta)}\,\omega_n(z).$$

Here $\omega_n(z) \in B$, vanishes at $z = a_1, \cdots, a_n$ and has modulus one n.e. on Γ. Just as in §3.2 we see that ω_n is the unique extremal function for problem A'. Since, on

(9) When K is the Szegö kernel, the right-hand side of (8) reduces to $K(a,a)^2$. *Added in proof*: We remark also that it is easily proved directly from (5) that ω is univalent in Ω. Hence it can be proved without difficulty that ω is the Riemann mapping function onto the unit disk.

the other hand, it is readily deduced that $\phi(z) = \omega_{a_1}(z) \cdots \omega_{a_n}(z)$ is an extremal (where $\omega_a(z)$ is an extremal for problem A) these functions are identical apart from a constant factor of modulus one. Hence we deduce, from (1) and 3.2(6):

$$(2) \qquad k_n(\zeta,\zeta) = K(\zeta,\zeta) \prod_{m=1}^{n} \left\{ 1 - \frac{|K(a_m,\zeta)|^2}{K(a_m,a_m)K(\zeta,\zeta)} \right\}.$$

Moreover, one deduces by the same reasoning as 3.2(4):

$$(3) \qquad k_n(z,\zeta) = \frac{\begin{vmatrix} K(z,\zeta) & K(z,a_1) & \cdots & K(z,a_n) \\ K(a_1,\zeta) & K(a_1,a_1) & \cdots & K(a_1,a_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(a_n,\zeta) & K(a_n,a_1) & \cdots & K(a_n,a_n) \end{vmatrix}}{\begin{vmatrix} K(a_1,a_1) & \cdots & K(a_1,a_n) \\ K(a_2,a_1) & \cdots & K(a_2,a_n) \\ \cdots & & \cdots \\ K(a_n,a_1) & \cdots & K(a_n,a_n) \end{vmatrix}}.$$

Substituting $z = \zeta$ and comparing with (2) gives an interesting determinant identity (which could also be proved algebraically by simple transformations, using 3.2(4).) We see from (3), since $k_n(\zeta,\zeta)$ is given as the quotient of two Gram determinants, that $k_n(\zeta,\zeta)$ is the square of the distance d from $K_\zeta(z)$ to the linear manifold spanned by $K_{a_1}(z), \cdots, K_{a_n}(z)$. Hence

THEOREM 3. *d^2 is given by the right-hand side of (2). Consequently, the necessary and sufficient condition on an infinite sequence $\{a_m\}$ that there exist a function in H (equivalently: a function in B) vanishing on the sequence $\{a_m\}$ but not identically, is*

$$\sum \frac{|K(a_m,\zeta)|^2}{K(a_m,a_m)} < \infty.$$

This condition holds for some $\zeta \in \Omega$ if and only if it holds for every ζ.

4. **Concluding remarks.** It would be of interest to obtain an analogous theory for a multiply-connected region. If we consider a sequential Hilbert space $\{a_n\}$, $-\infty < n < \infty$, the norm of a sequence being defined by $[\sum_{n=-\infty}^{\infty} |a_n|^2 \cosh(2n+1)\delta]^{1/2}$, and study closed subspaces invariant with respect to both left and right shifts, the equivalent function-theoretic problem is that of closed invariant subspaces (in the sense of the present paper) of $E_2(\Omega)$ where Ω is the annulus $e^{-\delta} < |z| < e^{\delta}$, E_2 denoting the Hilbert space normed by means of square integral with respect to arc length around the boundary(10). In cases of higher connectivity, there seems to be no simple interpretation in terms of "shift operators."

(10) *Added in proof.* A detailed study of invariant subspaces for the annulus has recently been carried out by D. E. Sarason in a 1963 dissertation at the University of Michigan.

In the case of the annulus, the invariant subspace consisting of all functions vanishing at some point a is not of the type covered by Theorem 1, but consists of all multiples of a function $\omega(z)$ whose modulus is constant on each boundary component (the constants being different). This function $\omega(z)$ is uniquely determined, apart from a constant factor, as the Riemann mapping function of the annulus on a unit disk from which a suitable concentric circular arc has been removed, such that $\omega(a) = 0$ (cf. Golusin [5, p. 421]). This suggests a possible "Beurling factorization" in the multiply-connected case: the "inner functions" are now bounded functions with (in general different) constant modulus on each boundary component. We have not, however, been able to establish this. To attack it by the methods of the present paper requires that the conclusion in axiom A4(ii) be replaced by: $f_1(t)\overline{g_1(t)} - f_2(t)\overline{g_2(t)}$ is constant n.e. on each boundary component. One then obtains, analogous to (2.1), functional equations connecting the r.k. of H and of S, but we have been unable to extract the desired information from these equations.

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BIBLIOGRAPHY

1. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337-404.

2. S. Bergman, *The kernel function and conformal mapping*, Math. Surveys No. 5, Amer. Math. Soc., Providence, R. I., 1950, 161 pp.

3. A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1948), 239–255.

4. P. Garabedian, *Schwarz' lemma and the Szegö kernel function*, Trans. Amer. Math. Soc. **67** (1949), 1–35.

5. G. M. Golusin, *Geometrische Funktionentheorie*, Deutscher Verlag der Wiss., East Berlin, 1957.

6. P. Halmos, *Shifts on Hilbert spaces*, J. Reine Angew. Math. **208** (1961), 102–112.

7. H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math. **99** (1958), 165-202.

8. P. Lax, *Translation invariant spaces*, Acta Math. **101**, (1959), 163–178.

9. I. I. Privalov, *Randeigenschaften Analytischer Funktionen*, authorized German translation, Deutscher Verlag der Wiss., East Berlin, 1956, 247 pp.

10. J. Rovnyak, *Ideals of square summable power series*, Math. Mag. **33** (1960), 265-270; correction, ibid. **34** (1960), 41–42.

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