METRICAL THEOREMS ON FRACTIONAL PARTS OF SEQUENCES

BY
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1. Introduction. Let \( C \) be the additive group of real numbers modulo 1, and let \( x \rightarrow \{x\} \) be the natural mapping from the reals onto \( C \). It is clear what we shall mean by an interval \( I \) in \( C \) and by the length \( l(I) \) of \( I \). Denote the distance of the real number \( \alpha \) to the closest integer by \( \| \alpha \| \). The image in \( C \) of the set of reals \( \xi \) satisfying \( \| \xi - \theta \| \leq \varepsilon \) with given \( \theta \) and \( 0 < \varepsilon < 1/2 \) is an example of an interval of \( C \) of length \( 2\varepsilon \).

Theorem 1. Let \( n \geq 1 \) and let \( P_1(q), \ldots, P_n(q) \) be nonconstant polynomials with integral coefficients. For each of the integers \( j = 1, \ldots, n \) let \( I_{j1} \supseteq I_{j2} \supseteq \cdots \) be a sequence of nested intervals in \( C \). Put \( \psi(q) = l(I_{1q}) \cdots l(I_{nq}) \) and

\[
\Psi(h) = \sum_{q=1}^{h} \psi(q).
\]

Put \( N(h;\alpha_1,\ldots,\alpha_n) \) for the number of integers \( q, 1 \leq q \leq h \), with

\[
\{\alpha_jP_j(q)\} \in I_{jq} \quad (j = 1, \ldots, n).
\]

Let \( \varepsilon > 0 \). Then

\[
N(h;\alpha_1,\ldots,\alpha_n) = \Psi(h) + O(\Psi(h)^{1/2+\varepsilon})
\]

for almost every \( n \)-tuple of real numbers \( \alpha_1, \ldots, \alpha_n \).

The theorem implies, for example, that the number of solutions of

\[
|ax - p - \theta| \leq q^{-1}
\]

in integers \( p \) and \( q, 1 \leq q \leq h \), is asymptotically equal to \( 2 \log h \) for every \( \alpha \notin \sigma(\theta) \) where \( \sigma(\theta) \) is a set of measure zero. To see this we only have to put \( n = 1, P(q) = q \) and to define intervals \( I_q \) as the images of the sets \( \| \xi - \theta \| \leq q^{-1} \).

On the other hand, let \( P(q) = a_0q^d + \cdots + a_d \) be a polynomial of degree \( d > 0 \) with integral coefficients, let \( \mu \) be real, and let \( M(h;\alpha) \) be the number of solutions in integers \( p, q, 1 \leq q \leq h, \) of

\[
|\alpha - p/P(q)| \leq q^{-\mu}.
\]
Then $M(h; \alpha)$ is bounded for almost every $\alpha$ if $\mu > d + 1$; $M(h; \alpha) \sim 2 |a_0| \log h$ if $\mu = d + 1$; and $M(h; \alpha) \sim 2 |a_0| h^{d+1-\mu(d+1-\mu)^{-1}}$ for almost every $\alpha$ if $\mu < d + 1$.

To see this, we remark that for $\mu > d$ and large $q$, (1.4) is equivalent to $\|\alpha P(q)\| \leq |P(q)| q^{-\mu}$. Thus our interval $I_q$ has length $\psi(q) = 2 |P(q)| q^{-\mu} = |2a_0 q^{d-\mu} + 2a_1 q^{d-\mu-1} + \cdots|$, and the theorem gives the result. For $\mu = d$, (1.4) becomes $|\alpha P(q) - p| \leq |a_0 + a_1 q^{-1} + \cdots|$, and $M(h; \alpha)$ becomes $2 |a_0| h$ plus (or minus) the number of solutions of $\|\alpha P(q)\| \leq |a_1 q^{-1} + \cdots|$ for $1 \leq q \leq h$, whence $M(h; \alpha) \sim 2 |a_0| h$ almost everywhere. Finally for $\mu < d$ our formula for $M(h; \alpha)$ is in fact true for every $\alpha$. The reader should have no difficulty in proving this elementary result.

There can be at most countably many $\alpha$'s such that $\{\alpha P_j(q)\}$ is an endpoint of $I_{jq}$ for some $q$, and hence we may assume $I_{jq}$ to be closed ($j = 1, \ldots, n$; $q = 1, 2, \ldots$). The intersections $J_j = \bigcap_q I_{jq}$ ($j = 1, \ldots, n$) are then nonempty. The case where $0 \in J_j$ for each $j$ is usually called the homogeneous case, the general case the inhomogeneous case.

Our theorem implies in particular that $N(h; \alpha_1, \ldots, \alpha_n)$ remains bounded almost everywhere if $\Psi(h)$ is bounded, while it will tend to infinity almost everywhere if $\Psi(h)$ tends to infinity. This had been proved by Khintchine [9] in the homogeneous case under the assumption that $P_j(q) = q$ ($j = 1, \ldots, n$) and that $q \psi(q)$ is decreasing. Szüsz [13] generalized Khintchine's result to the inhomogeneous case. Szüsz' method involves continued fractions and therefore applies only to the case $n = 1$. Before Szüsz, Cassels [2] had shown that Khintchine's conclusion is true for "almost every inhomogeneous case," that is, if $(I_{1q}, \ldots, I_{nq})$ is replaced by its translation by a vector $(\theta_1, \ldots, \theta_n)$ of reals mod 1 ($q = 1, 2, \ldots$), then the conclusion is true for almost every $\theta_1, \ldots, \theta_n$. Thus Cassel's result was "doubly metrical."

Erdös [5] proved for the homogeneous case with $n = 1$, $P(q) = q$, that $N(h; \alpha) \sim \Psi(h)$ almost everywhere, and the author [12] proved (1.3) in this case. Our generalization from the homogeneous to the inhomogeneous case is not trivial. We shall choose $\theta_j \in J_j$ ($j = 1, \ldots, n$) and use rational approximations to $\theta_j$. The generalization from linear to general polynomials also causes some difficulty.

Le Veque [10] proved a general theorem where polynomials $P(q)$ are replaced by general sequences $a(q)$ which have to satisfy a certain condition. However, this condition is not satisfied for $a(q) = q$, and it is difficult to decide whether it is satisfied for nonlinear polynomials.

It would be possible to replace (1.2) by $\{\{\alpha_1 P_1(q)\}, \ldots, \{\alpha_n P_n(q)\}\} \in H_n$, thus replacing products of intervals $I_{1q} \times \cdots \times I_{nq}$ by somewhat more general sets $H_q$ of $C \times \cdots \times C$.

(2) We use this opportunity to mention two errors in [12]: In Theorem 1 of [12] one has to assume that the functions $\psi_j(q)$ are bounded. Everywhere in §6 except in $\beta(Q, \theta)$, $\theta$ should be replaced by $\Theta = (\theta_1, \ldots, \theta_n)$. 

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[March]
In §10 we shall point out how one could prove a more general theorem where the expressions $a;P_j(q)$ are replaced by linear forms $\alpha_1 P_1(q_1) + \cdots + \alpha_m P_m(q_m)$. A special case of such a result is contained in Theorem 2 of [12].

**Theorem 2.** Let a sequence of positive integers $a_1(1) < a_2(2) < \cdots$ be given for $i = 1, \ldots, n$. Let $\theta$ be arbitrary but fixed, and put

$$
\sum_{h=1}^{n} \cdots \sum_{q_n=1}^{n} \left( q_1 \cdots q_n \right)^{\sum_{i=1}^{n} \alpha_i a_i(q_i) + \theta}^{-1}.
$$

Then one has for $\varepsilon > 0$ and almost every $\alpha_1, \ldots, \alpha_n$

(1.5) 

$$
(\log h)^{n+1} \leq \sum_{h=1}^{n} \cdots \sum_{q_n=1}^{n} \left( q_1 \cdots q_n \right)^{\sum_{i=1}^{n} \alpha_i a_i(q_i) + \theta}^{-1}.
$$

Using Theorem 2, together with an $n$-dimensional generalization of a result of Erdős and Turan [7, Theorem 3], we shall easily deduce

**Theorem 3.** Assume the hypotheses of Theorem 1 to be satisfied, and assume we deal with the special case $P_j(q) = q$ ($j = 1, \ldots, n$) and $I_1 = I_{j_2} = \cdots$ ($j = 1, \ldots, n$). Write $\psi$ for $\psi(1) = \psi(2) = \cdots$, and let $\varepsilon > 0$. Then

$$
N(h; \alpha_1, \ldots, \alpha_n) = h \psi + O(\log h)^{n+1} + \varepsilon
$$

for almost every $\alpha_1, \ldots, \alpha_n$.

Khintchine [8, §3], proved the surprisingly small error-term $O(\log h)^{1+\varepsilon}$ for $n = 1$, and hence our result is not best possible. However, Khintchine's method involves continued fractions and cannot easily be generalized to $n > 1$. It seems that Theorem 1 cannot much be improved for nonlinear polynomials. Behnke [1, Theorem XXV] showed for $n = 1$, $P(q) = q^2$ and $I_1 = I_2 = \cdots = I$, say, that the relation $D_\alpha(h) = \text{def sup}_I |N_h(q; \alpha) - 2h(I)| \leq \sqrt{h}$ is wrong for every $\alpha$.

2. Notation and simplification. Throughout, $[\alpha]$ will the integral part of the real number $\alpha$. $U$ will denote the unit interval $0 \leq \xi < 1$.

We shall prove the case $n = 1$ of Theorem 1 in §§2–8. In §9 we shall point out the necessary changes for $n > 1$.

The set of $\alpha$'s in $U$ where $\{\alpha P(q)\} \in I_q$ has measure $\psi(q)$. Assume now that $\Psi(h)$ is bounded. Given $\varepsilon > 0$ there is a $q_0$ such that $\sum_{q>q_0} \psi(q) < \varepsilon$, and the set of $\alpha$'s in $U$ such that $\{\alpha P(q)\} \in I_q$ for some $q > q_0$ has measure $< \varepsilon$. Hence $N(h; \alpha)$ is bounded for almost every $\alpha$.

From now on, we shall assume that $\Psi(h)$ tends to infinity.

Let $\theta \in I = \bigcap I_q$. Then each $I_q$ is union of $\theta$ and of two intervals $I_q^t$ and $I_q^s$, where $I_q^t$ is of the type $0 < \{\theta - \xi\} \leq \psi^t(q)$, where $I_q^s$ is of the type $0 < \{\xi - \theta\} \leq \psi^s(q)$, and where $\psi^t(q) + \psi^s(q) = \psi(q)$. ($I_q^t$ or $I_q^s$ may be empty.) Now $\Psi^t(h)$, $\Psi^s(h)$, $N^t(h; \alpha)$, $N^s(h; \alpha)$ can be defined in the obvious way. One has $\Psi(h) = \Psi^t(h) + \Psi^s(h)$ and $N(h; \alpha) = N^t(h; \alpha) + N^s(h; \alpha)$ for almost every $\alpha$. Hence it will suffice to prove the theorem for the case of intervals of type $I^t$ and the case of intervals of type $I^s$. 


Since the mapping $\xi \to -\xi$, $\theta \to -\theta$ transforms intervals of type $I'$ into intervals of type $I'$, we may restrict ourselves to intervals of type $I'$.

From now on, $I_q$ will denote the interval $0 < \{\xi - \theta\} \leq \psi(q)$.

Replacing $P(q)$ by $-P(q)$ and $\alpha$ by $-\alpha$ if necessary, we may assume that $P(q) > 0$, $P'(q) > 0$ for $q > q_0$. Making a translation by $q_0$ we may even assume $P(q) > 0$, $P'(q) > 0$ for $q > 0$.

The introduction of a parameter $k$ is essential for our proof. Put $\phi(k, x)$ for the number of integers $y$ between 1 and $x$, $1 \leq y \leq x$, such that $\gcd(x, y) \leq k$. $\phi(1, x)$ is the well-known Euler $\phi$-function.

Given $q \geq 1$ there are pairs of relatively prime integers $a, b$ such that
\begin{equation}
1 \leq a \leq q^{1/2} \quad \text{and} \quad |b - \theta/a| < a^{-1} q^{-1/2}.
\end{equation}
This follows from Dirichlet's principle. For every integer $q \geq 1$ we pick integers $a = a(q)$, $b = b(q)$ with these properties. We define $S(k, q)$ as the set of integers $p$ where
\begin{equation}
\gcd(pa(q) + b(q), P(q)) \leq k.
\end{equation}

The sets $S(k, q)$ have two important properties:

1. If $p \in S(k, q)$ and $p = p' \mod P(q)$, then $p' \in S(k, q)$.

2. The number $\phi^*(k, q)$ of integers of $S(k, q)$ in $1 \leq x \leq P(q)$ satisfies $\phi^*(k, q) \geq \phi(k, P(q))$.

To prove (2), put $P(q) = q_1 q_2$ where every prime factor of $q_1$ divides $a$ and where $q_2$ and $a$ are relatively prime. Now $\gcd(a, b) = 1$ yields $\gcd(pa + b, P(q)) = \gcd(pa + b, q_2)$ and $\phi^*(k, q) = q_1 \phi(k, q_2) \geq \phi(k, P(q))$.

We now put
\[
\beta(q, \alpha) = \begin{cases} 1 & \text{if } \alpha \in U \text{ and } \{\alpha\} \in I_q, \\ 0 & \text{otherwise,} \end{cases}
\]
\[
\gamma(q, \alpha) = \sum_p \beta(q, \alpha P(q) - p),
\]
\[
\gamma(k, q, \alpha) = \sum_{p \in S(k, q)} \beta(q, \alpha P(q) - p),
\]
\[
\Gamma(q) = \int_0^1 \gamma(q, \alpha) d\alpha,
\]
\[
\Gamma(k, q) = \int_0^1 \gamma(k, q, \alpha) d\alpha,
\]
\[
\Gamma(k, q, r) = \int_0^1 \gamma(k, q, \alpha) \gamma(k, r, \alpha) d\alpha,
\]
\[
A(k, q, r) = \Gamma(k, q, r) - \psi(q)\psi(r),
\]
It is easy to see that \( N(h; \alpha) = \sum_{q=1}^{h} \gamma(q, \alpha) \), and we define

\[
N(k; u, v; \alpha) = \sum_{q=u+1}^{v} \gamma(k, q, \alpha).
\]

One has

\[
\Gamma(q) = \sum_{p} \int_{0}^{1} \beta(q, P(q)x - p) dx = P(q) \int_{-\infty}^{\infty} \beta(q, P(q)x) dx = \psi(q),
\]

and similarly

\[
\Gamma(k, q) = \psi(q) \phi^*(k, q) P(q)^{-1}.
\]

Summing over \( q \) we find

\[
\int_{0}^{1} N(h; \alpha) dx = \Psi(h)
\]

and

\[
\int_{0}^{1} N(k; u, v; \alpha) dx = \sum_{q=u+1}^{v} \psi(q) \phi^*(k, q) P(q)^{-1}.
\]

3. Deduction of Theorem 1 from two propositions.

**Proposition 1.** Let \( \delta > 0 \). Then

\[
\sum_{q=1}^{h} (P(q) - \phi(k, P(q))) P(q)^{-1} \ll hk^{\delta-1} + h^\delta k^\delta.
\]

**Proposition 2.** For every \( \delta > 0 \)

\[
\sum_{q=1}^{h} \sum_{r=1}^{h} A(k, q, r) \ll \Psi^{1+\delta}(h) + \Psi(h)k^\delta.
\]

**Remark.** Here and later, the estimate \( \ll \) holds simultaneously in \( h \) and \( k \). That is, the constant implied by \( \ll \) depends only on \( \delta \).

We are going to show that Theorem 1 is a consequence of these two propositions. The propositions will be proved later.

**Lemma 1.** Let \( \omega_1(q), \omega_2(q), \omega_3(q) \) be positive bounded functions of positive integers \( q \), and put

\[
\Omega_i(h) = \sum_{q=1}^{h} \omega_i(q)
\]

\((i = 1, 2, 3)\).
Assume that \( \omega_1 \) and \( \omega_3 \) are decreasing, and that \( \Omega_2(r) \leq \Omega_3(r) \) for every \( r \). Then

\[
(3.3) \sum_{q=1}^{h} \omega_1(q) \omega_2(q) \ll \Omega_3(\lceil \Omega_1(h) \rceil).
\]

**Proof.** If \( \Omega_1 \) is bounded, then so is the sum in (3.3). Hence we assume \( \Omega_1 \) to be unbounded. Since \( \omega_1 \) is decreasing, and since \( \Omega_2 \leq \Omega_3 \), one finds by partial summation that \( \sum_{q=1}^{h} \omega_1(q) \omega_2(q) \leq \sum_{q=1}^{h} \omega_1(q) \omega_3(q) \).

To estimate the latter sum we may assume \( \omega_1(q) \leq 1 \). Put \( m_0 = 0 \) and for integral \( a > 0 \) put \( m_a \) for the largest \( m \) with \( \Omega_1(m) \leq a \). Then \( m_a \geq a \) and \( \omega_1(m_a + 1) + \cdots + \omega_1(m_a + 1) \leq 2 \). Putting \( b = \lceil \Omega_1(h) \rceil \) we obtain

\[
\sum_{q=1}^{h} \omega_1(q) \omega_3(q) \leq \sum_{a=0}^{b} (\omega_1(m_a + 1) \omega_3(m_a + 1) + \cdots + \omega_1(m_a + 1) \omega_3(m_a + 1)) \leq 2 \sum_{a=0}^{b} \omega_3(m_a + 1) \leq 2 \sum_{a=1}^{b+1} \omega_3(a) = 2 \Omega_3(\lceil \Omega_1(h) \rceil + 1).
\]

Denote by \( J_r \) the set of intervals \( (u, t], 0 \leq u = t \cdot 2^s < v = (t + 1)2^s \leq 2^r \) where \( r, s, t \) are non-negative integers. Every interval \( (0, w] \) where \( w \) is integral and \( w \leq 2^r \) is union of not more than \( \max(1, r) \) intervals of \( J_r \). Given an integer \( u > 0 \) put \( n_u \) for some integer satisfying \( \lceil \Psi(n_u) \rceil = u \), and put \( n_0 = 0 \). Since \( \psi(q) \leq 1 \) and since \( \Psi(h) \) tends to infinity, such an \( n_u \) will always exist. Put \( h_r = n_{2r} \).

For the remainder of this section, \( k \) and \( r \) will be connected by

\[
(3.4) k = 2^r.
\]

**Lemma 2.** Let \( \delta > 0 \). Then

\[
(3.5) 0 \leq \int_{0}^{1} (N(h_r, x) - N(k; 0, h_r, x)) dx \ll 2^{r+6}
\]

and

\[
(3.6) \sum_{(u, v) \in J_r} \int_{0}^{1} (N(k; n_u, n_v, x) - \Psi(n_u, n_v))^2 dx \ll 2^{r+r+6}.
\]

**Proof.** Formulae (2.5) and (2.6) yield

\[
S_r = \int_{0}^{1} (N(h_r, x) - N(k; 0, h_r, x)) dx = \sum_{q=1}^{h_r} \psi(q)(P(q) - \phi^*(k, q)) P(q)^{-1}
\]

\[
\leq \sum_{q=1}^{h_r} \psi(q)(P(q) - \phi(k, P(q))) P(q)^{-1}.
\]

We now put

\[
\omega_1(q) = \psi(q), \quad \omega_2(q) = (P(q) - \phi(k, P(q))) P(q)^{-1}, \quad \omega_3(q) = c(k^{q-1} + q^{k-1}k^q).
\]

Proposition 1 shows that Lemma 1 is applicable if \( c > 0 \) is chosen large enough. Under our conditions we actually obtain the bound \( 2 \Omega_3(\lceil \Omega_1(h) \rceil) + 1 \). Hence
This is true for every $\delta > 0$, and hence (3.5) is proved.

$$N(k; u, v; \alpha) - \Psi(u, v) = \sum_{q = u + 1}^{v} (\gamma(k, q, \alpha) - \psi(q)).$$

Hence by (2.3), (2.4) and the estimate just derived,

$$\int_{0}^{1} (N(k; u, v; \alpha) - \Psi(u, v))^{2} \, d\alpha$$

$$= \sum_{q = u + 1}^{v} \sum_{q' = u + 1}^{v} (\Gamma(k, q, q') - \Gamma(k, q) \psi(q') - \Gamma(k, q') \psi(q) + \psi(q) \psi(q'))$$

$$= \sum_{q = u + 1}^{v} \sum_{q' = u + 1}^{v} A(k, q, q') + 2 \sum_{q = u + 1}^{v} \sum_{q' = u + 1}^{v} \psi(q) \psi(q') (P(q) - \phi^{*}(k, q)) P(q)^{-1}$$

$$\ll \sum_{q = u + 1}^{v} \sum_{q' = u + 1}^{v} A(k, q, q') + \sum_{q' = u + 1}^{v} \psi(q')^{2} r^{\delta}.$$
Lemma 2 implies \( \mu_r \ll r^{-2} \). Every interval \((0, w]\), \( w \leq 2^r \), is union of at most max \((1, r)\) intervals of \( J_r \), hence \((0, n_w]\) is union of at most max \((1, r)\) intervals \((n_u, n_v]\) where \((u, v]\) \(\in J_r \). Thus \( N(k; 0, n_w; x) - \Psi(n_w) = \sum (N(k; n_u, n_v; x) - \Psi(n_u, n_v)) \), where the sum is over at most \(r + 1\) pairs \((u, v]\) \(\in J_r \). This relation together with (3.8) and Cauchy’s inequality gives for \( \alpha \in U \), \( \alpha \notin \sigma_r \)

\[
(N(k; 0, n_w; x) - \Psi(n_w))^2 \leq r^2(r + 1)^2 \frac{2^{-2r + \delta}}{x^2}.
\]

(3.9)

Lemma 3 is a consequence of (3.7) and (3.9).

**Proof of Theorem 1.** Since \( \sum r^{-2} \) is convergent, there exists for almost every \( \alpha \in U \) an \( r_0 = r_0(\alpha) \) such that \( \alpha \notin \sigma_r \) for \( r \geq r_0 \). Assume \( \alpha \) has such an \( r_0 \), and assume \( w > 2^{r_0} \). Choose \( r \) such that \( 2^{r - 1} \leq w < 2^r \). Then \( r > r_0 \), \( \alpha \notin \sigma_r \), and Lemma 3 implies

\[
N(n_w; \alpha) = \Psi(n_w) + O(r^2 2^{r/2 + \delta})
\]

(3.10)

\[
= \Psi(n_w) + O\left(\frac{w^{1/2 + \delta} \log^2 w}{r^2}\right)
\]

\[
= \Psi(n_w) + O\left(\frac{w^{1/2 + \delta} \log^2 \Psi(n_w)}{r^2}\right).
\]

Since \( \Psi(n_{w+1}) = \Psi(n_w) + O(1) \), (3.10) is true for arbitrary integers \( h \) and not only the \( n_w \)'s. And since \( \delta > 0 \) was arbitrary, we find

\[
N(h; \alpha) = \Psi(h) + O(\Psi(h)^{1/2 + \delta})
\]

for almost every \( \alpha \in U \). Hence (1.3) is true for almost every \( \alpha \).

**4. The number of solutions of** \( P(x) = 0 \) \text{ (mod} \ d) \. Put \( D(q) \) for the number of positive divisors of \( q \). As is well known,

\[
D(q) \ll q^\delta
\]

(4.1)

for every \( \delta > 0 \). Put \( z(d) = z_P(d) \) for the number of solutions of \( P(x) = 0 \) \text{ (mod} \ d) \. Here, as always, \( P(x) \) is a nonconstant polynomial with integral coefficients. Define the discriminant \( \Delta \) of \( P(x) \) in the usual way if \( P(x) \) is nonlinear, and put \( \Delta = \alpha_0 \) if \( P(x) = a_0x + a_1 \).

**Lemmas 4.** Let \( P(x) \) be a polynomial of degree \( f \) and with discriminant \( \Delta \neq 0 \). Then \( z_P(p^k) \leq f \Delta^2 \) for every prime-power \( p^k \).

**Proof.** For linear \( P(x) \) it is well known that \( z(m) \leq \text{g.c.d.}(m, \Delta) \leq \Delta \leq f \Delta^2 \). The case where \( P(x) \) is nonlinear and primitive, that is, where the coefficients of \( P(x) \) are relatively prime, is Theorem 54 of [11]. A proof can be found there. In the general nonlinear case one has \( P(x) = cQ(x) \) with primitive \( Q(x) \), whence \( z_P(p^k) \leq cz_Q(p^k) \leq cf \Delta^2 Q \leq f \Delta^2 p \).

**Corollary.** Let \( P(x) \) be a polynomial with no multiple factors. Let \( \delta > 0 \). Then
Proof. The set \( \tau \) of prime-powers \( p^k \) such that \( p^{k\alpha} \equiv \alpha^2 \mod \beta \) is finite. For every \( d \),
\[
z(d)d^{-\delta} \leq \prod_{p \in \tau} z(p^\alpha)p^{-k\beta} \ll 1.
\]

Given an integer \( g > 0 \) we define a function \( \xi(d) \) of positive integers \( d \) as follows:
\( \xi(d) \) is multiplicative, and \( \xi(p^{x+y}) = p^{x+1} \) if \( p \) is a prime and \( 1 \leq y \leq g \). Our function has the property that \( d \mid m^\xi \) implies \( \xi(d) \mid m \).

Lemma 5. Let \( P(x) \) be a nonconstant polynomial, \( g \) a positive integer and \( s > 1 \). Then the two sums
\[
(4.3) \sum_{d=1}^{\infty} z_p(d)d^{-s}
\]
and
\[
(4.4) \sum_{d=1}^{\infty} (\xi(d))^{-s}
\]
are convergent.

Proof. There is an integer \( m \) and a polynomial \( Q(x) \) without multiple factors such that \( P(x) \mid Q(x)^m \). Now \( P(x) \equiv 0 \mod d \) implies \( Q(x) \equiv 0 \mod (d) \), and hence
\[
z_p(d)d^{-s} \leq z_Q(m(d))(m(d))^{-1}d^{-s} = z_Q(p^s(x+y))p^{(1-s)(mx+y)-x-1}
\]
\[
\leq f\Delta^2 p^{-x(m(s-1)+1)-y(s-1)-1} \leq f\Delta^2 p^{-s-x-s}.
\]
This implies
\[
\sum_{e=1}^{\infty} z_p(p^e)p^{-es} \leq mf\Delta^2 p^{-s} \sum_{x=0}^{\infty} p^{-xs} \leq c_xp^{-s}.
\]

Since the product \( \prod_{p}(1 + cp^{-s}) \) over all primes \( p \) is convergent, the convergence of (4.3) follows.

The convergence of (4.4) is proved similarly.

5. Proof of Proposition 1. The Euler \( \phi \)-function \( \phi(x) = \phi(1, x) \) can be expressed \( \phi(x) = x \sum_{y \mid x} \mu(y)y^{-1} \), where \( \mu(y) \) is the Moebius function. Now
\[
\phi(k, P(q)) = \sum_{x \leq k; x \mid P(q)} \phi(P(q)x^{-1}) = \sum_{x \leq k; x \mid P(q)} P(q)x^{-1} \sum_{y \mid P(q)x^{-1}} \mu(y)y^{-1},
\]
hence
\[
T_{k,h} = \sum_{q=1}^{h} \phi(k, P(q))P(q)^{-1} = \sum_{q=1}^{h} \sum_{x \leq k; x \mid P(q)} x^{-1} \sum_{y \mid P(q)x^{-1}} \mu(y)y^{-1},
\]
\[
= \sum_{x \leq k; x \mid P(h)} x^{-1} \sum_{y \mid P(h)x^{-1}} \mu(y)y^{-1} \sum_{q \mid h \cdot xy \mid P(q)} 1.
\]
The number of \( q \leq h \) such that \( xy \mid P(q) \) equals \( hz(xy)(xy)^{-1} + O(z(xy)) \). Therefore

\[
T_{k,h} = h \sum_{x \leq k; x \leq P(h)} \sum_{y \leq P(h)x^{-1}} z(xy)(xy)^{-2} \mu(y) + O\left( \sum_{x \leq k} \sum_{y \leq P(h)} z(xy)(xy)^{-1} \right)
= hU_{k,h} + O(V_{k,h}),
\]
say. Putting \( xy = w \) and using (4.1) with \( \delta = \varepsilon/2 \) and Lemma 5 with \( s = 1 + \varepsilon/2 \), \( \varepsilon > 0 \), we find

\[
U_{k,h} = \sum_{w \leq k; w \leq P(h)} z(w)w^{-2} \sum_{y \mid w} \mu(y) + O\left( \sum_{w > k} z(w)w^{-2}D(w) \right) = 1 + O(k^{\varepsilon-1}).
\]

Similarly,

\[
V_{k,h} \leq \sum_{w \leq P(h)k} z(w)w^{-1}D(w) \leq P(h)^{s}k^{\varepsilon} \sum_{w = 1}^{\infty} (z(w)w^{-1-\varepsilon/2}D(w)w^{-\varepsilon/2}) \leq P(h)^{s}k^{\varepsilon}.
\]

Combining our formulae and observing that \( \varepsilon > 0 \) was arbitrary we obtain

\[
T_{k,h} = h + O(hk^{\varepsilon-1} + h^{s}k^{\varepsilon}),
\]
thereby proving the proposition.

We use the remainder of this section to prove four related lemmas.

**Lemma 6.** Let \( P(x) \) be a polynomial of degree \( f > 1 \), and let \( \varepsilon > 0 \). Then

\[
W_{h} = \sum_{q = 1}^{h} \frac{q - 1}{P(q)} \sum_{d \mid P(q); d < q^{1-\varepsilon}} d \sum_{r \leq q; d \mid P(r)} 1 \ll h.
\]

**Proof.** Choose \( \delta > 0 \) so small that \( 2\delta f \leq \varepsilon(1 - f^{-1}) \).

There is an integer \( g \geq 1 \) and a polynomial \( Q(x) \) with no multiple factors such that \( P(x) \mid Q(x)^{f} \). We may choose \( g \leq f \). Now \( d \mid P(r) \) implies \( \delta(d) \mid Q(r) \), hence the number of \( r \leq q \) with \( d \mid P(r) \) is not larger than \( (q(\delta(d))^{-1} + 1)z(\delta(d)) \) and therefore by the corollary to Lemma 4 not larger than

\[
\ll (q(\delta(d))^{-1} + 1) \leq (qd^{-1/\delta} + 1) \delta^{\varepsilon} \leq (qd^{-1/\delta} + 1) \delta^{\varepsilon}.
\]

Using \( D(P(q)) \ll q^{\delta^{\varepsilon}} \) we obtain

\[
W_{h} \ll \sum_{q = 1}^{h} q^{-f} \sum_{d \mid P(q); d < q^{f-\varepsilon}} (qd^{1-1/f+\delta} + d^{1+\delta})
\ll \sum_{q = 1}^{h} D(P(q))(q^{-f+1+(1-1/f+\delta)(f-\varepsilon)} + q^{-f+(1+\delta)(f-\varepsilon)}) \ll q^{2\delta f-\varepsilon(1-1/f)} \ll h.
\]

**Lemma 7.** Let \( P(x) \) be arbitrary and \( \delta > 0 \). Then

\[
\sum_{q = 1}^{h} \sum_{d \mid P(q)} d^{-\delta} \ll h.
\]

**Proof.** The part of the sum where \( d \geq q \) is not larger than
The part of the sum where $d < q$ is estimated by

$$\sum_{d=1}^{h} d^{-\delta} \sum_{d < q \leq h : d \mid P(q)} 1 \leq \sum_{d=1}^{h} d^{-\delta} h d^{-1} z(d) \leq h \sum_{d=1}^{\infty} z(d) d^{-1-\delta} \ll h.$$

**Lemma 8.** Write $D_k(x)$ for the number of positive divisors of $x$ which are not larger than $k$, and let $\delta > 0$. Then

$$\sum_{q=1}^{h} D_k(P(q)) \ll hk^\delta.$$

**Proof.** We break the sum into two parts, $\sum_{q=1}^{\min(k,h)} + \sum_{k < q \leq h}$, where the second part may be empty. For $q$ contributing to the first part of the sum, $D_k(P(q)) \leq D(P(q)) \ll k^\delta$, and we obtain the desired estimate. The second part equals

$$\sum_{k<q\leq h} D_k(P(q)) = \sum_{d \leq k} \sum_{k < q \leq h : d \mid P(q)} 1 \leq \sum_{d \leq k} h d^{-1} z(d) \ll \sum_{d=1}^{\infty} z(d) d^{-1-\delta} \ll hk^\delta.$$

**Lemma 9.** Write $D(x,y)$ for the number of common positive divisors of integers $x, y \neq 0$. Let $P_1(x), P_2(x)$ be polynomials with integral coefficients such that $P_1(x) \neq 0$ for $x > 0$. Then

$$(5.1) \quad X_{h_1,h_2} = \sum_{q_1=1}^{h_1} \sum_{q_2=1}^{h_2} D(P_1(q_1), P_2(q_2)) \ll h_1 h_2.$$  

This estimate holds simultaneously in $h_1, h_2$.

**Proof.** It is sufficient to prove (5.1) with $P_1(x), P_2(x)$ both replaced by the product $P_1(x)P_2(x)$. We may therefore assume $P_1(x) = P_2(x) = P(x)$, say. There is an integer $g > 0$ and a polynomial $Q(x)$ without multiple factors such that $P(x) \mid Q(x)^g$.

Let $\sigma$ be the set of positive divisors of $P(x)$ where $1 \leq x \leq \min(h_1, h_2)$. The number of elements of $\sigma$ is $\ll (\min(h_1, h_2))^{1+\delta}$ for every $\delta > 0$.

$$X_{h_1,h_2} \ll \sum_{d \in \sigma} \left( \sum_{q_1 \leq h_1 : d \mid P(q_1)} 1 \right) \left( \sum_{q_2 \leq h_2 : d \mid P(q_2)} 1 \right) \ll \sum_{d \in \sigma} \left( \sum_{q_1 \leq h_1 : \xi(d) \mid Q(q_1)} 1 \right) \left( \sum_{q_2 \leq h_2 : \xi(d) \mid Q(q_2)} 1 \right) \ll \sum_{d \in \sigma} (h_1(\xi(d))^{-1} + 1)(h_2(\xi(d))^{-1} + 1)z_0^{\xi(d)} \ll \sum_{d \in \sigma} (h_1(\xi(d))^{-1} + 1)(h_2(\xi(d))^{-1} + 1)(\xi(d))^{2\delta}.$$
Using the distributive law we can break this sum into four parts, and Lemma 5 implies that each part is $\ll h_1 h_2$.

6. Estimates for $A(k, q, r)$. In what follows, $d^* = d^*(q, r)$ will mean g.c.d. $(P(q), P(r))$. Put $B(k, q, r)$ for the number of pairs of integers $p, s, p \in S(k, q), s \in S(k, r), 0 \leq p < P(q)$, such that

$$|P(q)(s + \theta) - P(r)(p + \theta)| < \min (d^*, P(q)\psi(r)).$$

**Lemma 10.** For $r \leq q$, $A(k, q, r) \leq \psi(q) P(q)^{-1} B(k, q, r)$.

**Proof.** All the expressions $P(q)s - P(r)p$ are multiples of $d^*$. Write $C(l, k, q, r)$ for the number of pairs $p, s, p \in S(k, q), s \in S(k, r), 0 \leq p < P(q)$ such that $P(q)s - P(r)p = ld^*$. The congruence $P(r)p = ld^*$ (mod $P(q)$) has $d^*$ solutions in $p$, and therefore

$$C(l, k, q, r) \leq d^*.$$

By definition,

$$\Gamma(k, q, r) = \sum_{p \in S(k,q)} \sum_{s \in S(k,r)} \int_0^1 \beta(q, P(q)\alpha - p) \beta(r, P(r)\alpha - s) d\alpha.$$

We now make the substitution $P(q)\alpha' = P(q)\alpha - p - \theta$. Then $P(r)\alpha - s = P(r)\alpha' + \theta - (P(q)(s + \theta) - P(r)(p + \theta)) P(q)^{-1}$ and

$$\Gamma(k, q, r)$$

$$= \sum_{p \in S(k,q)} \sum_{s \in S(k,r)} \int_{-(p+\theta)}^{1-(p+\theta)} \beta(q, P(q)\alpha' + \theta) \beta(r, P(r)\alpha' + \theta$$

$$- (P(q)(s + \theta) - P(r)(p + \theta)) P(q)^{-1}) d\alpha'$$

$$= \sum_l C(l, k, q, r)$$

$$\cdot \int_{-\infty}^{\infty} \beta(q, P(q)\alpha + \theta) \beta(r, P(r)\alpha + \theta - ld^* + (P(q) - P(r))\theta) P(q)^{-1}) d\alpha$$

$$= \sum_l C(l, k, q, r) D(q, r, ld^* + (P(q) - P(r))\theta),$$

where $D(q, r, t) = \int_{-\infty}^{\infty} \beta(q, P(q)\alpha + \theta) \beta(r, P(r)\alpha + \theta - tP(q)^{-1}) d\alpha$.

For the following estimates we recall that $\beta(q, \xi + \theta)$ is the characteristic function of $0 < \xi \leq \psi(q)$. We note

$$\int_{-\infty}^{\infty} D(q, r, t) dt = \psi(q) \psi(r)$$

as well as $0 \leq D(q, r, t) \leq \psi(q) P(q)^{-1}$ and the fact that $D$ is zero outside the interval $(-P(q)\psi(r), P(r)\psi(q))$, hence in particular if $|t| \geq P(q)\psi(r)$. Furthermore, $D(q, r, t)$ is decreasing for $t > 0$, increasing for $t < 0$. Hence
\[
\Gamma(k, q, r) \leq d^* \sum_{l: |l|d^* + (p(q) - P(r)) \theta| \geq d^*} D(q, r, ld^* + (p(q) - P(r)) \theta) + \sum_{l: |l|d^* + (p(q) - P(r)) \theta| < d^*} C(l, k, q, r) D(q, r, ld^* + (p(q) - P(r)) \theta)
\]

(6.1)

\[
\leq d^* \int_{-\infty}^{\infty} D(q, r, \lambda d^* + (p(q) - P(r)) \theta) d\lambda + \psi(q) P(q)^{-1} B(k, q, r)
\]

= \psi(q) \psi(r) + \psi(q) P(q)^{-1} B(k, q, r),

and the lemma follows.

Put \( E_\delta(k, q, r) \) for the number of \( p \in S(k, q), s \in S(k, r), 0 \leq p < P(q) \) with \(|P(q)(s + \theta) - P(r)(p + \theta)| < P(q)q^{-1} d^*\).

**Lemma 11.** Let \( P(q) \) be a polynomial of degree \( f > 0 \), and let \( \varepsilon = 1 \) if \( f = 1 \), \( \varepsilon > 0 \) if \( f > 1 \). Let \( \delta > 0 \). Then

\[
\sum_{q=1}^{h} \sum_{r=1}^{q} \psi(q) P(q)^{-1} B(k, q, r) \leq \Psi(h)^{1+\delta}
\]

\[
+ \sum_{q=1}^{h} \sum_{r \leq q \text{ with } d^* < q^{f-\varepsilon}} \psi(q) P(q)^{-1} E_\delta(k, q, r).
\]

**Proof.** Choose \( \delta_1 > 0 \), \( \delta_2 > 0 \) such that \( \delta_1 + 1/\delta_2 < \delta \). We shall use the easily proved estimate

(6.3)

\[
B(k, q, r) \leq 2d^*.
\]

We consider four parts \( \Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4 \) of the sum we want to estimate.

\( \Sigma_1: d^* < q^{f-\varepsilon} \). We may assume \( f > 1 \), since \( d^* < q^{f-\varepsilon} \) is otherwise impossible.

\[
\Sigma_1 \leq 2 \sum_{q=1}^{h} \sum_{r \leq q \text{ with } d^* < q^{f-\varepsilon}} \psi(q) P(q)^{-1} d^*
\]

\[
\leq 2 \sum_{q=1}^{h} \psi(q) P(q)^{-1} \sum_{d|P(q); d < q^{f-\varepsilon}} d \sum_{r \leq q \text{ with } d|P(r)} 1.
\]

Using Lemma 6 and partial summation we obtain \( \Sigma_1 \leq \Psi(h) \).

\( \Sigma_2: d^* < (q/r)^{1/\delta_1} \).

\[
\sum_{r < qd^* - \delta_1} B(k, q, r) \leq \sum_{d|P(q)} \sum_{r < qd^* - \delta_1; d|P(r)} d \leq \sum_{d|P(q)} \sum_{x \leq P(qd^* - \delta_1); d|x} 1
\]

\[
\leq \sum_{d|P(q)} dP(q)d^{-1-f\delta_1},
\]

\[
\Sigma_2 \leq \sum_{q=1}^{h} \psi(q) \sum_{d|P(q)} d^{-f\delta_1} \leq \Psi(h)
\]

by Lemma 7 and partial summation.
\[ \sum_{3} : d^* < \Psi(q)^{\frac{3}{2}}. \]

\[ \sum_{r < q \text{ with } d^* < \Psi(q)} \delta_2 B(k, q, r) \leq \sum_{d | P(q) : d < \Psi(q)^{\frac{3}{2}}} \sum_{r \leq q : d | P(r)} d \]

\[ \leq \sum_{d | P(q) : d < \Psi(q)^{\frac{3}{2}}} d \sum_{x \leq q : d | x} 1 \]

\[ \leq \sum_{d | P(q) : d < \Psi(q)^{\frac{3}{2}}} d P(q) d^{-1} = P(q) \sum_{d | P(q) : d < \Psi(q)^{\frac{3}{2}}} 1. \]

Putting \( l = \Psi(h)^{\frac{3}{2}} \) we obtain

\[ \sum_{3} \leq \sum_{q = 1}^{h} \psi(q) D_1(P(q)). \]

Lemma 8 together with partial summation gives \( \sum_{3} \leq \Psi(h)^{1 + \frac{3}{2}}. \)

\[ \sum_{4} : d^* \geq q^{f - \varepsilon}, \ d^* \geq (q/r)^{1/\delta_1}, \ d^* \geq \Psi(q)^{\delta_2}. \]

Under these conditions, \( P(q) \psi(r) \leq P(q) \psi(qd^* \delta_1)^{(2)} = P(q) q^{-1} d^* \delta_1 q d^* - \delta_1 \psi(qd^* - \delta_1) \leq P(q) q^{-1} d^* \delta_1 \Psi(q) \leq P(q) q^{-1} d^* \delta_1^{1 + 1/\delta_2} \leq P(q) q^{-1} d^* \delta_1 \); therefore \( B(k, q, r) \leq E_4(k, q, r). \)

Obviously, \( \sum_{4} \) is bounded by the right-hand sum of (6.2).

7. Proof of Proposition 2 for nonlinear polynomials. In the case of polynomials of degree \( f > 1 \) we may use Lemma 6, which ceases to be true if \( f = 1 \). On the other hand, much of the preceding discussion could be simplified for \( f = 1 \).

We assume now \( f > 1 \).

We define \( (x, y; k) \) by

\[
(x, y; k) = \begin{cases} 
\text{g.c.d.} (x, y) \text{ if this divisor is } \geq xk^{-1}, \\
0 \text{ otherwise.}
\end{cases}
\]

**Lemma 12.** Assume that \( \delta > 0 \) is so small that \( f - 1/2 < (f - 1/4)(1 - \delta) \). Further assume \( q > q_0(P, \delta), \ r \leq q, \ d^* = \text{g.c.d.} (P(q), P(r)) \geq q^{f - 1/4} \). Then

(7.1) \[ E_4(k, q, r) \leq (P(q), P(r); k). \]

**Proof.** Put \( c = |a_0| + \cdots + |a_f| \), where the \( a_i \)'s are the coefficients of \( P(q) \).

Choose \( q_0 \) so large that the two inequalities

\[ q_0^{1/4} > 2c, \quad 2cq_0^{f - 1/2} < q_0^{(f - 1/4)(1 - \delta)} \]

hold, and let \( q \geq q_0 \).

The numbers \( a = a(q), \ b = b(q) \) satisfy

\[ \theta = b/a + R, \quad |R| < a^{-1} q^{-1/2}, \quad a \leq q^{1/2} \]

and

(7.2) \[ 2caq^{f - 1} \leq 2cq^{f - 1/2} < q^{(f - 1/4)(1 - \delta)} \leq d^{*1 - \delta}. \]

(3) For \( 0 \leq a < 1 \) define \( \psi(n - a) = \psi(n) \).
$E_d(k, q, r)$ is bounded by the number of pairs $p, s$, $0 \leq p < P(q)$, $p \in S(k, q)$ satisfying

$$|P(q)s - P(r)p + (P(q) - P(r))(b/a + R)| < P(q)q^{-1}d^{*\delta}.$$ 

This equation together with (7.2) yields

$$|P(q)(s + b/a) - P(r)(p + b/a)| < P(q)q^{-1}d^{*\delta} + |P(q) - P(r)||R|$$

(7.3)

$$< d^{*2} + g^{*1/2}a = d^{*}/a.$$

The left-hand side of (7.3) is an integral multiple of $d^{*}/a$, hence it must be zero.

(7.4) 

$P(q)(as + b) = P(r)(ap + b)$.

It remains to estimate the number of solutions of (7.4) in pairs $p, s$, $0 \leq p < P(q)$, $p \in S(k, q)$. Putting $P(q) = d^*P(q)^*$ we find that (7.4) implies

$$ap + b = 0 (mod P(q)^*)$$

Since $a$ and $b$ are relatively prime, this congruence has at most one solution in $p$ modulo $P(q)^*$, hence at most $d^*$ solutions in $0 \leq p < P(q)$. On the other hand, the congruence gives $g.c.d. (ap + b, P(q)) \geq P(q)* = P(q)d^{*-1}$, while $p \in S(k, q)$ implies $g.c.d. (ap + b, P(q)) \leq k$. Thus $E_4(k, q, r)$ is zero unless $d^* \geq P(q)k^{-1}$.

Lemma 12 is proved.

Proof of Proposition 2 ($f > 1$). We may assume that $\delta > 0$ is so small that $f - 1/2 < (f - 1/4)(1 - \delta)$. Combining Lemma 10, Lemma 11 with $\epsilon = 1/4$ and Lemma 12, we obtain

$$\sum_{q=1}^{h} \sum_{r=1}^{h} A(k, q, r) \ll \Psi(h)^{1+\delta} + \sum_{q=1}^{h} \sum_{r=1}^{q} \psi(q) P(q)^{-1}(P(q), P(r); k).$$

Using Lemma 8 and partial summation we find

$$\sum_{q=1}^{h} \sum_{r=1}^{q} P(q)^{-1}(P(q), P(r); k) \leq \sum_{q=1}^{h} P(q)^{-1} \sum_{d|P(q); d \leq P(q)k^{-1}} d \sum_{r=1}^{1} dP(q)d^{-1}$$

$$= \sum_{q=1}^{h} dP(q) \ll h k^{\delta}$$

and

$$\sum_{q=1}^{h} \sum_{r=1}^{q} \psi(q) P(q)^{-1}(P(q), P(r); k) \ll \Psi(h)k^{\delta}.$$
Lemma 13. Let \( P(x) \) be a linear polynomial, let \( 0 < \delta < 1/4 \), \( q \geq 1 \) and \( d^* \mid P(q) \). Then

\[
\sum_{r \leq q: (P(q), P(r)) = d^*} E_\delta(k, q, r) \ll q d^*^{-1/4} + d^* + q^{1/2} + \begin{cases} q & \text{if } d^* \mid P(q)k^{-1}, \\ 0 & \text{otherwise.} \end{cases}
\]

Remark. The constant involved in the symbol \( \ll \) depends on \( P(x) \) only.

Proof. Put \( c = \left| a_0 \right| + \left| a_1 \right| \) where \( P(x) = a_0x + a_1 \). The relation \( |P(q)s - P(r)p + (P(q) - P(r))\theta| < P(q)q^{-1}d^* \delta \leq cd^*^{1/4} \) in the definition of \( E_\delta(k, q, r) \) implies \( \| (P(q) - P(r))d^*^{-1}\theta \| < cd^*^{-3/4} \). Given \( r \) such that the last inequality holds, there are at most \( 2c \) integers \( l \) with

\[
\|(P(q) - P(r))d^*^{-1}\theta - l\| < cd^*^{-3/4}.
\]

Given \( r \) and \( l \), \( P(q)s - P(r)p = ld^* \) has at most \( d^* \) solutions in \( p \), \( 0 \leq p < P(q) \). Putting

\[
F(q, r) = \begin{cases} 1 & \text{if } \| (P(q) - P(r))d^*^{-1}\theta \| < cd^*^{-3/4}, \\ 0 & \text{otherwise,} \end{cases}
\]

we thus find

\[
E_\delta(k, q, r) \ll d^*F(q, r).
\]

Assume now that \( r \) runs through those values between 1 and \( q \) where \( d^* \mid P(r) \). Then \( P(q) - P(r) \) runs through some or all of the numbers 0, \( d^* \), \( 2d^* \), ..., \( [a_0q/d^*]d^* = q^*d^* \). Thus if we put \( G(q, d^*) \) for the number of integers \( x \) in \( 0 \leq x \leq q^* \) satisfying

\[
\| x\theta \| < cd^*^{-3/4},
\]

then

\[
\sum_{r \leq q: d^* \mid P(r)} F(q, r) \leq G(q, d^*).
\]

We now distinguish three cases: \( A \), \( B \) and \( C \).

A. \( 2ca(q) \leq d^*^{1/4} \). How often does (8.3) hold when \( x \) runs through an interval \( m < x \leq m + a(q) \)? Putting \( \theta = b/a + R \) and \( x = m + y \), the inequality becomes \( \| m\theta + yb/a + yR \| < cd^*^{-3/4} \) and this implies \( \| m\theta + yb/a \| < cd^*^{-3/4} + aR < cd^*^{-3/4} + q^{-1/2} \). The number of solutions of (8.3) for \( x \) in an interval of length \( a \) is therefore \( \leq (d^*^{-3/4} + q^{-1/2})a + 1 \leq d^*^{-3/4}a + 1 \). Hence

\[
G(q, d^*) \ll (d^*^{-3/4}a + 1)(qa^{-1} + 1) \ll q^*d^*^{-1/4} + q^{1/2}d^*^{-3/4} + 1,
\]

and

\[
d^*(G(q, d^*)) \ll qad^*^{-1/4} + d^*.
\]

(8.1) now follows from (8.2), (8.4) and the last inequality.
B. $2ca(q) < d*^{1/4}$, $2aq* | R | \geq 1$. Putting $\theta = b/a + R$ again, we rewrite (8.3) as $\| xb/a + xR \| < cd*^{-3/4}$. This implies that

\begin{equation}
| m/a + xR | < cd*^{-3/4}
\end{equation}

for some integer $m$. For fixed $m$ the number of solutions in $x$ of (8.5) is at most $2cd*^{-3/4} | R |^{-1} + 1$. On the other hand, $x \leq q*$, whence $| m | \leq cad*^{-3/4} + aq* | R |$. Thus

$$G(q, d*) \leq (d*^{-3/4} | R |^{-1} + 1)(ad*^{-3/4} + aq* | R | + 1)$$

$$\leq a^2q*d*^{-3/2} + ad*^{-3/4} + aq*d*^{-3/4} + q*q^{-1/2} + 2aq*d*^{-3/4} + 1$$

$$\leq a^2q*d*^{-3/4} + q*q^{-1/2} + 1$$

and

$$d*G(q, d*) \leq qd*^{-1/4} + q^{1/2} + d*.$$ 

C. $2ca(q) < d*^{1/4}, 2aq* | R | < 1$. $E_\delta(k, q, r)$ is bounded by the number of solutions in integers $p, s, 0 \leq p < P(q)$, $p \in S(k, q)$, of

$$| P(q)(s + b/a) - P(r)(p + b/a) + (P(q) - P(r))R | < cd*^{1/4}.$$ 

Now for $r \leq q$, $d* | P(r)$, one has $| P(q) - P(r) | \leq q*d*$, and we obtain the inequality

$$| P(q)(s + b/a) - P(r)(p + b/a) | \leq cd*^{1/4} + q*d* | R | < d*/2a + d*/2a = d*/a.$$ 

Just as in the proof of Lemma 12 we may conclude that (7.4) holds, and we obtain (7.1). The number of $r \leq q$ with $d* | P(r)$ is $\leq qd*^{-1}$, and therefore

$$\sum_{r \leq q, d* \mid P(r)} E_\delta(k, q, r) \leq \begin{cases} q & \text{if } d* \geq P(q)k^{-1}, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof of Proposition 2 ($f = 1$).** We may assume $0 < \delta < 1/4$. By applying Lemma 10 and Lemma 11 with $\epsilon = 1$ we obtain

$$\sum_{q=1}^{h} \sum_{r=1}^{h} A(k, q, r) \leq \Psi(h)^{1+\delta} + \sum_{q=1}^{h} \sum_{r=1}^{q} \psi(q)P(q)^{-1}E_\delta(k, q, r).$$

By Lemma 13,

$$\sum_{q=1}^{h} \sum_{r=1}^{h} P(q)^{-1}E_\delta(k, q, r)$$

$$\leq \sum_{q=1}^{h} q^{-1} \sum_{d* \mid P(q)} (qd*^{-1/4} + d* + q^{1/2}) + \sum_{q=1}^{h} \sum_{d* \mid P(q); d* \geq P(q)k} -1$$

$$\leq \sum_{q=1}^{h} \sum_{d* \mid P(q)} d^{-1/4} + \sum_{q=1}^{h} q^{-1/2}D(P(q)) + \sum_{q=1}^{h} D_q(P(q))$$

$$\leq h + h + hk^\delta \leq hk^\delta.$$
Here we used (4.1) and Lemmas 7 and 8 to estimate the last three sums. Proposition 2 now follows by partial summation.

9. The higher dimensional case. Most of the arguments used for the case $n = 1$ carry over immediately to $n > 1$, but some of them have to be modified.

We may assume that $I_{j_0}$ is of the type $0 < \xi_j - \theta_j \leq \psi_j(q)$. For each of the integers $j = 1, \ldots, n$ we can now define $a_j(q)$, $b_j(b)$, $S_j(k,q)$, $\beta_j(q,a_j)$, $\gamma_j(q,a_j)$, $\cdots$, $\Gamma_j(k,q,r)$. For given $q, r$ we write $d^n_j$ for the greatest common divisor of $P_j(q)$ and $P_j(r)$, and we may now define $B_j(k,q,r)$, $\cdots$, $E_{j_0}(k,q,r)$, $F_j(q,r)$, $G_j(q,d^n_j)$. We put $\beta(q, a_1, \cdots, a_n) = \prod_j \beta_j(q,a_j)$, $\gamma(q,a_1, \cdots, a_n) = \prod_j \gamma_j(q,a_j)$, $\cdots$, $\Gamma(k,q,r) = \prod_j \Gamma_j(k,q,r)$, and we define $A(k,q,r)$ as in paragraph 2.

**Proposition 1a.** Let $\delta > 0$. Then

$$\sum_{q=1}^h (P_1(q) \cdots P_n(q) - \phi(k, P_1(q)) \cdots \phi(k, P_n(q)))^{-1} \leq h \delta^{-1} + h^\delta \delta.$$

**Proposition 2a.** Let $\delta > 0$. Then (3.2) holds.

The argument of paragraph 3 can be used to deduce the general theorem from these propositions.

Proposition 1a follows from Proposition 1 and

$$P_1(q) \cdots P_n(q) - \phi(k, P_1(q)) \cdots \phi(k, P_n(q)) = \sum_{j=1}^n P_1 \cdots P_{j-1}(P_j - \phi(k, P_j))\phi(k, P_{j+1}) \cdots \phi(k, P_n).$$

(6.1) now becomes

$$\Gamma_j(k,q,r) \leq \psi_j(q)\psi_j(r) + \psi_j(q)P_j(q)^{-1}B_j(k,q,r),$$

and therefore for $r \leq q$

$$A(k,q,r) \leq \sum_{m=1}^n \sum_{\Delta_m} H(k,q,r;m,\Delta_m),$$

where $\Delta_m$ runs through all divisions of the integers 1, \ldots, $n$ into two classes $i_1, \cdots, i_m$ and $j_1, \cdots, j_{n-m}$, and where

$$H(k,q,r;m,\Delta_m) = \psi(q) \prod_{s=1}^m (P_s^{-1}(q)B_s(k,q,r)) \prod_{t=1}^{n-m} \psi_t(r).$$

For reasons of symmetry it will suffice to estimate $H(k,q,r;m,\Delta^o_m)$, where $\Delta^o_m$ is the division with $i_1 = 1, \cdots, i_m = m$. We shall use

(9.1) $B_i(k,q,r) \leq 2d^n_i$ \hspace{1cm} (i = 1, \ldots, n).

**Lemma 14.** Let $m > 1$. Then

$$\sum_{q=1}^h \sum_{r=1}^q H(k,q,r;m,\Delta^o_m) \ll \Psi(h).$$
Proof. We use the estimate
\[ H(k, q, r; m, \Delta_n^q) \ll \psi(q) P_1(q)^{-1} P_2(q)^{-1} d_1^* d_2^*. \]

By Schwartz' inequality,
\[
Y_h = \sum_{q=1}^{h} \sum_{r=1}^{q} P_1(q)^{-1} d_1^* P_2(q)^{-1} d_2^* \leq \left( \sum_{q=1}^{h} \sum_{r=1}^{q} P_1(q)^{-2} d_1^{*2} \right)^{1/2} \left( \sum_{q=1}^{h} \sum_{r=1}^{q} P_2(q)^{-2} d_2^{*2} \right)^{1/2}.
\]

Now
\[
\sum_{q=1}^{h} \sum_{r=1}^{q} P(q)^{-2} d_1^{*2} \leq \sum_{q=1}^{h} P(q)^{-2} \sum_{d \mid P(q)} d^2 \sum_{r \leq q} 1 \leq \sum_{q=1}^{h} P(q)^{-2} \sum_{d \mid P(q)} d^2 P(q) d^{-1} = \sum_{q=1}^{h} \sum_{d \mid P(q)} d^{-1} \ll h
\]

by Lemma 7. Hence \( Y_h \ll h \), and Lemma 14 follows by partial summation.

Everything can be completed as in the case \( n = 1 \) once we have shown

**Lemma 11a.** Let \( \epsilon = 1 \) if the degree \( f_1 \) of \( P_1(x) \) equals 1, \( \epsilon > 0 \) if \( f_1 > 1 \). Let \( \delta > 0 \). Then

\[
\sum_{q=1}^{h} \sum_{r=1}^{q} H(k, q, r; 1, \Delta_n^q) \ll \Psi(h)^{1+\delta} + \sum_{q=1}^{h} \sum_{r \leq q \text{ with } d_1^* q \leq r} \psi(q) P_1(q)^{-1} E_1(k, q, r).
\]

**Proof.** Choose \( \delta_1 > 0, \delta_2 > 0 \) such that \( \delta_1 + 1/\delta_2 < \delta \). We write \( \chi_1(q) = \psi_2(q) \cdots \psi_n(q) \) and put

\[
\Psi_1(h) = \sum_{q=1}^{h} \psi_1(q), \quad X_1(h) = \sum_{q=1}^{h} \chi_1(q).
\]

Since both \( \psi_1(q) \) and \( \chi_1(q) \) are decreasing, one has

\[
(9.2) \quad h \Psi(h) \geq \Psi_1(h) X_1(h).
\]

\( H(k, q, r; 1, \Delta_n^q) \) equals \( \psi(q) P_1(q)^{-1} B_1(k, q, r) \chi_1(r) \).

We consider four parts of the sum we want to estimate. \( \Sigma_1 \) consists of terms with \( d_1^* < q^{1-\epsilon} \), \( \Sigma_2 \) of terms with \( d_1^* < (q/r)^{1/\delta_1} \), \( \Sigma_3 \) of terms where \( d_1^* < \Psi_1(q)^{\delta_2} \), and \( \Sigma_4 \) consists of the remaining terms, that is, terms where \( d_1^* \geq q^{1-\epsilon} \).
For the parts $\Sigma_1$, $\Sigma_2$, $\Sigma_4$ we estimate $H(\cdots) \leq \psi(q)P_1(q)^{-1}B_1(k,q,r)$ and proceed as in paragraph 6. The difficulty lies in estimating $\Sigma_3$.

Let $d\mid P_1(q)$ and denote the numbers $r$ having $r \leq q$ and $d\mid P_1(r)$ by $r_1 < r_2 < \cdots < r_J$. One has $j \leq P_1(q)d^{-1}$ and $r_j \leq c(jd)^{1/f_1} \geq c'djP_1(q)^{-1}q$ for large $q$. Hence

\[
\sum_{r \leq q \text{ with } d^* \leq \Psi_1(q)} B_1(k,q,r) \chi_1(r) \leq \sum_{d\mid P_1(q)} \delta_d \sum_{r \leq q \text{ with } d\mid P_1(r)} \chi_1(r) \\
\leq \sum_{d\mid P_1(q)} \delta_d \sum_{r \leq q} \chi_1(q)P_1(q)q^{-1}d^{-1} \\
= \chi_1(q)P_1(q)q^{-1}D_f(P_1(q))
\]

with $f = f(q) = \Psi_1(q)^{3/2}$. Hence

\[(9.3) \quad \sum_3 \leq \sum_{q=1}^{h} \psi(q)q^{-1}X_1(q)D_f(P_1(q)).\]

We estimate the last sum in three parts, which overlap somewhat.

$T_1 : \Psi(q) \geq q^{1/4d_2}$. Unless this part is empty, there is a largest $q \leq h$ in $T_1$ say $q_1$. By (4.1),

\[
T_1 \leq \sum_{q=1}^{q_1} \psi(q)D(P_1(q)) \leq \sum_{q=1}^{q_1} \psi(q)q^{3/4d_2} \leq \Psi(q_1)q_1^{3/4d_2} \\
\leq \Psi(q_1)^{1+\delta} \leq \Psi(h)^{1+\delta}.
\]

$T_2 : \chi_1(q) \leq q^{1-1/4d_2}$. Again using (4.1) we obtain

\[
T_3 = \sum_{q\in \sigma} g(q)D_f(P_1(q)) = \sum_{d \leq f(h_1)} \sum_{q \in \sigma, d\mid P_1(q), f(q) \geq d} g(q).
\]

Let $x_1, \ldots, x_d(d)$ be the solutions of $P_1(x) \equiv 0 (\mod d)$. Since $g(q)$ is decreasing, one has

\[
\sum_{q \in \sigma, f(q) \geq d, q = x_i (\mod d)} g(q) \leq g(q_i) + d^{-1} \sum_{f(q) \geq d} g(q),
\]

where $q_i$ is the smallest $q = x_i (\mod d)$ such that $q \in \sigma$ and $f(q) \geq d$. Since $f(q) \leq q^{1/2}$, one finds $q_i \geq d^2$. Therefore

\[
\sum_{q \in \sigma, f(q) \geq d, d\mid P_1(q)} g(q) \leq z(d)g(d^2) + z(d)d^{-1} \sum_{f(q) \geq d} g(q).
\]
Observing \( z(d)g(d^2) \leq dg(d^2) \leq \sum_{q=(d-1)^2+1}^{d^2} g(q) \) we obtain

\[
T_3 \leq \sum_{d \leq f(h_1)} \left( \sum_{q=(d-1)^2+1}^{d^2} g(q) + z(d)d^{-1} \sum_{f(q) \geq d} g(q) \right)
\]

\[
\leq \sum_{q=1}^{f_1(h_1)} \psi(q) + \sum_{q=1}^{h} g(q) \sum_{d \leq f(q)} z(d)d^{-1}
\]

\[
\leq \Psi(h) + \sum_{q=1}^{h} \psi(q)\Psi_1(q)\delta
\]

by Lemma 5. (9.2) finally yields

\[
T_3 \ll \Psi(h) + \sum_{q=1}^{h} \psi(q)(q^{-1}X_1(q))^{1-\delta}\Psi(q)^{\delta} \ll \Psi(h)^{1+\delta}
\]

Lemma 11a is proved.

10. Linear forms. We restrict ourselves to the case of one form only.

**Proposition 3.** Let \( P_1(q_1), \ldots, P_n(q_n) \) be nonconstant polynomials, \( n+1 \), and let \( I_{q_1, \ldots, q_n} \) be intervals of \( C \) \( (q_i = 1, 2, \ldots ; i = 1, \ldots, n) \). We assume that the length of \( I_{q_1, \ldots, q_n} \) is \( \psi_1(q_1)\psi_2(q_2) \cdots \psi_n(q_n) \), where \( \psi_i(x) \) are decreasing functions \( (i = 1, \ldots, n) \), and we put

\[
\Psi_i(h) = \sum_{q_i=1}^{h} \psi(q_i).
\]

We write \( M(h_1, \ldots, h_n; \alpha_1, \ldots, \alpha_n) \) for the number of solutions of \( \{ \alpha_1P_1(q_1) + \cdots + \alpha_nP_n(q_n) \} \in I_{q_1, \ldots, q_n} \), where \( 1 \leq q_i \leq h_i \) \( (i = 1, \ldots, n) \). Let \( \epsilon > 0 \). Then for almost all \( \alpha_1, \ldots, \alpha_n \),

\[
M(h_1, \ldots, h_n; \alpha_1, \ldots, \alpha_n) = \Psi_1(h_1) \cdots \Psi_n(h_n) + O(\Psi_1(h_1) \cdots \Psi_n(h_n))^{1/2+\epsilon}.
\]

This estimate holds simultaneously for \( h_1, \ldots, h_n \).

**Proof.** We restrict ourselves to a few hints. The reader might compare paragraph 6 of [12]. We assume \( n > 1 \).

We put \( \beta(q_1, \ldots, q_n, \xi) \) equal to 1 if \( \{ \xi \} \in I_{q_1, \ldots, q_n} \) and \( \xi \in U, \beta(\cdots) = 0 \) otherwise. \( \Gamma(q_1, \ldots, q_n; r_1, \ldots, r_n) \) stands for the integral

\[
\int_0^1 \cdots \int_0^1 \left( \sum_p \left( \beta(q_1, \ldots, q_n, \sum \alpha_iP_i(q_i) - p) \right) \left( \sum_s \beta(r_1, \ldots, r_n, \sum \alpha_iP_i(r_i) - s) \right) \right) \, d\alpha_1 \cdots d\alpha_n,
\]

and \( A(q_1, \ldots, q_n; r_1, \ldots, r_n) \) for

\[
\Gamma(q_1, \ldots, r_n) - \psi_1(q_1) \cdots \psi_n(q_n)\psi_1(r_1) \cdots \psi_n(r_n).
\]

**Proposition 2b.** \( \sum_{q_1=1}^{h_1} \cdots \sum_{r_n=1}^{h_n} A(q_1, \ldots, r_n) \ll \Psi_1(h_1) \cdots \Psi_n(h_n). \)
To deduce Proposition 3 from Proposition 2b we put

\[ M(h_1, \ldots, h_n; k_1, \ldots, k_n; \alpha_1, \ldots, \alpha_n) \]

for the number of \( q_1, \ldots, q_n, h_i < q_i \leq k_i \) (\( i = 1, \ldots, n \)) such that \( \{ \sum \alpha_i P_i(q_i) \} \in I_{q_1, \ldots, q_n} \) and we put \( \Psi(h, k) = \sum_{h < q \leq k} \psi(q) \). We choose integers \( m_i = m_i(r_1, \ldots, r_n) \) such that \( \lfloor 2^{r_1 + \ldots + r_n} \rfloor \Psi(m_i) \rfloor = u \) The following two lemmas are now used.

**Lemma 2b.** Let \( \delta > 0 \). Then one has for \( T = r_1 + \ldots + r_n \)

\[
\sum_{(u_1, v_1) \in J} \ldots \sum_{(u_n, v_n) \in J} \int_0^1 (M(m_{u_1}^1, \ldots, m_{u_n}^n; m_{v_1}^1, \ldots, m_{v_n}^n; \alpha_1, \ldots, \alpha_n)
- \Psi_1(m_{u_1}^1, m_{v_1}^1) \Psi_n(m_{u_n}^n, m_{v_n}^n))^2 \, d\alpha_1 \ldots d\alpha_n
\leq 2^{(r_1 + \ldots + r_n)(1 + \delta)} .
\]

**Lemma 3b.** Let \( \delta > 0 \). There are subsets \( \sigma_{r_1, \ldots, r_n} (r_i = 1, 2, \ldots; i = 1, \ldots, n) \) of \( U \times \ldots \times U \) with measures

\[
\mu_{r_1, \ldots, r_n} \leq r_1^{-2} \ldots r_n^{-2}
\]
such that

\[
M(m_{w_1}^1, \ldots, m_{w_n}^n; \alpha_1, \ldots, \alpha_n) = \Psi_1(m_{w_1}^1) \Psi_n(m_{w_n}^n) + O(r_1^2 \ldots r_n^2 (r_1 + \ldots + r_n)(1/2 + \delta))
\]
for every \( w_1, \ldots, w_n \) with \( w_i \leq 2^{r_1 + \ldots + r_n} \) (\( i = 1, \ldots, n \)) and \( (\alpha_1, \ldots, \alpha_n) \) in \( U \times \ldots \times U \) but not in \( \sigma_{r_1, \ldots, r_n} \).

To prove Proposition 2b we require

**Lemma 10b.** A. If the matrix

\[
\begin{pmatrix}
P_1(q_1), \ldots, P_n(q_n) \\
P_1(r_1), \ldots, P_n(r_n)
\end{pmatrix}
\]

has rank 2, then

\[
A(q_1, \ldots, r_n) = 0.
\]

B. If the matrix has rank 1, then

\[
A(q_1, \ldots, r_n) \leq \psi_1(q_1) \cdots \psi_n(q_n) P_1(q_1)^{-1} B_1(q_1, \ldots, r_n),
\]
where \( B_1(q_1, \ldots, r_n) \) is the number of solutions of \( \lfloor P_1(q_1)(s + \theta') - P_1(r_1)(p + \theta) \rfloor < d^*_i \) in integers \( p, s, 0 \leq p < P_1(q_1) \), where \( \theta, \theta' \) are the left endpoints of \( I_{q_1, \ldots, q_n}, I_{r_1, \ldots, r_n}, \) respectively, and where \( d^*_i = \text{g.c.d.} (P_1(q_1), P_1(r_1)) \).

We leave the proof of A to the reader. As for B, we make the substitution \( \alpha_2 = \xi_2, \ldots, \alpha_n = \xi_n, \sum \alpha_i P_i(q_i) = \xi_1 P_1(q_1) \), hence \( \sum \alpha_i P_i(r_i) = \xi_1 P_1(r_1) \). When \( \xi_2, \ldots, \xi_n \) is fixed, \( \xi_1 \) ranges in an interval of length 1, and \( \Gamma \) equals
\[
\int_0^1 \left( \sum_{i} \beta(q_1, \ldots, q_n, \xi_1 P_1(q_1) - p) \right) \left( \sum_{s} \beta(r_1, \ldots, r_n, \xi_1 P_1(r_1) - s) \right) d\xi_1.
\]

This one-dimensional integral can be estimated by the method of paragraph 6.

The proof of Proposition 2b now proceeds as follows. We may restrict ourselves to terms \( r_1 \leq q_1 \). For fixed \( q_1, \ldots, q_n \), let \( \Delta = \text{g.c.d.}(P_1(q_1), \ldots, P_n(q_n)) \) and \( P_i(q_i) = P_i(q_i)^*\Delta \). In view of Lemma 10b we may restrict ourselves to \( r_1 \leq q_1 \) where \( P_1(q_1) \) is of the type \( IP_1(q_1)^* \), whence \( d_1^* = P_1(q_1)^*(\Delta, I) \). Since \( B(q_1, \ldots, r_n) \leq 2d_1^* \), one has

\[
\sum_{q_1 = 1}^{b_1} \cdots \sum_{q_n = 1}^{b_n} A(q_1, \ldots, r_n) \leq 4 \sum_{q_1 = 1}^{h_1} \cdots \sum_{q_n = 1}^{h_n} \psi_1(q_1) \cdots \psi_n(q_n) \sum_{l=1}^{A} P_1(q_1)^{-1}(P_1(q_1)^*(\Delta, I))
\leq 4 \sum_{q_1 = 1}^{h_1} \cdots \sum_{q_n = 1}^{h_n} \psi_1(q_1) \cdots \psi_n(q_n) D(\Delta)
\leq 4 \left( \sum_{q_1 = 1}^{h_1} \sum_{q_2 = 1}^{h_2} \psi_1(q_1) \psi_2(q_2) D(P_1(q_1), P_2(q_2)) \right)^{\Psi_3(h_3) \cdots \Psi_n(h_n)}.
\]

Using Lemma 9 and partial summation both for the sum over \( q_1 \) and over \( q_2 \) we obtain

\[
\sum_{q_1 = 1}^{h_1} \sum_{q_2 = 1}^{h_2} \psi_1(q_1) \psi_2(q_2) D(P_1(q_1), P_2(q_2)) \ll \Psi_1(h_1) \Psi_2(h_2).
\]

11. Theorem 2. To prove the lower bound in (1.5) we shall need

Proposition 4. Let \( a(1) < a(2) < \cdots \) be a sequence of positive integers and put \( M_a(h; \alpha) \) for the number of \( q \leq h \) such that \( \{xa(q)\} \in I \). Then for \( \varepsilon > 0 \) and almost all \( \alpha \) the inequality

\[
|M_a(h; \alpha) - h\lambda(I)| < h^{1/2} \log^{5/2 + \varepsilon} h
\]

holds for all intervals \( I \) and all \( h > h_1 \), where \( h_1 \) depends only on \( \alpha \) and \( \varepsilon \) (but not on \( I \)).

Proof. This proposition is a special case of Theorem 1 of [3] and of Theorem 1 of [6].

Proof of Theorem 2. We use the abbreviation

\[
\left\| \sum_{i=1}^{n} a_i(q_i) \right\| = \left\| \sum_{i=1}^{n} a_i(q_i) + \theta \right\|.
\]

Put \( \delta = \varepsilon/(n + 1) \). Using an idea of Littlewood [4, Appendix A], we consider the integral

\[
J(q_1, \ldots, q_n) = \int_0^1 \cdots \int_0^1 \left( \| \sum_{i=1}^{n} a_i(q_i) \| \log \| \sum_{i=1}^{n} a_i(q_i) \| \right)^{1+\delta} d\alpha_1 \cdots d\alpha_n.
\]
This integral has a finite value independent of \( q_1, \ldots, q_n \). Hence the sum

\[
\sum_{q_1=1}^{\infty} \cdots \sum_{q_n=1}^{\infty} (q_1 \log^{1+\delta} q_1 \cdots q_n \log^{1+\delta} q_n)^{-1} J(q_1, \ldots, q_n)
\]

is convergent and

\[
(11.1) \quad \sum_{q_1=1}^{\infty} \cdots \sum_{q_n=1}^{\infty} (q_1 \log^{1+\delta} q_1 \cdots q_n \log^{1+\delta} q_n) \log \| \sum \| |1+\delta|-1
\]

is convergent for almost all \( \alpha_1, \ldots, \alpha_n \).

It is easy to see that the inequality

\[
(11.2) \quad \| \sum \| \leq (q_1 \cdots q_n)^{-2}
\]

has only a finite number of solutions in integers \( q_1, \ldots, q_n \) for almost every \( \alpha_1, \ldots, \alpha_n \).

We are going to show that the upper estimate for \( \sum(h; \alpha_1, \ldots, \alpha_n) \) in (1.5) is true for every \( \alpha_1, \ldots, \alpha_n \) such that (11.1) is convergent and (11.2) has only finitely many solutions. There is a constant \( c > 0 \) such that \( \| \sum \| \geq c^{-1} (q_1 \cdots q_n)^{-2} \), whence

\[
|\log \| \sum \| | \leq 2 \log (q_1 \cdots q_n) + \log c.
\]

We obtain

\[
\sum(h; \alpha_1, \ldots, \alpha_n) \leq (\max_{q_i \leq h} (\log^{1+\delta} q_1 \cdots \log^{1+\delta} q_n | \log \| \sum \| |1+\delta)))
\]

\[
\sum_{q_1=1}^{\infty} \cdots \sum_{q_n=1}^{\infty} (q_1 \log^{1+\delta} q_1 \cdots q_n \log^{1+\delta} q_n) \log \| \sum \| |1+\delta|-1
\]

\[
\leq (\log h)^{(1+\delta)n} (\log h^{1+\delta}) \leq (\log h)^{n+1+\delta}.
\]

We now turn to the proof of the lower bound in (1.5). We are going to apply Proposition 4 to the sequence \( a(q) = a_1(q) \). For almost all reals \( \alpha \) and \( h \geq h_1(\alpha) \) one has \( |M_f(h; \alpha) - h l(l)| < h^{3/4} \). Let \( \alpha \) have this property. Denote the number of \( q \leq h \) such that \( \| \alpha_1 a_1(q) + \eta \| \leq \gamma \) by \( M_{\gamma,n}(h; \alpha_1) \). Then

\[
|M_{\gamma,n}(h; \alpha_1) - 2\gamma h| < h^{3/4} \quad \quad (h \geq h_1(\alpha_1), 0 \leq \gamma \leq 1/2, \eta \text{ arbitrary}).
\]

Let \( k_0 = k_0(h) \) be the largest integer with \( 2^{k_0+1} \leq h^{1/4} \). Then \( k_0 \geq 0 \) for \( h \geq h_2(\alpha_1) = \max(h_1(\alpha_1), 2^0) \). The number \( N_{k,n}(h; \alpha_1) \) of \( q \leq h \) such that

\[
(11.3) \quad 2^{-k-1} \| \alpha_1 a_1(q) + \eta \| \leq 2^{-k}
\]

satisfies \( N_{k,n}(h; \alpha_1) \geq 2^{-k} h - 2h^{3/4} \geq 2^{-k-1} h \) for every \( k \in 0 \leq k \leq k_0 \).

By considering the parts of the sum where (11.3) is satisfied for \( k = 0, \ldots, k_0 \) we obtain

\[
\sum_{q=1}^{h} \| \alpha_1 a_1(q) + \eta \| ^{-1} \geq \sum_{k=0}^{k_0} 2^k 2^{-k-1} h > \frac{1}{2} h k_0(h) \geq c_1(\alpha_1) h \log h.
\]

Partial summation yields
This inequality holds for arbitrary \( \eta \). By writing \( \eta = \alpha_2 a_2(q_2) + \cdots + \alpha_n a_n(q_n) + \theta \) and taking the sum over \( q_2, \ldots, q_n \) one finds
\[
\sum (h; \alpha_1, \ldots, \alpha_n) \geq c_3(\alpha_i) \log^{n+1} h.
\]

Remark. Our method could be used to show the following: The left inequality of (1.5) is true for arbitrary \( \alpha_1, \ldots, \alpha_{n-1} \) and \( \alpha_n \in \sigma(\alpha_1, \ldots, \alpha_{n-1}) \), where \( \sigma(\cdots) \) is a set containing almost all numbers. The other inequality of (1.5) holds for \( n \)-tuples such that \( \alpha_n \in \tau \) where \( \tau \) is independent of \( \alpha_1, \ldots, \alpha_{n-1} \) and contains almost all numbers.

12. Theorem 3. We define a function \( n(k_1, \ldots, k_n) \) as follows. \( n(0, \ldots, 0) = 0 \), and if \( k_1, \ldots, k_{i_m} \) are those \( k_i \)'s which are different from zero, then \( n(k_1, \ldots, k_n) = |k_1 \cdots k_{i_m}|^{-1} \). In our applications \( k_1, \ldots, k_n \) will always be integers. Write \( \exp \xi \) for \( e^{2\pi i \xi} \).

Generalized theorem of Erdös and Turan. There are absolute constants \( c_n, n = 1, 2, \ldots \) with the following properties.

Let \( n \geq 1, h \geq 1, \) and let vectors \( (\alpha_{i_1 q}, \ldots, \alpha_{i_n q}) \) be given \( (q = 1, \ldots, h) \). Put
\[
\omega(k_1, \ldots, k_n) = \left| \sum_{q=1}^{h} \exp \left( \sum_{i=1}^{n} \alpha_{i q} k_i \right) \right|.
\]

Let \( I_1, \ldots, I_n \) be intervals of \( C \) of lengths \( k(I_j) = \psi_j \) and put \( \psi = \prod \psi_j \). Write \( N \) for the number of \( q, 1 \leq q \leq h, \) such that simultaneously \( \{\alpha_{i q}\} \in I_j (j = 1, \ldots, n) \).

Let \( m \) be a positive integer. Then
\[
|N - \psi l| \leq c_n \left( h m^{-1} + \sum_{k_1, \ldots, k_n : |k_j| \leq m} \pi(k_1, \ldots, k_n) \omega(k_1, \ldots, k_n) \right).
\]

This theorem is a generalization to \( n \) dimensions of a result of Erdös and Turan [7, Theorem 3]. We shall not give a proof, since the argument in [7] can easily be extended to our situation.

Proof of Theorem 3. Put \( \alpha_{j q} = \alpha_{j q} (j = 1, \ldots, n; q = 1, 2, \ldots) \).
\[
\omega_{\delta}(k_1, \ldots, k_n) = \left| \sum_{q=1}^{h} \exp \left( \sum_{k=1}^{n} k_q \alpha_{q} \right) \right| \leq \left\| k_1 \alpha_1 + \cdots + k_n \alpha_n \right\|^{-1}.
\]

Theorem 3 is an immediate consequence of the generalized Erdös-Turan Theorem with \( m = h \) and the fact that
\[
\sum_{k_1, \ldots, k_n : |k_j| \leq m} \pi(k_1, \ldots, k_n) \left\| k_1 \alpha_1 + \cdots + k_n \alpha_n \right\|^{-1} \leq (\log m)^{n+1+s}
\]
for almost every \( \alpha_1, \ldots, \alpha_n \). This fact follows from Theorem 2.
References


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