ON THE CHARACTERIZATION OF SPECTRAL OPERATORS[1]

BY

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Introduction. This paper is mainly concerned with the characterization problem for bounded spectral operators (see [5]). We restrict the discussion to scalar type operators with real spectrum, which we call pseudo-hermitian operators (p. h.).

The criteria are given in terms of certain properties of the exponential group generated by the operator. The simplest criterion of this kind is for Hilbert space: a bounded linear operator S is p.h. if and only if $\| e^{itS} \| \leqq M < \infty$ for all real t. This statement is false for Banach spaces, and even for reflexive Banach spaces. In the latter case, we give additional necessary conditions on the group e^{itS}, which together with its uniform boundedness are both necessary and sufficient for S to be p.h.

Another useful criterion is the following: a bounded linear operator S on a reflexive Banach space is p.h. if and only if for every f in $L_1(R)$ (R the real line), we have $\| \int_R f(t) e^{itS} dt \| \leqq M \| \hat{f} \|_\infty$, where the norm on the left is the operator norm, $\hat{f}$ is the Fourier transform of f and $\| \cdot \|_\infty$ is the sup norm.

Unlike the general theorems in [5] about the spectrality of an operator, the characterizations obtained in this paper are easily applied to analytically given operators on concrete Banach spaces. An example is discussed in §4. Applications are discussed at the end of §5. §2 deals with an improvement of a theorem of Foguel [7] about the resolutions of the identity of sums and products of commuting spectral operators. §1 contains preliminaries. The characterization problem for spectral operators is discussed in §§3 and 5.

1. **Preliminaries.** The term "operator" is used for bounded linear operators on X into X, where X is some fixed Banach space. We refer to [5] for definitions and properties of spectral operators. If A is a Banach algebra with unit and $x \in A$, then $\sigma(x)$, $\rho(x)$, $r(x)$ and $R(\lambda; x)$ denote respectively the spectrum, the resolvent set, the spectral radius, and the resolvent of x. We write $r^+(x) = \sup(\operatorname{Re} \lambda; \lambda \in \sigma(x))$ and $s^+(x) = \sup(\operatorname{Im} \lambda; \lambda \in \sigma(x))$. Similarly, $r^-(x)$ and $s^-(x)$ are defined with inf instead of sup. Complementation is denoted by a prime. The commutativity of $x, y \in A$ is expressed by the symbol $x \smile y$.

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Definition 1. A *pseudo-hermitian* operator is a spectral operator of scalar type with real spectrum.

The terminology is motivated by the fact that a pseudo-hermitian operator S on a *Hilbert* space is similar to a hermitian operator. This follows from a well-known result of Mackey's [11] stating the existence of a nonsingular operator Q such that the resolution of the identity $E(\cdot)$ of S satisfies the equation $E(\delta) = QF(\delta)Q^{-1}$ for each Borel set δ of the complex plane, where $F(\cdot)$ is a self-adjoint spectral measure.

For arbitrary Banach spaces, the pseudo-hermitian operators play, among the spectral operators of scalar type, the same role as the hermitian operators among the normal operators in Hilbert spaces. This is shown in the following

Theorem 1. *Let S be a spectral operator of scalar type and let $E(\cdot)$ be its resolution of the identity. Then there exists a unique pair (A,B) of pseudo-hermitian operators such that $S = A + iB$ and*

(1) *$A \smile B$ and any operator which commutes with S commutes with A and B.*

(2) $\sigma(A) = \mathrm{Re}\,\sigma(S)$ *and* $\sigma(B) = \mathrm{Im}\,\sigma(S)$.

(3) *$E(\cdot)$ is the product measure $E_A \times E_B$, where E_A and E_B are the spectral measures induced on the real axis by the resolutions of the identity of A and B respectively.*

Proof. Let $\mathscr{B}$ and $\mathscr{B}_r$ denote the Borel fields of the complex plane and of the real line R respectively. We define on $\mathscr{B}_r$ two operator-valued functions E_A and E_B by

$$E_A(\delta) = E(\delta \times R) \text{ and } E_B(\delta) = E(R \times \delta)$$

for each $\delta \in \mathscr{B}_r$.

The following properties are immediate:

(a) E_A and E_B are spectral measures on R;

(b) they are uniformly bounded on $\mathscr{B}_r$, and their uniform bounds are less than or equal to the uniform bound of E;

(c) they are countably additive on $\mathscr{B}_r$ in the strong operator topology;

(d) their support lies in $[r^-(S), r^+(S)]$ and in $[s^-(S), s^+(S)]$ respectively.

Define now

$$A = \int_R \xi dE_A(\xi) \text{ and } B = \int_R \eta dE_B(\eta).$$

A and B are well-defined operators. Since we have

$$(*) \qquad A = \int \mathrm{Re}\,\lambda dE(\lambda) \text{ and } B = \int \mathrm{Im}\,\lambda dE(\lambda)$$

(where the integration is over the complex plane), A and B are the functions of S corresponding to the continuous functions $\mathrm{Re}\,\lambda$ and $\mathrm{Im}\,\lambda$ (respectively) in the operational calculus for scalar operators. Therefore, (1) follows, and furthermore A and B are scalar operators by Lemma 6 in [4]. Using (*), we also have

$$A + iB = \int (\operatorname{Re}\lambda + i\operatorname{Im}\lambda)dE(\lambda) = \int \lambda dE(\lambda) = S.$$

We prove (2) by applying Theorem 16 in [4] (with the function $f(\lambda) = \operatorname{Re}\lambda$). We have

$$\sigma(A) = \bigcap\{\overline{\operatorname{Re}\delta}; \delta \in \mathscr{B}, E(\delta) = I\}.$$

But $E(\sigma(S)) = I$ by Theorem 1 in [4]. Hence

$$\sigma(A) \subset \overline{\operatorname{Re}\sigma(S)} = \operatorname{Re}\sigma(S).$$

Similarly $\sigma(B) \subset \operatorname{Im}\sigma(S)$.

Now, since $A \smile B$, we have $\sigma(S) \subset \sigma(A) + i\sigma(B)$; but $\sigma(A)$ and $\sigma(B)$ are real (see above); therefore $\operatorname{Re}\sigma(S) \subset \sigma(A)$ and $\operatorname{Im}\sigma(S) \subset \sigma(B)$. This completes the proof of (2).

Next, let δ_1 and δ_2 be in $\mathscr{B}_r$. We have:

$$E_A(\delta_1)E_B(\delta_2) = E(\delta_1 \times R)E(R \times \delta_2) = E((\delta_1 \times R) \cap (R \times \delta_2)) = E(\delta_1 \times \delta_2).$$

Thus E is the product measure $E_A \times E_B$ (proving (3)).

To prove the uniqueness, suppose that $S = A_1 + iB_1$, where A_1 and B_1 are pseudo-hermitian operators satisfying (1)–(3). For $\delta \in \mathscr{B}_r$, we obtain from (3):

$$E(\delta \times R) = E_{A_1}(\delta)E_{B_1}(R).$$

Since $\sigma(B_1) \subset R$ by (2), we have $E_{B_1}(R) = I$, and therefore

$$E_{A_1}(\delta) = E(\delta \times R) = E_A(\delta) \text{ for each } \delta \in \mathscr{B}_r.$$

It follows that $A_1 = A$; similarly $B_1 = B$. This proves the uniqueness. Q.E.D.

Let S, A, and B be as in Theorem 1; A and B will be called the *real part* and the *imaginary part* of S (respectively). We also use the notation:

$$A = \operatorname{Re} S, \quad B = \operatorname{Im} S, \quad \bar{S} = A - iB.$$

Obviously, the operators S, $\bar{S}$, $\operatorname{Re} S$, and $\operatorname{Im} S$ commute and are in the second commutant of S. The operator S is pseudo-hermitian if and only if $S = \bar{S}$.

Given two scalar operators S_1 and S_2, the operator $\operatorname{Re}(S_1 + S_2)$ is not necessarily defined, since $S_1 + S_2$ may fail to be spectral, even when $S_1 \smile S_2$ (see [9]). Nevertheless, under sufficiently strong conditions, $\operatorname{Re}(S_1 + S_2)$, $\operatorname{Im}(S_1 + S_2)$, $\operatorname{Re}(S_1S_2)$, etc., are all defined and satisfy the same computational rules as for the complex analog. More precisely, we have (for weakly complete Banach spaces):

THEOREM 2. *Let S_1 and S_2 be two commuting scalar operators. Suppose that* *the Boolean algebra of projections generated by the resolutions of the identity of* S_1 *and* S_2 *is uniformly bounded. Then*

$$\begin{aligned} \operatorname{Re}(S_1+S_2) &= \operatorname{Re} S_1 + \operatorname{Re} S_2; \quad \operatorname{Im}(S_1+S_2) = \operatorname{Im} S_1 + \operatorname{Im} S_2; \\ \overline{S_1+S_2} &= \bar{S}_1 + \bar{S}_2; \quad \operatorname{Re}(S_1 S_2) = \operatorname{Re} S_1 \cdot \operatorname{Re} S_2 - \operatorname{Im} S_1 \cdot \operatorname{Im} S_2; \\ \operatorname{Im}(S_1 S_2) &= \operatorname{Re} S_1 \cdot \operatorname{Im} S_2 + \operatorname{Re} S_2 \cdot \operatorname{Im} S_1 \text{ and } \overline{S_1 S_2} = \overline{S_1} \cdot \overline{S_2}. \end{aligned}$$

Proof. Under the conditions of the theorem, it is well known [7] that $S_1 + S_2$ and $S_1 S_2$ are spectral of scalar type. Therefore, $\operatorname{Re}(S_1 + S_2)$, $\operatorname{Im}(S_1 + S_2)$, etc., are meaningful, according to Theorem 1.

Let $A = \operatorname{Re}(S_1 + S_2)$, $A_i = \operatorname{Re}(S_i)$ $(i = 1,2)$, $B = \operatorname{Im}(S_1 + S_2)$, $B_i = \operatorname{Im} S_i$ $(i = 1,2)$. We have $S_1 + S_2 = A + iB$; $S_k = A_k + iB_k$ $(k = 1,2)$. Thus $A + iB = (A_1 + A_2) + i(B_1 + B_2)$(2) or $A - (A_1 + A_2) = i(B_1 + B_2 - B)$. The spectrum of the operator on the left-hand side is real, while that of the operator on the right-hand side is pure imaginary. Therefore

$$\sigma(A - [A_1 + A_2]) = \sigma(B - [B_1 + B_2]) = \{0\}.$$

Now, by Theorem 17 in [4], the full algebra $\mathscr{A}$ generated by the scalar operators S_1 and S_2 and by their resolution of the identity is equivalent to the algebra $C(\mathscr{M})$ of all continuous complex-valued functions on the maximal ideal space $\mathscr{M}$ of $\mathscr{A}$. Therefore $\mathscr{A}$ is semi-simple, and it follows that $A = A_1 + A_2$ and $B = B_1 + B_2$. The other relations are proved in the same way.

We shall see in the sequel (§3) that the pseudo-hermitian operators can be characterized in a "closed" analytic way. This fact motivates the interest in this special class of spectral operators, since the known characterizations for the whole class of spectral operators (see [5]) are very difficult to apply to concrete problems. Furthermore, by Theorem 8 in [4] and Theorem 1 above, the knowledge of the *class* of pseudo-hermitian operators implies a global knowledge of the *class* of spectral operators. More precisely, the class of spectral operators is the subclass of $\{A + iB + N;\ A, B$ pseudo-hermitian and N generalized nilpotent$\}$, for which A, B, and N commute, and the Boolean algebra of projections generated by the resolutions of the identity of A and B is uniformly bounded.

Before going into the characterization problem for pseudo-hermitian operators we apply Theorems 1 and 2 to improve a result of Foguel's [7].

2. **Convolutive properties of the resolution of the identity**(3). The following theorem was proved by S. R. Foguel [7], Theorem 7, for weakly complete Banach spaces.

Let T_1 and T_2 be two commuting spectral operators on the Banach space X. Suppose that the Boolean algebra of projections generated by their resulotions

(2) The uniqueness claim of Theorem 1 does not apply directly, because we do not know a priori that $A_1 + A_2$ and $B_1 + B_2$ satisfy condition (3) in Theorem 1.

(3) In this section X is a weakly complete Banach space.

of the identity, $E_1(\cdot)$ and $E_2(\cdot)$, is uniformly bounded. Then $T_1 + T_2$ and T_1T_2 are spectral operators, and their resolutions of the identity $G_1(\cdot)$ and $G_2(\cdot)$ satisfy:

$$G_1(\delta)x = \int E_2(\delta - \lambda)dE_1(\lambda)x \tag{1}$$

and

$$G_2(\delta)x = \int E_2(\delta/\lambda)dE_1(\lambda)x \tag{2}$$

for each Borel set δ of the complex plane and for each x in X for which

$$G_i(\partial\delta)x = 0 \qquad (i = 1, 2), \tag{3}$$

where $\partial\delta$ denotes the boundary of δ.

We show (Theorem 3) that the identities (1) and (2) are valid without the restriction (3).

In particular, it then follows that Theorems 1 and 2 in [7] are corollaries of Theorem 3 in this section.

We consider only the case of a sum; the corresponding theorem for a product of spectral operators is proved in an analogous way.

THEOREM 3. *Let T_1 and T_2 be two commuting spectral operators on the Banach space X. Suppose that the Boolean algebra of projections generated by their resolutions of the identity, $E_1(\cdot)$ and $E_2(\cdot)$, is uniformly bounded (thus implying that $T = T_1 + T_2$ is spectral, by Theorem 7 in [7]). Let $E(\cdot)$ be the resolution of the identity of T. Then $E(\cdot)$ is the convolution of $E_1(\cdot)$ and $E_2(\cdot)$ (i.e., $E(\delta) = \int E_2(\delta - \lambda)dE_1(\lambda)$ in the strong operator topology, for each $\delta \in \mathscr{B}$).*

The statement of the corresponding theorem for a product is obtained by replacing the $+$ and $-$ signs in Theorem 3 by "$\cdot$" and ":" respectively.

Proof. The theorem is an immediate corollary of the following lemmas.

LEMMA 1. *Let the hypothesis be as in Theorem 3. Define*

$$F(\delta)x = \int E_2(\delta - \lambda)dE_1(\lambda)x$$

for each $\delta \in \mathscr{B}$ and $x \in X$. Then $F(\cdot)$ is a uniformly bounded spectral measure on $\mathscr{B}$, countably additive in the strong operator topology and commuting with T. Furthermore, the support of the restriction $F/E(\delta)X$ of F to $E(\delta)X$ is in the closure $\bar{\delta}$ of δ (for each $\delta \in \mathscr{B}$).

LEMMA 2. *Let E and F be two spectral measures which are weakly (and hence strongly) countably additive. Suppose that*

$$\text{Support}\,\{F(\cdot)E(\delta)\} \subset \bar{\delta}$$

for each $\delta \in \mathscr{B}$. Then $E = F$.

Proof of Lemma 1. $F(\delta)$ is well defined in the strong operator topology. A direct check shows that it is a spectral measure on $\mathscr{B}$. Since

$$F(\delta)x = \int \int c_{\delta-\lambda}(\xi) dE_2(\xi) dE_1(\lambda)x,$$

where $c_{\delta-\lambda}$ is the characteristic function of $\delta - \lambda$, it follows that $F(\cdot)$ is countably additive in the strong operator topology and uniformly bounded on $\mathscr{B}$ if the product measure $E_2 \times E_1$ is uniformly bounded on $\mathscr{B} \times \mathscr{B}$; but this follows from the assumption that the Boolean algebra of projections generated by $E_1(\cdot)$ and $E_2(\cdot)$ is uniformly bounded. The commutativity of F and T is trivial. Hence, $F(\delta) \smile E(\varepsilon)$ for all $\delta, \varepsilon \in \mathscr{B}$, and $F \mid E(\delta)X$ is meaningful. The statement support $\{F \mid E(\delta)X\} \subset \bar{\delta}$ will now be proved.

Let S, S_k be the scalar parts of T, T_k respectively $(k = 1,2)$. For each $\delta \in \mathscr{B}$, we denote:

$$X_\delta = E(\delta)X; \quad S_\delta = S \mid E(\delta)X; \quad S_{k\delta} = S_k \mid E(\delta)X.$$

Since $T_1 \smile T_2$, also $S_k \smile S$ and therefore $S_k \smile E(\delta)$. Hence S_δ and $S_{k\delta}$ are well-defined elements of $B(X_\delta)$, and $S_\delta = S_{1\delta} + S_{2\delta}$. These operators are spectral, and their resolution of the identity are the restrictions $E_\delta(\cdot)$ and $E_{k\delta}(\cdot)$ of $E(\cdot)$ and $E_k(\cdot)$ to X_δ.

By the definition of spectral operators, $\sigma(S_\delta) \subset \bar{\delta}$. Therefore, if Γ_δ is a finite union of simple Jordan contours which contains $\bar{\delta}$ in its interior, then

$$\frac{1}{2\pi i} \int_{\Gamma_\delta} R(\mu; S_\delta)\, d\mu = I_\delta, \tag{1}$$

where I_δ is the restriction of the identity operator I to X_δ. It follows that

$$E(\delta) = \frac{1}{2\pi i} \int_{\Gamma_\delta} R(\mu; S_\delta) d\mu \cdot E(\delta) \quad (\text{for all } \delta \in \mathscr{B}), \tag{2}$$

the equality being nowbetween elements of $B(X)$.

Since $S_\delta = S_{1\delta} + S_{2\delta}$ where $S_{1\delta} \smile S_{2\delta}$, we have:

$$R(\mu; S_\delta) = \int_{\sigma(S_{1\delta})} R(\mu - \lambda; S_{2\delta}) dE_{1\delta}(\lambda), \tag{3}$$

for $\mu \notin \sigma(S_{1\delta}) + \sigma(S_{2\delta})$ (see remark following Corollary 7 in [10]).

Case 1. *S_1 and S_2 are pseudo-hermitian.* In this case, S, S_δ and $S_{k\delta}$ $(k = 1,2)$ have real spectrum, and, therefore, (3) is valid for all nonreal μ.

Let δ be an interval (open, closed or half-closed) on R: say $\delta = (\xi_1, \xi_2)$. Then, for any $\varepsilon > 0$, Γ_δ may be chosen as the rectangle with vertices $\xi_2 + \varepsilon + i\varepsilon$, $\xi_1 - \varepsilon + i\varepsilon$, $\xi_1 - \varepsilon - i\varepsilon$, and $\xi_2 + \varepsilon - i\varepsilon$ (in this order). The representation (3) is valid for all the points on Γ_δ, except perhaps for $\xi_1 - \varepsilon$ and $\xi_2 + \varepsilon$. Denote by Γ_δ^* the open contour $\Gamma_\delta^* = \Gamma_\delta - \{\xi_1 - \varepsilon, \xi_2 + \varepsilon\}$. Since $R(\mu; S_\delta)$ is continuous on Γ_δ, we have:

$$\frac{1}{2\pi i}\int_{\Gamma_\delta} R(\mu;S_\delta)d\mu = \frac{1}{2\pi i}\int_{\Gamma_\delta^*} R(\mu;S_\delta)d\mu$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\delta^*}\int_{\sigma(S_{1\delta})} R(\mu-\lambda;S_{2\delta})dE_{1\delta}(\lambda)d\mu$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\delta^*}\int_{\sigma(S_{1\delta})}\int_{\sigma(S_{2\delta})} \frac{1}{\mu-\lambda-\nu}dE_{2\delta}(\nu)dE_{1\delta}(\lambda)d\mu$$

$$= \int_{\sigma(S_{1\delta})}\left\{\int_{\sigma(S_{2\delta})}\left(\frac{1}{2\pi i}\int_{\Gamma_\delta^*}\frac{d\mu}{\mu-(\lambda+\nu)}\right)dE_{2\delta}(\nu)\right\}dE_{1\delta}(\lambda).$$

But

$$\frac{1}{2\pi i}\int_{\Gamma_\delta^*}\frac{d\mu}{\mu-(\lambda+\nu)} = \begin{cases} 1 & \text{if } \lambda+\nu \in (\xi_1-\varepsilon,\ \xi_2+\varepsilon), \\ \frac{1}{2} & \text{if } \lambda+\nu = \xi_1-\varepsilon \text{ or } \xi_2+\varepsilon, \\ 0 & \text{if } \lambda+\nu \notin [\xi_1-\varepsilon,\ \xi_2+\varepsilon]. \end{cases}$$

Hence

$$\text{(4)} \qquad \frac{1}{2\pi i}\int_{\Gamma_\delta} R(\mu;S_\delta)d\mu = \int_{\sigma(S_{1\delta})} E_{2\delta}((\xi_1-\varepsilon,\xi_2+\varepsilon)-\lambda)dE_{1\delta}(\lambda) + Q$$

where

$$\text{(5)} \qquad Q = \frac{1}{2}\int_{\sigma(S_{1\delta})}[E_{2\delta}(\{\xi_1-\varepsilon-\lambda\}) + E_{2\delta}(\{\xi_2+\varepsilon-\lambda\})]dE_{1\delta}(\lambda)$$

(the integrals are understood in the strong operator topology). The first operator on the right of (4) is a projection G in X_δ (this is shown by checking directly that $G^2 = G$). Using (1), $Q = I_\delta - G$; hence Q is a projection in X_δ, i.e., $Q^2 = Q$. Starting from (5), we compute directly Q^2, and we obtain $Q^2 = \frac{1}{2}Q$. Hence $Q = 0$ on X_δ, and we conclude from (2) and (4) that

$$\text{(6)} \qquad E(\delta)x = \int_{\sigma(S_{1\delta})} E_{2\delta}((\xi_1-\varepsilon,\xi_2+\varepsilon)-\lambda)dE_{1\delta}(\lambda)\,E(\delta)x$$

for $\delta = (\xi_1,\xi_2)$ or $(\xi_1,\xi_2]$, etc., and $x \in X$. Equivalently,

$$\text{(7)} \qquad E(\delta)x = \int_R E_2((\xi_1-\varepsilon,\xi_2+\varepsilon)-\lambda)dE_1(\lambda)\,E(\delta)x.$$

Case 2. *The general case.* By Theorem 1, we have $S = A + iB$ and $S_k = A_k + iB_k$ $(k = 1,2)$, where A, A_k, B and B_k are commuting pseudo-hermitian operators. By Theorem 2, $A = A_1 + A_2$ and $B = B_1 + B_2$. Let $\delta = (\xi_1,\xi_2)$ or $(\xi_1,\xi_2]$, etc., $\sigma = (\eta_1,\eta_2)$ or $(\eta_1,\eta_2]$, etc., $\varepsilon > 0$ and $\varepsilon_1 > 0$.

Then, by commutativity and (3) in Theorem 1, we obtain from (7):

$$\begin{aligned}E(\delta \times \sigma)x &= E_A(\delta)E_B(\sigma)x \\ &= \int\int_{R\times R} E_{A_2}((\xi_1 - \varepsilon, \xi_2 + \varepsilon) - \xi)E_{B_2}((\eta_1 - \varepsilon_1, \eta_2 + \varepsilon_1) - \eta) \\ &\qquad \cdot d(E_{A_1} \times E_{B_1})(\xi,\eta) \cdot E(\delta \times \sigma)x \\ &= \int E_2((\delta \times \sigma)^0 - \lambda)dE_1(\lambda)\cdot E(\delta \times \sigma)x,\end{aligned}$$

where $(\delta \times \sigma)^0$ is any open rectangle including $\overline{\delta \times \sigma}$. The integration is on any Borel set including $\sigma(S_1)$. Equivalently, we have proved that for any rectangle δ, and any open rectangle δ^0 including $\bar{\delta}$, the following relation holds:

$$E(\delta) = F(\delta^0)E(\delta). \tag{8}$$

Now, if $\lambda \notin \bar{\delta}$, let $a = \operatorname{dist}\{\lambda; \bar{\delta}\}$ $(a \neq 0)$, and let V be the open square with center λ and diameter a. Then V is included in $\delta_1^0 - \delta_2^0$, where δ_1^0 and δ_2^0 are the open rectangles concentric with δ, with diameters $\operatorname{diam}(\delta) + 2a$ and $\operatorname{diam}(\delta) + a/4$ respectively. By (8), $E(\delta) = F(\delta_1^0)E(\delta) = F(\delta_2^0)E(\delta)$, and therefore, $F(\delta_1^0 - \delta_2^0)E(\delta) = 0$; hence, also $F(V)E(\delta) = 0$. This shows that λ is *not* in the support of $F(\cdot)E(\delta)$, proving that

$$\operatorname{support}\{F(\cdot)E(\delta)\} \subset \bar{\delta} \tag{9}$$

for any *rectangle* δ.

Next, let δ be any Borel set in the plane. If $\lambda \notin \bar{\delta}$, we may choose an open square V centered at λ and a covering of δ by disjoint *rectangles* $\delta_k (k = 1, 2, \cdots, n)$ such that (i) $\delta \subset \bigcup_{k=1}^n \delta_k$ and (ii) V and $\bigcup_{k=1}^n \delta_k$ are *separated.*

We obtain:

$$F(V)E(\delta) = F(V)E(\delta)\cdot F(V)E\left(\bigcup_{k=1}^n \delta_k\right) = F(V)E(\delta)\cdot \sum_{k=1}^n F(V)E(\delta_k).$$

The rectangles V and δ_k $(k = 1, 2, \cdots, n)$ are separated; hence, by (9), $F(V)E(\delta_k) = 0$, and we conclude that $F(V)E(\delta) = 0$. Thus λ is not in the support of $F(\cdot)E(\delta)$. This proves that (9) *is valid for* any Borel set δ in $\mathscr{B}$. Equivalently, support $\{F \mid E(\delta)X\} \subset \bar{\delta}$. Q.E.D.

Proof of Lemma 2. By the assumption on support,

$$F(\delta_1)E(\delta_2) = 0 \tag{10}$$

whenever $\delta_1, \delta_2 \in \mathscr{B}$ are separated (i.e., $\bar{\delta}_1$ and $\bar{\delta}_2$ lie in two disjoint open sets).

Let $\delta \in \mathscr{B}$ and $\delta_n = \delta + 2^{-n}\varepsilon$ where ε is the open unit disk. We have $\delta_{n+1} \subset \delta_n$ and $\lim_n \delta_n = \bigcap_{n=1}^\infty \delta_n = \bar{\delta}$.

Now each δ_n' is separated from δ, so that, by (10), $F(\delta)E(\delta_n') = 0$. Since E is countably additive in the strong operator topology, we obtain:

$$F(\delta)E((\bar{\delta})')x = F(\delta)E(\lim_n \delta_n')x = \lim_n F(\delta)E(\delta_n')x = 0 \text{ for all } x \in X.$$

Thus $F(\delta)E((\bar{\delta})')=0$, so that

$$F(\delta)E(\delta')=F(\delta)E(\partial\delta\cap\delta') \tag{11}$$

(where $\partial\delta$ denotes the boundary of δ; $\delta\in\mathscr{B}$). Next, let $\delta_n=\{\lambda\in\delta;\operatorname{dist}\{\lambda;\delta'\}\geqq 2^{-n}\}$. We have: $\lim_n\delta_n=\bigcup_{n=1}^{\infty}\delta_n=$ interior (δ). Again, δ_n and δ' are separated, so that $F(\delta')E(\delta_n)=0$ by (10), and therefore

$$F(\delta')E(\operatorname{int}\delta)x=F(\delta')E(\lim_n\delta_n)x=\lim_n F(\delta')\cdot E(\delta_n)x=0.$$

Hence

$$F(\delta')E(\delta)=F(\delta')E(\partial\delta\cap\delta). \tag{12}$$

Now, we use the evident identity:

$$F(\delta)-E(\delta)=F(\delta)E(\delta')-F(\delta')E(\delta)\qquad(\text{for all }\delta\in\mathscr{B}). \tag{13}$$

We obtain, by (11), (12), and (13),

$$F(\delta)-E(\delta)=F(\delta)E(\partial\delta\cap\delta')-F(\delta')E(\partial\delta\cap\delta). \tag{14}$$

By the same argument as above, we reduce (14) to:

$$F(\delta)-E(\delta)=F(\partial\delta\cap\delta)E(\partial\delta\cap\delta')-F(\partial\delta\cap\delta')E(\partial\delta\cap\delta). \tag{15}$$

Now, if δ is open, then $\partial\delta\cap\delta=\varnothing$, and therefore, (15) implies $F(\delta)=E(\delta)$. If δ is closed, then $\partial\delta\cap\delta'=\varnothing$, and we obtain again $F(\delta)=E(\delta)$. By strong σ-additivity, the equality extends to any Borel set of the complex plane. Q.E.D.

3. **Characterization of pseudo-hermitian operators.** If S is a pseudo-hermitian operator, then $e^{-2\pi i\xi S}$(ξ real) is the Fourier-Stieltjes transform of the resolution of the identity E of S (considered as an operator-valued measure on the Borel sets of the real line). Formally, E is thus the inverse Fourier-Stieltjes transform (IFST) of $e^{-2\pi i\xi S}$. But this operator-valued function is defined for any bounded linear operator S, by means of the operational calculus for *analytic* functions. This suggests the possibility of characterizing the pseudo-hermitian operators as those operators for which the IFST of $e^{-2\pi i\xi S}$ *exists in a certain sense*. This is the underlying idea in the following discussion. The connection thus obtained between spectral theory and Fourier analysis shows the importance of the specialization to the study of pseudo-hermitian operators.

The considerations above are made rigorous by means of Theorem 2.1.3 in [2].

Let X be a *reflexive* Banach space; let $B(X)$ be the Banach algebra of bounded linear operators of X into X. We denote by $C(R)$ the space of all bounded continuous complex-valued functions on the real line R, normed by the supremum of the absolute value of the functions.

For $N>0$, $\varepsilon\geqq 0$, $\eta\in R$ and $S\in B(X)$, define:

$$G_N(\eta;\varepsilon) = \int_{R_1} \exp\left\{-\left[\left(\frac{\xi}{N}\right)^2 + 2\varepsilon|\xi| - 2i\xi\eta\right]\pi\right\} e^{-2\pi i\xi S} d\xi.$$

This is a well-defined $B(X)$-valued function of η; we consider N and ε as parameters varying in the ranges indicated above. The operator S is fixed in our discussion.

THEOREM 4. *S is a pseudo-hermitian if and only if the conditions* (a)–(c) *hold.*

(a) $\| e^{-2\pi i\xi S} \| \leqq M < \infty$ *for all* ξ *in* R.

(b) *For each* $\gamma \in C(R)$ *and* $\varepsilon \geqq 0$, $\int_R \gamma(\eta)\, G_N(\eta;\varepsilon)d\eta$ *converges* (*when* $N \to \infty$) *in the weak operator topology to an element* $B(\gamma;\,\varepsilon)$ *of* $B(X)$, *and* $\| B(\gamma;\varepsilon) \| \leqq K \| \gamma \|$ ($K > 0$ *may depend on* ε, *but not on* γ).

(c) *As a function of* ε ($\varepsilon \geqq 0$), $B(e^{-2\pi i v\xi};\varepsilon)$ *is continuous from the right at* $\varepsilon = 0$, *in the weak operator topology* (*for each real* v).

The present section is devoted to the proof of Theorem 4.

We first consider Fourier transforms of functions in $L_2 \otimes A$, where A is now an arbitrary Banach space. $L_2 \otimes A$ denotes the Banach space of (the equivalence classes of)(4) A-valued strongly measurable functions $f(\xi)$, $\xi \in R$, for which $\int_R \|f(\xi)\|^2 d\xi < \infty$, with the norm $\|f\| = (\int_R \|f(\xi)\|^2 d\xi)^{1/2}$.

On the other hand, we consider the integral

$$F(\eta) = \int_R \frac{e^{-2\pi i\xi\eta} - 1}{-2\pi i\xi} f(\xi)d\xi. \tag{1}$$

It converges in the strong topology of A, and defines an A-valued function. Suppose now that $F(\eta)$ is a.e. weakly differentiable, and let $f^{\wedge}(\eta)$ be its weak derivative (defined a.e.).

We have:

$$x^*(f^{\wedge}(\eta)) = \frac{d}{d\eta} x^*(f(\eta)) = \frac{d}{d\eta} \int_R \frac{e^{-2\pi i\xi\eta} - 1}{-2\pi i\xi} x^*(f(\xi))d\xi$$

for a.e. η in R.

But the last expression is equal (a.e.) to $x^*(f)^{\wedge}(\eta)$ (see [16, p. 250]). Thus:

$$x^*(f^{\wedge}(\eta)) = x^*(f)^{\wedge}(\eta) \qquad \text{a.e.} \tag{2}$$

This motivates the following

DEFINITION 2. Let $f \in L_2 \otimes A$. We say that f has an L_2-Fourier transform if there exists an A-valued function $f^{\wedge}(\eta)$ (defined a.e. on R) such that equation (2) is satisfied for all x^* in a determining set $\Gamma \subset A^*$ (see [8, p.34]) and a.e. η on R.

The L_2-Fourier transform of f is the (uniquely determined a.e.) A-valued function $f^{\wedge}$, whenever it exists.

(4) Two functions are in the same equivalence class if they differ only on a set of Lebesgue measure 0.

The difficulty involved in the generalization of the concept of L_2-Fourier transforms to functions in $L_2 \otimes A$ is related with the fact that the Parseval and Plancherel theorems are not valid in general when A is not the complex field (see remark following Lemma 4 below).

We do not intend to go deeply into such problems. Definition 2 is a "working definition." Neither do we try to prove general "existence theorem" for L_2-Fourier transforms of functions in $L_2 \otimes A$. We just note that the discussion preceding Definition 2 shows that the weak differentiablity a.e. of $F(\eta)$ (defined by Equation (1)) is a sufficient condition for the existence of the L_2-Fourier transform of f. Furthermore, in this case, $f\,\hat{}$ is a.e. equal to the weak derivative of F. In particular, if A is weakly complete, we conclude from the differentiability of $x^*(F(\eta))$ (see [16, pp.248–250]) that $F(\eta)$ is weakly differentiable and hence that every function in $L_2 \otimes A$ has an L_2-Fourier transform, which is given by the weak derivative of F (a.e.).

The lemma below gives another sufficient condition for the existence of L_2-Fourier transforms of functions in $L_2 \otimes A$ (this is all that we need in the sequel).

LEMMA 3. *Let $f \in L_2 \otimes A$, and suppose that the integral*

$$\int_{-\omega}^{\omega} e^{-2\pi i\xi\eta} f(\xi)\,d\xi \qquad (\omega > 0)$$

converges weakly to an element of A for a.e. $\eta \in R$, when $\omega \to \infty$. Then f has an L_2-Fourier transform, which is equal a.e. to the weak limit of the integral above.

The proof is straightforward and is therefore omitted.

LEMMA 4. *Let A be a Banach algebra with unit. Let $x \in A$ be such that either $s^+(x) < 0$ or $s^-(x) > 0$. Then $(1/2\pi i)R(\xi;x)$ is in $L_2 \otimes A$, its L_2-Fourier transform exists and is given a.e. by*

$$\begin{cases} 0 & \text{for } \eta < 0 \\ -e^{-2\pi i\eta x} & \text{for } \eta \geqq 0 \end{cases} \qquad \text{if } s^+(x) < 0$$

and by

$$\begin{cases} e^{-2\pi i\eta x} & \text{for } \eta < 0 \\ 0 & \text{for } \eta \geqq 0 \end{cases} \qquad \text{if } s^-(x) > 0.$$

Proof. If either $s^+(x) < 0$ or $s^-(x) > 0$, the real line lies in $\rho(x)$. Therefore $(1/2\pi i)R(\xi; x)$ is continuous on R and its norm is $O(|\xi|^{-1})$ for $|\xi| \to \infty$; hence this function is in $L_2 \otimes A$.

For $\omega > 0$, let Γ_ω^+ be the boundary of the rectangle with vertices ω, $\omega + i\omega$, $-\omega + i\omega$ and $-\omega$ (in this order), with the base omitted. Let Γ_ω^- be the reflection of Γ_ω^+ in the real axis, with positive direction.

If $s^+(x) < 0$, we may choose ω large enough (say $\omega > \omega_0$) so that $\sigma(x)$ is in the interior of the contour $\Gamma_\omega^- \cup [-\omega,\omega]$. By Cauchy's Theorem and the operational calculus in A, we have:

$$\int_{-\omega}^{\omega} e^{-2\pi i\xi\eta}\cdot\frac{1}{2\pi i}R(\xi;x)d\xi = \frac{1}{2\pi i}\int_{\Gamma_\omega^+} e^{-2\pi i\lambda\eta}R(\lambda;x)d\lambda \qquad (0<\omega<\infty)$$

$$= -e^{-2\pi i\eta x} + \frac{1}{2\pi i}\int_{\Gamma_\omega^-} e^{-2\pi i\lambda\eta}R(\lambda;x)d\lambda \qquad (\omega_0<\omega).$$

Estimating the integrals over Γ_ω^+ and Γ_ω^-, we find that their strong limit for $\omega\to\infty$ is 0 for $\eta<0$ and $\eta>0$ respectively.

It follows that the integral $\int_{-\omega}^{\omega} e^{-2\pi i\xi\eta}\cdot(1/2\pi i)R(\xi;x)d\xi$ converges strongly (for $\omega\to\infty$) to 0 for $\eta<0$ and to $-e^{-2\pi i\mu x}$ for $\eta>0$. Using Lemma 3 we get the first part of Lemma 4; the second part is proved in the same way.

REMARKS. (1) If either $r^+(x)<0$ or $r^-(x)>0$, a similar result holds for the function $(1/2\pi)R(i\xi;x)$. Its L_2-Fourier transform exists, and is a.e. given by

$$\begin{cases} e^{-2\pi i\eta x} & \eta<0 \\ 0 & \eta\geqq 0. \end{cases} \quad \text{if } r^+(x)<0.$$

An analogous formula is true for $r^-(x)>0$.

(2) The Parseval equality is not valid in general in $L_2\otimes A$. Consider for example the Banach algebra of 2×2 matrices over the complex field, with norm

$$\|(x_{ij})\| = \sum_{i,j}|x_{ij}|.$$

Let

$$x = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix},$$

so that $r^+(x)<0$. Taking $f(\xi)=(1/2\pi)R(i\xi;x)$, we have $\|f(\xi)\| = (9/4+4/\pi)^{1/2}$, while

$$\|f^\wedge\| = \left(\frac{13}{4}\right)^{1/2} \neq \|f\|$$

(the norms are the $L_2\otimes A$ norms).

LEMMA 5. *Condition* (b) *holds, for* $\varepsilon>0$, *for any* $S\in B(X)$ *with real spectrum.*

Proof. Given $\varepsilon>0$ and $S\in B(X)$ with real spectrum, we define:

$$(1) \qquad F_\varepsilon(\xi) = \frac{1}{2\pi i}\{R(\xi;S+i\varepsilon I)-R(\xi;S-i\varepsilon I)\} \qquad (\xi\in R).$$

Since $\sigma(S\pm i\varepsilon I)=\sigma(S)\pm i\varepsilon$ and $\sigma(S)$ is real, the real line is in the resolvent set of $S\pm i\varepsilon I$, and therefore $F_\varepsilon(\xi)$ is a well-defined $B(X)$-valued function. It is obviously in $L_2 \otimes B(X)$ and according to Lemma 4, its L_2-Fourier transform exists and is given a.e. by:

$$F_\varepsilon^\wedge(\eta) = \begin{cases} e^{-2\pi i\eta(S+i\varepsilon I)} & \text{for } \eta < 0, \\ e^{-2\pi i\eta(S-i\varepsilon I)} & \text{for } \eta \geqq 0. \end{cases}$$

(Notice that $s^-(S + i\varepsilon I) = \varepsilon < 0$ and $s^+(S - i\varepsilon I) = -\varepsilon < 0$.) Equivalently,

(2) $$F_\varepsilon^\wedge(\eta) = e^{-2\pi\varepsilon|\eta|} \cdot e^{-2\pi i\eta S}.$$

Now, $F_\varepsilon(\xi) = \varepsilon/\pi R(\xi - i\varepsilon; S) R(\xi + i\varepsilon; S)$, by the "First Resolvent Equation" [8, p. 126].

Therefore $\| F_\varepsilon(\xi) \|$ is $O(\xi^{-2})$ for $|\xi| \to \infty$, and it follows that $F_\varepsilon(\xi)$ is in $L_1 \otimes B(X)$ (as well as in $L_2 \otimes B(X)$). Thus, for $x \in X$, $x^* \in X^*$, $x^*F_\varepsilon(\xi)x \in L_1(R) \cap L_2(R)$, and therefore, the L_2-Fourier transform of $x^*F_\varepsilon(\xi)x$ coincides a.e. with the usual L_1-Fourier transform. According to Definition 2, we conclude that $e^{-2\pi\varepsilon|\eta|} \cdot x^*e^{-2\pi i\eta S}x$ is the usual L_1-Fourier transform of $x^*F_\varepsilon(\xi)x \in L_1(R)$.

By Theorem 2.1.3 in [2], it follows that the integral

$$\int_R \exp\left(-\pi\left[\left(\frac{\xi}{N}\right)^2 - 2i\xi\eta\right]\right) \cdot (e^{-2\pi\varepsilon|\xi|} \cdot x^*e^{-2\pi i\xi S}x)\,d\xi$$

converges in L_1-norm to $x^*F_\varepsilon(\eta)x$ when $N \to \infty$.

Equivalently:

(3) $$\lim_{N\to\infty} x^*G_N(\eta;\varepsilon)x = x^*F_\varepsilon(\eta)x \text{ in } L_1\text{-norm}.$$

Hence:

(4) $$\lim_{N\to\infty} \int_R \gamma(\eta) G_N(\eta;\varepsilon)\,d\eta = \int_R \gamma(\eta) F_\varepsilon(\eta)\,d\eta$$

in the weak operator topology, for each $\gamma \in C(R)$ and $\varepsilon > 0$. We write

(5) $$B(\gamma;\varepsilon) = \int_R \gamma(\eta) F_\varepsilon(\eta)\,d\eta.$$

We have $B(\gamma;\varepsilon) \in B(X)$, and $\| B(\gamma;\varepsilon) \| \leqq \| F_\varepsilon \|_1 \cdot \| \gamma \|$, where $\| F_\varepsilon \|_1$ is the $L_1 \otimes B(X)$ norm of F_ε (i.e., $\int_R \| F_\varepsilon(\eta) \| \,d\eta$), which is nite for $\varepsilon > 0$. Q.E.D.

REMARK. Let S be an operator with real spectrum. Let $\varepsilon > 0$. Now, for v real, take $\gamma(\xi) = e^{-2\pi i v\xi}$ in (5). We obtain from (2) that

$$B(e^{-2\pi i v\xi};\, \varepsilon) = F_\varepsilon^\wedge(v) = e^{-2\pi\varepsilon|v|} \cdot e^{-2\pi i v S}.$$

Therefore:

$$\lim_{\varepsilon\to 0+} B(e^{-2\pi i v\xi};\, \varepsilon) = e^{-2\pi i v S}$$

in the uniform operator topology. We conclude that if condition (b) holds, then condition (c) holds *if and only if* $B(e^{-2\pi i v\xi};\ 0) = e^{-2\pi i vS}$.

In the next lemma, we give formulas for the computation of $s^+(x)$ and $s^-(x)$ for an element x of a Banach algebra.

LEMMA 6. *Let A be a Banach algebra with unit e. Let $x \in A$. Then*:

$$s^+(x) = \lim_{\eta \to \infty} \frac{\log \| e^{-2\pi i \eta x} \|}{2\pi\eta}$$

and

$$s^-(x) = \lim_{\eta \to -\infty} \frac{\log \| e^{-2\pi i \eta x} \|}{2\pi\eta}$$

meaning that the limits exist, and their values are $s^+(x)$ and $s^-(x)$ when $\eta \to \infty$ and $\eta \to -\infty$ respectively(5).

Proof. For $\varepsilon > 0$ and $\xi \in R$, let

$$H_\varepsilon(\xi) = \frac{1}{2\pi i}\{R(\xi; x - i[s^-(x) - \varepsilon]e) - R(\xi; x - i[s^+(x) + \varepsilon]e)\}.$$

Since $s^-(x - i[s^-(x) - \varepsilon]e) = \varepsilon > 0$ and $s^+(x - i[s^+(x) + \varepsilon]e) = -\varepsilon < 0$, we may apply Lemma 4 and conclude that $H_\varepsilon(\xi)$ is in $L_2 \otimes A$ and has an L_2-Fourier transform, which is given a.e. by

$$H_\varepsilon^\wedge(\eta) = e^{-2\pi i \eta x} \cdot \phi_\varepsilon(\eta) \tag{1}$$

where

$$\phi_\varepsilon(\eta) = \begin{cases} e^{-2\pi\eta(s^-(x)-\varepsilon)}, & \eta < 0, \\ e^{-2\pi\eta(\ {}^+(x)+\varepsilon)}, & \eta \geqq 0. \end{cases} \tag{2}$$

Now, as in the proof of Lemma 5, we check, by means of the "First Resolvent Equation," that $H_\varepsilon(\xi)$ is also in $L_1 \otimes A$, and it follows that

$$e^{-2\pi i \eta x} \cdot \phi_\varepsilon(\eta) = \int_R e^{-2\pi i \eta \xi} H_\varepsilon(\xi) d\xi;$$

hence

$$\| e^{-2\pi i \eta x} \| \cdot \phi_\varepsilon(\eta) \leqq \int_R \| H_\varepsilon(\xi) \| d\xi = K < \infty.$$

Taking logarithms and dividing by $2\pi\eta$, we obtain (using (2)):

(i) *For $\eta < 0$*:

$$\frac{\log \| e^{-2\pi i \eta x} \|}{2\pi\eta} \geqq s^-(x) - \varepsilon + \frac{\log K}{2\pi\eta};$$

hence

(5) Equivalently $s^+(x) = \log r(e^{-ix})$ and $s^-(x) = -\log r(e^{ix})$.

$$\liminf_{\eta\to-\infty} \frac{\log \| e^{-2\pi i\eta x}\|}{2\pi\eta} \geqq s^-(x) - \varepsilon.$$

This being true for each $\varepsilon > 0$, we obtain

$$\liminf_{\eta\to-\infty} \frac{\log \| e^{-2\pi i\eta x}\|}{2\pi\eta} \geqq s^-(x). \tag{3}$$

(ii) *For* $\eta > 0$: we obtain by the same method:

$$\limsup_{\eta\to\infty} \frac{\log \| e^{-2\pi i\eta x}\|}{2\pi\eta} \leqq s^+(x). \tag{4}$$

Now, for λ varying in $\sigma(x)$, $e^{-2\pi i\eta\lambda}$ varies in $\sigma(e^{-2\pi i\eta x})$ (by the Spectral Mapping Theorem), and therefore $|e^{-2\pi i\eta\lambda}| \leqq \|e^{-2\pi i\eta x}\|$; hence:

$$2\pi\eta\cdot \operatorname{Im}\lambda \leqq \log \| e^{-2\pi i\eta x}\|. \tag{5}$$

Thus, for $\eta < 0$, we have $\operatorname{Im}\lambda \geqq \log \| e^{-2\pi i\eta x}\|/2\pi\eta$; this being true for each $\lambda \in \sigma(x)$ and each $\eta < 0$, we obtain:

$$s^-(x) = \inf_{\lambda\in\sigma(x)} \operatorname{Im}\lambda \geqq \limsup_{\eta\to-\infty} \frac{\log \| e^{-2\pi i\eta x}\|}{2\pi\eta} \geqq \liminf_{\eta\to-\infty} \frac{\log \| e^{-2\pi i\eta x}\|}{2\pi\eta} \geqq s^-(x) \quad \text{(by(3))}.$$

Thus we have equality everywhere, proving the existence of

$$\lim_{\eta\to-\infty} \frac{\log \| e^{-2\pi i\eta x}\|}{2\pi\eta}$$

and showing that its value is $s^-(x)$. Similarly, we get from (5) and (4) (for $\eta > 0$):

$$s^+(x) = \sup_{\lambda\in\sigma(x)} \operatorname{Im}\lambda \leqq \liminf_{\eta\to\infty} \frac{\log \| e^{-2\pi i\eta x}\|}{2\pi\eta} \leqq \limsup_{\eta\to\infty} \frac{\log \| e^{-2\pi i\eta x}\|}{2\pi\eta} \leqq s^+(x).$$

Hence the limit exists and $\lim_{\eta\to\infty} \log \| e^{-2\pi i\eta x}\|/2\pi\eta = s^+(x)$. Q.E.D

REMARK. Since $r^+(x) = s^+(ix)$ and $r^-(x) = s^-(ix)$, it follows from Lemma 6 that the limits of $\log \| e^{-2\pi i\eta x}\|/2\pi\eta$, when $\eta\to\infty$ or $\eta\to-\infty$, exist, and are equal to $r^+(x)$ and $r^-(x)$ respectively.

COROLLARY 1. *$\sigma(x)$ is real if and only if*

$$\liminf_{|\eta|\to\infty} \frac{\log \| e^{-2\pi i\eta x}\|}{2\pi\eta} = 0\text{(6)}.$$

Proof. $\sigma(x)$ is real if and only if $s^+(x) = s^-(x) = 0$.

COROLLARY 2. *If $\| e^{-2\pi i\eta x}\| \leqq M < \infty$ for all $\eta \in R$, then $\sigma(x)$ is real.*

Proof. Apply Corollary 1.

(6) Equivalently, $\sigma(x)$ is real if and only if $r(e^{ix}) = r(e^{-ix}) = 1$.

LEMMA 7. *Let X be a reflexive Banach space, and suppose that $S \in B(X)$ satisfies condition* (b) *for $\varepsilon = 0$. Then there exists a unique uniformly bounded operator-valued function $F(\cdot)$ on the Borel sets of the real line such that*:

(i) *$F(\cdot)$ is finitely additive*;

(ii) *$F(\cdot)x$ is countably additive for each x in X*; *and*

(iii) *$B(\gamma;0)x = \int_R \gamma(\xi)\,dF(\xi)x$ for each x in X.*

Proof. We notice first that $G_N(\eta;\varepsilon)$ is meaningful for any $\varepsilon \geqq 0$. Indeed, it follows from Lemma 6 that, given $\delta > 0$, there exists $\xi_0 = \xi_0(\delta) > 0$ such that

$$e^{2\pi(s^+(S)-\delta)\xi} < \| e^{-2\pi i\xi S} \| < e^{2\pi(s^+(S)+\delta)\xi} \quad \text{for } \xi > \xi_0$$

and

$$e^{2\pi(s^-(S)+\delta)\xi} < \| e^{-2\pi i\xi S} \| < e^{2\pi(s^-(S)-\delta)\xi} \quad \text{for } \xi < -\xi_0.$$

It follows that the integral defining $G_N(\eta;\varepsilon)$ converges in the uniform operator topology.

Next, the integral $\int_R \gamma(\eta)\, G_N(\eta;\varepsilon)d\eta$ exists (in the uniform operator topology) for each $\gamma \in C(R)$.

Therefore

$$x^*\int_R \gamma(\eta)G_N(\eta;\varepsilon)d\eta x = \int_R \gamma(\eta)x^*G_N(\eta;\varepsilon)xd\eta$$

for each $x \in X$ and $x^* \in X^*$. Hence

$$x^*B(\gamma;0)x = \lim_{N\to\infty}\int_R \gamma(\eta)\,x^*G_N(\eta;0)\,xd\eta.$$

It follows that

$$x^*B(\alpha_1\gamma_1 + \alpha_2\gamma_2;\ 0)x = \alpha_1 x^*B(\gamma_1;\ 0)x + \alpha_2 x^*B(\gamma_2;\ 0)x$$

for all $\gamma_1, \gamma_2 \in C(R)$ and all α_1, α_2 complex numbers. Condition (b) implies also:

$$|x^*B(\gamma;0)x| \leqq (K\|x\|\,\|x^*\|)\,\|\gamma\|. \tag{1}$$

We conclude that $x^*B(\gamma;0)x$ is a bounded linear functional on $C(R)$. By the Riesz Representation Theorem, there exists a unique finite regular complex Borel measure $\mu(\cdot\,;x,x^*)$ on the Borel field $\mathscr{B}_r$ of R such that

$$x^*B(\gamma;0)x = \int_R \gamma(\eta)\,d\mu(\eta;x,x^*), \tag{2}$$

and

$$\|x^*B(\cdot\,;0)x\| = \|\mu(\cdot\,;x,x^*)\| \tag{3}$$

for every $\gamma \in C(R)$ with compact support; in equation (3), the norm on the left is the norm of $x^*B(\cdot\,;0)x$ as a bounded linear functional on $C(R)$; the norm on the right is the total variation of μ. By (1) and (3), we obtain for each $\delta \in \mathscr{B}_r$:

$$(4)\qquad \begin{aligned} |\mu(\delta;x,x^*)| &\leqq \|\mu(\cdot\,;x,x^*)\| = \|x^*B(\cdot\,;0)x\| \\ &= \sup_{\|\gamma\|=1} |x^*B(\gamma;0)x| \leqq K\|x\|\,\|x^*\|. \end{aligned}$$

We conclude from (4), (2) and the uniqueness of the Riesz representation that $\mu(\delta;\cdot,\cdot)$ is a bounded bilinear form on $X \times X^*$ for each fixed $\delta \in \mathscr{B}_r$. The space X being reflexive, it follows that there exists a unique bounded linear operator $F(\delta) \in B(X)$ such that

$$(5)\qquad \mu(\delta;x,x^*) = x^*F(\delta)x$$

for each $\delta \in \mathscr{B}_r$, $x \in X$ and $x^* \in X^*$. From (4) and (5) it follows that $F(\cdot)$ is a uniformly bounded finitely additive $B(X)$-valued measure on $\mathscr{B}_r$. By (5), $x^*F(\cdot)x$ is countably additive on $\mathscr{B}_r$; therefore $F(\cdot)$ is countably additive in the strong operator topology (see [8, p. 75]).

Thus, the integral $\int_R \gamma(\eta)\,dF(\eta)$ is meaningful in the the strong operator topology. We have:

$$x^*\int_R \gamma(\eta)\,dF(\eta)x = \int_R \gamma(\eta)\,d\mu(\eta;x,x^*) = x^*B(\gamma;0)x,$$

for each x in X, x^* in X^* and γ in $C(R)$ with compact support. Hence $B(\gamma;0) = \int_R \gamma(\eta)\,dF(\eta)$ for each γ in $C(R)$ with compact support; this equation extends to all γ in $C(R)$ because $F(\cdot)$ is uniformly bounded. The uniqueness of $F(\cdot)$ with the properties (i)–(iii) follows from the uniqueness of the Riesz representation. Q.E.D.

COROLLARY 3. *Let $S \in B(X)$ have real spectrum and suppose that condition* (b) *holds for $\varepsilon = 0$* (*by Lemma* 5, *condition* (b) *holds for $\varepsilon > 0$ since S has real spectrum*; *thus condition* (b) *holds for $\varepsilon \geqq 0$*). *Let $F(\cdot)$ be the measure defined in Lemma* 7. *Then the Fourier-Stieltjes transform of $F(\cdot)$ is $e^{-2\pi i v S}$ if and only if condition* (c) *holds.*

Proof. We have noted already that the hypothesis of the corollary implies that condition (b) holds. Therefore, by the remark following Lemma 5, condition (c) holds if and only if $B(e^{-2\pi i v\xi};0) = e^{-2\pi i v S}$. But, by Lemma 7, $B(e^{-2\pi i v\xi};0)$ is the Fourier-Stieljes transform of $F(\cdot)$. Q.E.D.

LEMMA 8. *Let S and $F(\cdot)$ be as in Corollary* 3. *Suppose furthermore that condition* (c) *holds. Then $F(\cdot)$ is a spectral measure on $\mathscr{B}_r$.*

Proof. By Lemma 7, all we have to prove is that $F(R) = I$ and that $F(\delta_1 \cap \delta_2) = F(\delta_1)F(\delta_2)$ for all δ_1 and δ_2 in $\mathscr{B}_r$.

By Corollary 3, we have:

$$(1)\qquad \int_R e^{-2\pi i v\xi}\,dF(\xi)x = e^{-2\pi i v S}x$$

for all v in R and x in X.

Taking $v = 0$, we obtain $F(R) = I$.

Next, if v_1 and v_2 are real numbers, we have:

$$\int_R e^{-2\pi i(v_1+v_2)\xi}\, dF(\xi)x = e^{-2\pi i(v_1+v_2)S}x = e^{-2\pi i v_1 S}\cdot e^{-2\pi i v_2 S}x$$
$$= \int_R e^{-2\pi i v_1 \xi}\, dF(\xi)\cdot \int_R e^{-2\pi i v_2 \xi}\, dF(\xi)x.$$

Using this relation, we obtain

$$\int_R \gamma_1(\xi)\gamma_2(\xi)\, dF(\xi)x = \int_R \gamma_1(\xi)\, dF(\xi) \int_R \gamma_2(\xi)\, dF(\xi)x$$

for functions γ_1 and γ_2 in $C(R)$ of the form $\sum_{k=1}^n \alpha_k e^{-2\pi i v^k \xi}$, where $\alpha_1, \cdots, \alpha_n$ are complex and $v_1, \cdots, v_n$ are real numbers.

By the uniform boundedness of $F(\cdot)$, it follows therefore that the map $\gamma \to \int_R \gamma(\xi) dF(\xi)$ is multiplicative over the uniformly closed linear subspace A of $C(R)$ generated by the set $\{e^{-2\pi i v\xi}; v \in R\}$. Since A is closed under complex conjugation, we apply the Stone-Weierstrass Theorem, and using again the uniform boundedness of $F(\cdot)$ and the bounded convergence theorem for integrals, we conclude that the map is also multiplicative over the class of functions which are (pointwise) limits of uniformly bounded sequences of functions in $C(R)$. In particular, this is true for characteristic functions of intervals. It follows that $F(\alpha_1)F(\alpha_2) = F(\alpha_1 \cap \alpha_2)$ for intervals; since $F(\cdot)$ is countably additive in the strong operator topology, this relation extends to all α_1 and α_2 in $\mathscr{B}_r$. Q.E.D.

Proof of Theorem 4. (1) *Necessity.* Let $E(\cdot)$ be the resolution of the identity of the pseudo-hermitian operator S, and let $F(\cdot)$ be the restriction of $E(\cdot)$ to $\mathscr{B}_r$. Then

$$e^{-2\pi i\eta S} = \int_R e^{-2\pi i\eta\xi}\, dF(\xi).$$

Hence $\| e^{-2\pi i\eta S}\| \leqq K$ and condition (a) holds.

Since $\sigma(S)$ is real, condition (b) holds for $\varepsilon > 0$ by Lemma 5. We prove that condition (b) holds also for $\varepsilon = 0$. For each x in X and x^* in X^*, $x^* e^{-2\pi i\eta S}x$ is the Fourier-Stieltjes transform of the countably additive complex measure $x^*F(\cdot)x$ with finite total variation. By Theorem 2.1.3 in [2], it follows that the indefinite integral of $x^*G_N(\eta;0)x$ is "Bernoulli-convergent" to $x^*F(\cdot)x$ when $N \to \infty$, i.e., its total variation is uniformly bounded for $N > 0$ and

$$\lim_{N\to\infty} \int_R \gamma(\eta)x^*G_N(\eta;0)x\,d\eta = \int_R \gamma(\eta)\, dx^*F(\eta)x$$

for all γ in $C(R)$. Equivalently, the integral $\int_R \gamma(\eta)G_N(\eta;0)d\eta$ converges (when $N \to \infty$) to $\int_R \gamma(\eta)dF(\eta) = B(\gamma;0)$ in the weak operator topology. We also have

$$\| B(\gamma;0) \| = \left\| \int_R \gamma(\eta) dF(\eta) \right\| \leqq K \| \gamma \|,$$

and condition (b) is proved.

By the remark following Lemma 5, condition (c) holds if and only if

$$B(e^{-2\pi i v \xi};0) = e^{-2\pi i v S} \text{ for } v \text{ real.}$$

But this follows from the facts that

$$B(\gamma;0) = \int_R \gamma(\eta) dF(\eta) \text{ and } e^{-2\pi i v S} = \int_R e^{-2\pi i v \eta} dF(\eta).$$

This completes the proof of the necessity.

(2) *Sufficiency*. By Corollary 2, condition (a) implies that the spectrum of S is real. Then, by Lemmas 7 and 8, using conditions (b) and (c), we conclude that there exists a unique *uniformly bounded spectral measure* $F(\cdot)$ on $\mathscr{B}_r$, countably additive in the strong operator topology, such that

$$B(\gamma;0) = \int_R \gamma(\xi) dF(\xi) \tag{1}$$

for all γ in $C(R)$. Furthermore, by Corollary 3, the Fourier-Stieltjes transform of $F(\cdot)$ is $e^{-2\pi i v S}$,

$$e^{-2\pi i v S} = \int_R e^{-2\pi i v \xi} dF(\xi) \text{ for all real } v. \tag{2}$$

Thus, $F(\cdot)$ commutes with $e^{-2\pi i v S}$ for all real v. Since

$$\lim_{v \to 0} (-2\pi i v)^{-1} (e^{-2\pi i v S} - I) = S$$

(in the uniform operator topology), it follows that $F(\cdot)$ commutes with S. Therefore, the Banach algebra generated by $\{F(\delta);\ \delta \in \mathscr{B}_r\}$ and S is commutative and has an identity (since $I = F(R)$). Let $\mathscr{M}$ be its maximal ideal space. Since $F(\cdot)$ is a spectral measure, the support of $F(\cdot)(m)$ is either void or consists of a single point ξ_m on R (for each fixed m in $\mathscr{M}$). Thus, by (2),

$$e^{-2\pi i v S}(m) = \int_R e^{-2\pi i v \xi} d(F(\xi)(m)) = e^{-2\pi i v \xi_m}$$

in the second case and 0 in the first. But

$$e^{-2\pi i v S}(m) = e^{-2\pi i v S(m)}$$

for all real v; it follows that only the second case is possible and $S(m) = \xi_m$.

By Lemma 2 in [4], we obtain (for each δ in $\mathscr{B}_r$):

(3) $$\sigma(S|F(\delta)X) = \{S(m); m \in \mathcal{M}, F(\delta)(m) = 1\} = \{\xi_m; m \in \mathcal{M}, F(\delta)(m) = 1\}.$$

If ξ_m is not in $\bar{\delta}$, then $F(\delta)(m) = 0$. Thus $\xi_m \in \bar{\delta}$ for each $m \in \mathcal{M}$ such that $F(\delta)(m) = 1$. Hence, by (3), we conclude that

$$\sigma(S|F(\delta)X) \subset \bar{\delta}.$$

If we extend now the measure $F(\cdot)$ to the Borel sets of the complex plane by defining $E(\delta) = F(\delta \cap R)$ for each Borel set δ of the plane, we then obtain a resolution of the identity $E(\cdot)$ for S. This shows that *S is a spectral operator with real spectrum*. Thus, by Theorem 1 in [4], the support of $E(\cdot)$ (or $F(\cdot)$) is $\sigma(S)$. Therefore, by (2), we have:

$$(-2\pi i v)^{-1}(e^{-2\pi i v S} - 1) = \int_{\sigma(S)} (-2\pi i v)^{-1}(e^{-2\pi i v \xi} - 1)dF(\xi).$$

The expression on the left-hand side converges to S (when $v \to 0$) in the uniform operator topology. *Since $\sigma(S)$ is compact*. the integral on the right-hand side converges to $\int_{\sigma(S)} \xi dF(\xi)$ in the strong operator topology. We conclude that $S = \int_{\sigma(S)} \xi dF(\xi)$. This completes the proof of Theorem 4.

REMARKS. In applying Theorem 4 to concrete problems, we may use the following facts in order to simplify the verification of the conditions (a)–(c).

(1) Condition (a) may be replaced by the condition

(a′) $$\liminf_{|\xi| \to \infty} (2\pi\xi)^{-1} \log \| e^{-2\pi i \xi S} \| = 0,$$

which is equivalent to the condition that $\sigma(S)$ be real (see Corollary 1).

(2) Condition (b) for $\varepsilon > 0$ follows from condition (a) (or (a′)). This is a consequence of Lemma 5 and Corollaries 1 and 2. Therefore, if (a) or (a′) hold, we only have to check condition (b) for $\varepsilon = 0$.

(3) Once we have obtained $B(\gamma; 0)$ (by checking condition (b) for $\varepsilon = 0$), condition (c) can be verified by showing that

$$B(e^{-2\pi i v \xi}; 0) = e^{-2\pi i v S} \text{ for real } v.$$

(4) In Hilbert space, condition (a) alone is sufficient for S to be p.h. This is false in general (see §5).

4. **Examples.** (a) We apply Theorem 4 to a concrete example. For $1 < p < \infty$, let S be the operator defined on $L_p[0, 1]$ by:

(1) $$(Sf)(x) = xf(x) + \int_0^x (e^{x-t} - 1)f(t)dt.$$

For $\xi \in R$, we find that

$$(e^{-2\pi i\xi S}f)(x) = e^{-2\pi i\xi x}f(x) + \int_0^x K(x,t;\xi)f(t)dt, \tag{2}$$

where

$$K(x,t;\xi) = \frac{2\pi i\xi}{2\pi i\xi - 1}(e^{-2\pi i\xi x}e^{x-t} - e^{-2\pi i\xi t}).$$

We then obtain the estimate

$$\| e^{-2\pi i\xi S} \| \leqq e + 2 \text{ for all real } \xi. \tag{3}$$

Next, we show that S satisfies condition (b) in Theorem 4. According to the remark following the proof of Theorem 4, we have only to check condition (b) for $\varepsilon = 0$. Let $\gamma \in C(R), f \in L_p[0,1]$ and $x \in [0,1]$. By (2), we have:

$$\begin{aligned}
&\left[\int_R \gamma(\eta)G_N(\eta;0)d\eta f\right](x) \\
&= \int_R \gamma(\eta)\int_R e^{-\pi(\xi/N)^2 + 2\pi i\xi(\eta - x)}d\xi d\eta f(x) \\
&\quad + \int_0^x \left\{\int_R \gamma(\eta)\int_R e^{-\pi(\xi/N)^2 + 2\pi i\xi\eta}K(x,t;\xi)d\xi d\eta\right\} f(t)dt \\
&= J_N^1(x) + J_N^2(x).
\end{aligned}$$

Now

$$J_N^1(x) = \int_R \gamma(\eta + x)Ne^{-\pi(N\eta)^2}d\eta f(x).$$

Therefore, by Theorem 1.1.1 in [2], J_N^1 converges to γf in the weak topology of $L_p[0,1]$.

Next

$$\begin{aligned}
J_N^2(x) &= \int_0^t \left\{\int_R \gamma(\eta + x)\int_R e^{-\pi(\xi/N)^2 + 2\pi i\xi\eta} \cdot \frac{2\pi i\xi}{2\pi i\xi - 1}d\xi d\eta\right\} e^{x-t}f(t)dt \\
&\quad - \int_0^x \left\{\int_R \gamma(\eta + t)\int_R e^{-\pi(\xi/N)^2 + 2\pi i\xi\eta} \cdot \frac{2\pi i\xi}{2\pi i\xi - 1}d\xi d\eta\right\} f(t)dt.
\end{aligned}$$

We write $2\pi i\xi/(2\pi i\xi - 1) = 1 - 1/(1 - 2\pi i\xi)$ and observe that $1/(1 - 2\pi i\xi)$ is the Fourier transform of the function in $L_1(R)$ defined by:

$$h(\eta) = \begin{cases} e^{\eta} & \text{for } \eta \leqq 0, \\ 0 & \text{for } \eta > 0. \end{cases}$$

Therefore, using Theorem 2.1.3 in [2], we obtain:

$$\lim_{N\to\infty} J_N^2 = \int_0^x (\gamma(x) - \int_R \gamma(\eta + x)h(\eta)d\eta)e^{x-t}f(t)dt - \int_0^x \Big(\gamma(t) - \int_R \gamma(\eta + t)h(\eta)d\eta\Big)f(t)dt$$

in the weak topology of $L_p[0,1]$, and we conclude that

$$\lim_{N\to\infty} \left(\int_R \gamma(\eta)G_N(\eta;0)d\eta f\right)(x) = \gamma(x)f(x) + \int_0^x L(x,t;\gamma)f(t)dt \tag{4}$$

where

$$L(x,t;\gamma) = \gamma(x)e^{x-t} - \gamma(t) - e^{-t}\int_t^x \gamma(\eta)e^{\eta}d\eta. \tag{5}$$

Hence:

$$\lim_{N\to\infty} \int_R \gamma(\eta)G_N(\eta;0)d\eta = B(\gamma;0) \tag{6}$$

in the weak operator topology (over $L_p[0,1]$), where

$$[B(\gamma;0)f](x) = \gamma(x)f(x) + \int_0^x L(x,t;\gamma)f(t)dt, \qquad f \in L_p[0,1]. \tag{7}$$

Finally, we find the estimate:

$$\| B(\gamma;0) \| \leqq (2e+1)\| \gamma \|.$$

According to the remark following the proof of Theorem 4, we check condition (c) by showing that

$$B(e^{-2\pi i\nu\xi};0) = e^{-2\pi i\nu S} \text{ for all real } \nu.$$

But this follows at once from (7), (2) and the identity

$$L(x,t;e^{-2\pi i\nu\xi}) = K(x,t;\nu).$$

We conclude that S satisfies conditions (a)–(c) in Theorem 4. *Therefore S is pseudo-hermitian.* In particular, as an operator over $L_2[0,1]$, S is similar to a hermitian operator.

For completeness, we determine also the resolution of the identity F of S. Using Lemma 7 and (7), we obtain:

$$\left[\int_R \gamma(\xi)dF(\xi)f\right](x) = \int_R \gamma(\xi)d\mu_x(\xi)(Tf)(x) - \int_R \gamma(\xi)(Tf)(\xi)c_{[0,x]}(\xi)d\xi,$$

where $\mu_x(\cdot)$ is the measure defined on the Borel sets of R by

$$\mu_x(\delta) = \begin{cases} 1 & \text{if } x \in \delta, \\ 0 & \text{if } x \notin \delta \end{cases}$$

and T is the (nonsingular) operator defined on $L_p[0,1]$ by

(8) $$(Tf)(x) = f(x) + \int_0^x e^{x-t}f(t)dt.$$

It follows that

(9) $$[F(\delta)f](x) = \mu_x(\delta)(Tf)(x) - \int_{\delta\cap[0,x]} (Tf)(\xi)d\xi$$

for each Borel set δ on R, f in $L_p[0,1]$ and x in $[0,1]$. Equations (8) and (9) define the resolution of the identity $F(\cdot)$ of the pseudo-hermitian operator S.

Now we have:

$$(Sf)(x) = \left(\int \xi dF(\xi)f\right)(x) = x(Tf)(x) - \int_0^x \xi(Tf)(\xi)d\xi.$$

Since T is nonsingular, every f in $L_p[0,1]$ can be written as $T^{-1}g$ with $g \in L_p[0,1]$; hence:

$$(ST^{-1}g)(x) = xg(x) - \int_0^x \xi g(\xi)d\xi.$$

Applying T on both sides of the equation, we obtain

$$(TST^{-1}g)(x) = xg(x),$$

proving that S is similar to the "multiplication by x" operator on $L_p[0,1]$ (for $1 < p < \infty$). We notice that the same holds for $p = 1$ (direct check!). Furthermore, the similarity mapping is performed by means of the operator T defined in (8).

(b) The Spectral Theorem for hermitian operators in Hilbert space follows from Theorem 4. Indeed, conditions (a)–(c) are satisfied if S is hermitian: first $e^{-2\pi i\eta S}$ is unitary, and therefore $\|e^{-2\pi i\eta S}\| = 1$; i.e., condition (a) is satisfied. Next, let $x \in X$; let $\alpha_1, \cdots, \alpha_n$ be complex and $\eta_1, \cdots, \eta_n$ be real numbers ($n \geqq 1$). Then

$$\begin{aligned}\sum_{i,j=1}^{n} (e^{-2\pi i(\eta_i-\eta_j)S}x, x)\alpha_i\bar{\alpha}_j &= \sum_{i,j=1}^{n} (\alpha_i e^{-2\pi i\eta_i S}x, \alpha_j e^{-2\pi i\eta_j S}x) \\ &= \left\| \sum_{i=1}^{n} \alpha_i e^{-2\pi i\eta_i S}x \right\|^2 \geqq 0;\end{aligned}$$

i.e., the function $f_x(\eta) = (e^{-2\pi i\eta S}x, x)$ is positive definite; hence it is the Fourier transform of a positive measure μ_x, by Bochner's Theorem. We have

(1) $$\|\mu_x\| = \operatorname{var} \mu_x = f_x(0) = \|x\|^2.$$

Now the function $(e^{-2\pi i\eta S}x, y)$ is the Fourier transform of the complex measure

$$\mu_{xy} = \frac{1}{4}(\mu_{x+y} - \mu_{x-y} + i\mu_{x+iy} - i\mu_{x-iy}).$$

For a fixed Borel set δ on R, $\mu_{xy}(\delta)$ is a complex bilinear form such that $\mu_{xx}(\delta) = \mu_x(\delta) \geqq 0$. Therefore, by Schwartz inequality and (1):

$$(2) \qquad |\mu_{xy}(\delta)| \leqq \mu_x(\delta)^{1/2} \cdot \mu_y(\delta)^{1/2} \leqq \|x\| \|y\|.$$

There exists therefore an operator-valued set function $F(\delta)$ such that $\mu_{xy}(\delta) = (F(\delta)x, y)$. This equation shows furthermore that $F(\cdot)$ is countably additive in the strong operator topology. Now, by Theorem 2.1.3 in [2], the integral $(\int_R \gamma(\eta)G_N(\eta;0)d\eta x, y)$ converges (when $N \to \infty$) to $\int_R \gamma(\eta)d(F(\eta)x, y)$; i.e.,

$$\lim_{N\to\infty} \int_R \gamma(\eta)G_N(\eta;0)d\eta = \int_R \gamma(\eta)dF(\eta) \equiv B(\gamma;0)$$

in the weak operator topology. Furthermore, by (2),

$$|(B(\gamma;0)x, y)| = \left|\int_R \gamma(\eta)d\mu_{xy}(\eta)\right| \leqq \|\gamma\| \|x\| \|y\|,$$

i.e.,

$$\|B(\gamma;0)\| \leqq \|\gamma\|.$$

Finally, condition (c) follows from:

$$(B(e^{-2\pi i v\xi};0)x, y) = \int_R e^{-2\pi i v\xi} d\mu_{xy}(\xi) = (e^{-2\pi i vS}x, y).$$

5. **Other characterizations of pseudo-hermitian operators.** The characterization of pseudo-hermitian operators given in Theorem 4 has a "constructive" meaning. Indeed, $B(\gamma;0)$ is the operator $\gamma(S)$ defined in the operational calculus for scalar operators; it is constructed by a limiting process, using the group $\{e^{-2\pi i\xi S};\ \xi \text{ real}\}$. The existence of $B(\gamma;0)$ is the essential part of conditions (b) and (c); in other words, pseudo-hermitian operators are characterized in Theorem 4 by means of their operational calculus, which is furthermore obtained without knowing the resolution of the identity for S (and not even its existence). Since the knowledge of $B(\gamma;0) = \gamma(S)$ for all $\gamma \in C(R)$ is equivalent to the knowledge of the resolution of the identity for S, we might say that the latter is also given constructively through Theorem 4 (see example 1 in §4).

In this section, some nonconstructive characterizations of pseudo-hermitian operators are given. Their merit lies in their simplicity.

The first result is for *Hilbert spaces.*

THEOREM 5. *A bounded linear operator S on the Hilbert space X is pseudo-hermitian* (*i.e., similar to a hermitian operator*) *if and only if the group* $\{e^{-2\pi i\xi S};\ \xi\ \textit{real}\}$ *is uniformly bounded* (*i.e.,* $\|e^{-2\pi i\xi S}\| \leqq M < \infty$ *for all* $\xi \in R$).

Proof. We have only to prove the sufficiency. By a theorem of B. Sz.-Nagy [13], the uniform boundedness of the group $\{e^{i\xi S};\ \xi \in R\}$ implies the existence of a nonsingular operator Q such that $\{Q^{-1}e^{i\xi S}Q;\ \xi \in R\}$ is a group of *unitary* operators. By Stone's Theorem, the latter group has an infinitesimal generator iA, where A is *selfadjoint*. Now A is obviously equal to $Q^{-1}SQ$, and is therefore bounded. Hence A is hermitian and $S = QAQ^{-1}$ is similar to a hermitian operator, i.e., S is p.h.

REMARK. Wermer's result [15] on sums and products of commuting spectral operators in Hilbert space is an immediate corollary of Theorem 5 (for the case of real spectrum). Indeed, we may restrict our attention to the scalar parts, i.e., to p.h. operators. For the sum, this follows from the inequality

$$\| e^{it(S+T)} \| = \| e^{itS}e^{itT} \| \leqq \| e^{itS} \| \, \| e^{itT} \|,$$

using then Theorem 5. For the product, we write $2ST = (S+T)^2 - S^2 - T^2$, and since the square of a spectral operator is spectral, the result follows from the corresponding result for the sum.

Since Wermer's Theorem is false in Banach space (even if the space is reflexive, see [9] and [12]), it follows that Theorem 5 is not true in that situation, i.e., condition (a) in Theorem 4 is not sufficient by itself, even in reflexive Banach space. In the latter case, Theorem 4 gives additional necessary conditions which, together with condition (a) (or the weaker condition (a′)), are both necessary and sufficient for S to be p.h.

Notice that if the condition in Theorem 5 is satisfied with $M = 1$, then S is hermitian (and conversely). This follows at once from Sz.-Nagy's proof in [13].

Our next characterizations for pseudo-hermitian operators in *reflexive* Banach spaces are derived from the Bochner-Schoenberg Theorem [3; 14].

THEOREM 6. *Let S be a bounded linear operator on a reflexive*(7) *Banach space X. The following statements are equivalent*:

(1) *S is pseudo-hermitian.*

(2) *For every $f \in L_1(R)$, we have*

$$\left\| \int_R f(\xi) e^{-2\pi i \xi S} d\xi \right\| \leqq M \|\hat{f}\|_\infty,$$

where the norm on the left is the operator norm, $\hat{f}$ is the Fourier transform of f and $\| \ \|_\infty$ is the sup norm.

(3) *For every real vector $(\xi_1, \cdots, \xi_n)$ and every complex vector $(c_1, \cdots, c_n)$, $n = 1, 2, \cdots$, we have*

$$\left\| \sum_{k=1}^{n} c_k e^{-2\pi i \xi_k S} \right\| \leqq M \sup_{t \in R} \left| \sum_{k=1}^{n} c_k e^{-2\pi i \xi_k t} \right|.$$

(7) From now till the end of the paper, the space is assumed to be *reflexive*.

(4) *For every* $x \in X$ *and* $x^* \in X^*$ *with unit norm, the integral* $\int_R |x^*\{R(\xi - i\varepsilon; S) - R(\xi + i\varepsilon; S)\}x| d\xi$ *is uniformly bounded as* $\varepsilon \to 0+$.

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are immediate.

(2) $\Rightarrow$ (1). We first notice that the integral $\int_R f(\xi)e^{-2\pi i\xi S}d\xi$ converges in the uniform operator topology. Indeed, if $m > n$ are integers and $f \in L_1(R)$, let $f_{n,m}(\xi) = f(\xi)$ for $n \leqq \xi \leqq m$ and $f_{n,m}(\xi) = 0$ otherwise. Then

$$\left\| \int_n^m f(\xi)e^{-2\pi i\xi S}d\xi \right\| = \left\| \int_R f_{n,m}(\xi)e^{-2\pi i\xi S}d\xi \right\|$$

$$\leqq M \left\| \hat{f}_{n,m} \right\|_\infty \leqq M \int_n^m |f(\xi)| d\xi \underset{n,m\to\infty}{\to} 0.$$

We conclude that (2) is equivalent to

$$\left| \int_R f(\xi)(x^*e^{-2\pi i\xi S}x)d\xi \right| \leqq M \| x \| \| x^* \| \| \hat{f} \|_\infty$$

for all $f \in L_1(R)$, $x \in X$ and $x^* \in X^*$. By Schoenberg's Theorem, this implies the existence of a measure $\mu(\cdot; x, x^*)$ on the Borel sets of R such that $\| \mu(\cdot; x, x^*) \| \leqq M \| x \| \| x^* \|$ and $x^*e^{-2\pi i\xi S}x = \int_R e^{-2\pi i\xi\eta} d\mu(\eta; x, x^*)$ for all $x \in X$ and $x^* \in X^*$.

It follows (as in Lemma 7) that there exists a unique uniformly bounded (by M) operator-valued function $F(\cdot)$ on $\mathscr{B}_r$ such that (i) $F(\cdot)$ is finitely additive; (ii) $F(\cdot)x$ is countably additive and (iii) $e^{-2\pi i\xi S}x = \int e^{-2\pi i\xi\eta} dF(\eta)x$ for all $x \in X$.

We conclude (as in Lemma 8) that $F(\cdot)$ is a *spectral* measure on $\mathscr{B}_r$. The proof is then completed as that of Theorem 4 (sufficiency part from equation (2) to the end).

(3) $\Rightarrow$ (1). The proof is analogous to that of the implication (2) $\Rightarrow$ (1), using Bochner's Theorem [3].

(1) $\Rightarrow$ (4). Let E be the resolution of the identity for S. Let $v(\cdot; x, x^*)$ be the variation of the complex measure $x^*E(\cdot)x$. For $x \in X$ and $x^* \in X^*$ with unit norm, we have $v(R : x, x^*) \leqq 4M$ where M is the uniform bound of E, and therefore:

$$\int_R |x^*\{R(\xi - i\varepsilon, S) - R(\xi + i\varepsilon; S)\}x| d\xi$$

$$= 2\varepsilon \int_R \left| \int_{\sigma(S)} \frac{dx^*E(\eta)x}{(\xi - \eta)^2 + \varepsilon^2} \right| d\xi$$

$$\leqq \int_{\sigma(S)} \left(\int \frac{2\varepsilon d\xi}{(\xi - \eta)^2 + \varepsilon^2} \right) dv(\eta; x, x^*) = 2\pi v(R; x, x^*)$$

$$\leqq 8\pi M.$$

(4) $\Rightarrow$ (1). We first notice that the reality of $\sigma(S)$ is implicitly required by statement (4).

The Fourier transform of $x^*\{R(\xi - i\varepsilon; S) - R(\xi + i\varepsilon; S)\}x$ is

$$2\pi i e^{-2\pi\varepsilon|\xi|}\, x^* e^{-2\pi i \xi S} x$$

(see proof of Lemma 5). Using Fubini's Theorem, we obtain for all $f \in L_1(R)$:

$$\int_R f(\xi) e^{-2\pi\varepsilon|\xi|} x^* e^{-2\pi i\xi S}\, x d\xi = \frac{1}{2\pi i}\int_R x^*\{R(\xi - i\varepsilon; S) - R(\xi + i\varepsilon; S)\} x \hat{f}(\xi) d\xi.$$

By Lebesgue Convergence Theorem and condition 4, the left-hand side converges to $x^* \int_R f(\xi) e^{-2\pi i\xi S} d\xi x$ when $\varepsilon \to 0+$, while the right-hand side is uniformly bounded (as $\varepsilon \to 0+$) by $M\|\hat{f}\|_\infty$ (for all $x \in X$, $x^* \in X^*$ with unit norm). Hence (4) $\Rightarrow$ (2) and therefore (4) $\Rightarrow$ (1). Q.E.D.

REMARK. Let $L_1^{\wedge}(R)$ denote the space $L_1(R)$ normed with the spectral norm, $\|f\|_s = \|\hat{f}\|_\infty$. Let T be the linear operator defined by $Tf = \int_R f(t) e^{itS} dt$. Then statement 2 is equivalent to the statement: T is a *bounded* linear operator on $L_1^{\wedge}(R)$ into $B(X)$. The operator norm of T will be called the *pseudo-variation* of S, and will be denoted by $pv(S)$. Thus: $pv(S) = \sup \| \int_R f(t) e^{itS} dt \|$, where the sup is taken over all f in $L_1(R)$ with $\|f\|_s = 1$. Notice that $pv(S)$ is defined for any S in $B(X)$, and we have $0 < pv(S) \leqq \infty$; $pv(S)$ is finite if and only if S is pseudo-hermitian.

We also denote $b(S) = \sup_{t\in R} \|e^{itS}\|$. Some elementary properties of the pseudo-variation are listed below.

LEMMA 9. *The pseudo-variation is invariant under real translation and dilation*: $pv(S + rI) = pv(S)$ *for all real* r, *and* $pv(rS) = pv(S)$ *for all real* $r \neq 0$.

We have $1 \leqq b(S) \leqq \infty$; *if* $b(S) < \infty$, *then* $b(S) \leqq pv(S)$, *so that* $1 \leqq pv(S) \leqq \infty$. *In particular, if* $pv(S)$ *is finite* (*i.e., if* S *is p.h.*), *then* $1 \leqq b(S) \leqq pv(S) < \infty$.

Proof. The map $f \to f_r$, *where* $f_r(t) = f(t)e^{irt}$, is an isometry of $L_1^{\wedge}(R)$ onto itself. Hence:

$$\begin{aligned} pv(S + rI) &= \sup \left\| \int_R f_r(t) e^{itS} dt \right\| \quad (f \in L_1(R), \|f\|_s = 1) \\ &= \sup \left\| \int_R g(t) e^{itS} dt \right\| \quad (g \in L_1(R), \|g\|_s = 1) \\ &= pv(S). \end{aligned}$$

The invariance of the pseudo-variations under real dilation is proved in a similar way.

The inequality $1 \leqq b(S)$ follows at once from the group property of e^{itS}. Suppose now that $b(S) < \infty$, and for $\varepsilon > 0$, let t_0 be such that $\| e^{it_0 S} \| \geqq b(S) - \varepsilon$.

Using the Hahn-Banach Theorem, we choose a unit vector $u^* \in B(X)^*$ such that $u^* e^{it_0 S} = \| e^{it_0 S} \|$. Since $\|f\|_s \leqq \|f\|_1$ for $f \in L_1(R)$, we have

$$\{f \in L_1(R);\ \|f\|_1 \leqq 1\} \subset \{f \in L_1(R);\ \|f\|_s \leqq 1\}.$$

Denoting the former set by A and the latter set by B and using the isometry of L_1^* and L_∞, we obtain:

$$\begin{aligned} b(S) - \varepsilon \leqq \| e^{it_0 S} \| &= u^* e^{it_0 S} \leqq \sup_{t \in R} |u^* e^{itS}| \\ &= \sup_{f \in A} \left| \int_R f(t) u^* e^{itS} dt \right| \leqq \sup_{f \in A} \left\| \int_R f(t) e^{itS} dt \right\| \\ &\leqq \sup_{f \in B} \left\| \int_R f(t) e^{itS} dt \right\| = pv(S). \end{aligned}$$

Q.E.D.

THEOREM 7. *Let S_a be a net of operators converging strongly to the operator S. Suppose that the pseudo-variations of S_a are uniformly bounded. Then S (as well as all S_a) is pseudo-hermitian.*

Proof. Using the Uniform Boundedness Theorem (see, e.g., Hille-Phillips [8, Theorem 2.5.5]), we check easily that $\exp(itS_a)$ converges pointwise to e^{itS} in the strong operator topology.

Let $K = \sup_a pv(S_a)$. By Lemma 9, we have $\| \exp(itS_a) \| \leqq K$ for all real t. Now, for $f \in L_1(R)$ and $A > 0$, let $f_A(t) = f(t)$ for $-A \leqq t \leqq A$ and $f_A(t) = 0$ otherwise. Using the Lebesgue Dominated Convergence Theorem for nets (see, e.g., Dunford-Schwartz [6, p. 124, Theorem 7]), we obtain:

$$\int f_A(t) e^{itS} x dt = \lim_a \int f_A(t) \exp(itS_a) x dt$$

(in the strong topology of X, for all $x \in X$).

But

$$\left\| \int f_A(t) \exp(itS_a) x dt \right\| \leqq K \| \hat{f}_A \|_\infty \| x \|.$$

Hence $\| \int f_A(t) e^{itS} dt \| \leqq K \| \hat{f}_A \|_\infty$, and since $f_A \to f$ in L_1-norm when $A \to \infty$, we get finally:

$$\left\| \int f(t) e^{itS} dt \right\| \leqq K \| \hat{f} \|_\infty, \quad \text{for all } f \in L_1(R).$$

By Theorem 6, this shows that S is p.h. (and also that $pv(S) \leqq \sup_a pv(S_a)$).

REMARK. Let S be p.h. and let E be its resolution of the identity. Write $M = \sup(\| E(\delta) \|;\ \delta \in \mathscr{B}_r)$. Then it is easily seen that

$$M \leqq pv(S) \leqq 4M.$$

This implies that the hypothesis of uniform boundedness of the pseudo-variations of S_a is equivalent to that of uniform boundedness of the resolutions of the identity for S_a. This means that our Theorem 7 is equivalent to Theorem 2.3 in Bade [1] (for the case of real spectrum). However, our statement has the advantage of being entirely given in terms of "analytic" conditions on the S_a *themselves*, without any allusion to the generally unknown resolutions of the identity.

We consider now the relation between $f(S)$ and $f(S_a)$ for bounded Borel functions f on R.

Using essentially the notations of §4, we write:

$$G_N(t,S) = \int_R \exp(-\pi(u/N)^2 + 2\pi itu)e^{-2\pi iuS}du.$$

A straightforward application of Theorem 1.1.1 in [2] and of the Lebesgue Dominated Convergence Theorem gives:

LEMMA 10. *Let S be p.h. and let E be its resolution of the identity. Let f be a bounded Borel function on R with set of discontinuities K. Then for each* $x \in X$ *such that* $E(\bar{K})x = 0$, *we have*:

$$f(S)x = \lim_{N\to\infty} \int_R f(t)G_N(t,S)xdt \quad (\textit{strongly}).$$

In particular, if $E(\bar{K}) = 0$, *then* $f(S)$ *is the strong limit of* $\int_R f(t)G_N(t,S)dt$ (*as* $N \to \infty$). *Furthermore, if* S_a *and S are as in Theorem 7 and* $E(\bar{K}) = 0$, *then*

$$f(S_a) = \lim_{N\to\infty} \int_R f(t)G_N(t,S_a)dt \quad \textit{uniformly in } a.$$

Notice the following

COROLLARY. *For S and E as above and* $\delta \in \mathscr{B}_r$, *we have*:

$$E(\delta)x = \lim_{N\to\infty} \int_R e^{-\pi(u/N)^2}\hat{c}_\delta(u)e^{2\pi iuS}xdu,$$

provided that $E(\partial\delta)x = 0$ (c_δ *and* $\partial\delta$ *are the characteristic function and the boundary of* δ *respectively*).

The following theorem is easily proved by using Lemma 10 (the proof is omitted):

THEOREM 8. *Let* S_a *be a net of operators converging strongly to the operator S. Suppose that the pseudo-variations of* S_a *are uniformly bounded. Let then* E_a *and E be the resolutions of the identity for* S_a *and S respectively. Then for every bounded Borel function f on R,* $f(S_a)x$ *converges strongly to* $f(S)x$ *provided* *that* $E(\bar{K})x = 0$, *where* K *is the set of discontinuities of* f. *In particular,* $E_a(\delta)x$ *converges strongly to* $E(\delta)x$ *if* $E(\partial\delta)x = 0$.

According to the last remark, Theorem 8 is equivalent to Theorem 2.6 in Bade [1].

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