A SURFACE IN S^3 IS TAME IF IT CAN BE DEFORMED INTO EACH COMPLEMENTARY DOMAIN

BY

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1. Introduction. In [2] R. H. Bing showed that a closed, connected surface $M$ in $E^3$ is tame if it can be homeomorphically approximated from either side—that is, if for each component $U$ of $E^3 - M$ and each positive number $\varepsilon$ there is a homeomorphism of $M$ into $U$ which moves no point as much as $\varepsilon$. This criterion was used in [3] in showing that a closed, connected surface in $E^3$ is tame if its complement is 1-ULC (uniformly locally simply connected). This result in turn is used in §3 of this paper to show that a closed, connected surface $M$ in $E^3$ is tame provided that it can be deformed into either of its complementary domains by a homotopy which begins with the identity and at each subsequent stage takes $M$ into its complement. In §4 the following situation is considered. Suppose $M$ is a closed, connected surface which is tamely embedded in $S^3$ and $f$ is a map of $S^3$ onto itself which is a homeomorphism relative to $M$ (i.e., $f|_M$ is a homeomorphism and $f(S^3 - M) = S^3 - f(M)$). In Theorem 2 it is shown that $f(M)$ is tame. In Theorem 3 it is shown that, if in addition $M$ is unknotted (i.e., the closure of each complementary domain is a cube with handles) then $f|_M$ can be extended to a homeomorphism of $S^3$ onto itself. In §5 examples are given to show that Theorem 3 fails if $M$ is allowed to be knotted.

Throughout this paper $E^n$ will denote Euclidean $n$-dimensional space, $S^n$ its one point compactification, and $I$ the unit interval. $H_n(X)$ will denote the $n$-dimensional singular homology of $X$ with coefficients in the additive group $Z$ of integers.

All manifolds are assumed to be separable metric and without boundary unless otherwise indicated. A closed manifold is compact and without boundary. The word surface is used interchangeably with 2-manifold. If $M$ is a manifold with boundary, then $Bd(M)$ and $Int(M)$ denote, respectively, the boundary of $M$ and $M - Bd(M)$. The distance function is denoted by $d$.

If $X$ is a subset of a triangulated manifold $M$, then $X$ is said to be tame in $M$ if there is a homeomorphism of $M$ onto itself which takes $X$ onto a polyhedron. Otherwise $X$ is said to be wild.

A cube with $n$ handles is a 3-manifold with boundary which can be obtained

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from the 3-cell by choosing a collection of $2n$ mutually exclusive disks on its boundary and identifying them in pairs in an orientation preserving way. Two cubes with handles are homeomorphic if and only if they have the same number of handles. The boundary of a cube with $n$ handles is a closed, orientable surface of genus $n$.

A metric space $X$ is said to be 1-ULC (uniformly locally simply connected) if for each positive number $\varepsilon$ there is a positive number $\delta$ such that if $D$ is a disk and $f$ is a map of $\text{Bd}(D)$ into a subset of $X$ of diameter less than $\delta$ then $f$ can be extended to a map of $D$ into a subset of $X$ of diameter less than $\varepsilon$.

A map $f$ is said to be essential provided that it is not homotopic to a constant. The restriction of a map $f$ to a subset $A$ of its domain is denoted by $f|A$. If $H: X \times I \to Y$ is a homotopy, we use $H_t: X \to Y$ to denote the map defined by $H_t(x) = H(x,t)$.

2. Preliminaries. The following lemmas will be used in the body of this work.

**Lemma 1.** Suppose $D$ and $E$ are polyhedral disks in $\mathbb{E}^3$ such that $D \cap \text{Bd}(E) = E \cap \text{Bd}(D) = 0$. Let $\eta$ be a non-negative real-valued continuous function defined on $E$ such that for $x \in \text{Int}(E), \eta(x) > 0$. Then there is a piecewise linear homeomorphism $h$ of $\mathbb{E}^3$ onto itself such that for every $x \in E, d(x, h(x)) < \eta(x)$ and such that each component of $D \cap h(E)$ is a simple closed curve in $\text{Int}(D)$.

**Proof.** By [10, Theorem 1] there is a piecewise linear homeomorphism $h_1$ of $\mathbb{E}^3$ onto itself which takes $E$ onto a triangular disk $E'$ in the plane $x_3 = 0$. Let $E''$ and $E'''$ be subdisks of $E'$ which are concentric with $E'$ such that $E'' \subset \text{Int}(E') \subset E''' \subset \text{Int}(E)$. Let $\varepsilon = \min\{\eta(x) \mid x \in h_1^{-1}(E')\}$. Note that $\varepsilon > 0$. There is a positive number $\delta$ such that for $x \in E''$ and $y \in E'''$ with $d(x,y) < \delta$ then $d(h_1^{-1}(x), h_1^{-1}(y)) < \varepsilon$. We also assume that $\delta$ is smaller than $d(E'' - E''', h_1(D))$. There is a number $c, 0 < c < \delta$, such that the plane $x_3 = c$ contains no vertex of $h_1(D)$. Let $\phi$ be the continuous real-valued function defined on the $x_3 = 0$ plane which is 0 outside $E''$, $c$ on $E'''$, and which is linear on the segment of each ray from the center of $E'$ which lies in $E'' - E'''$. We define $h_2: \mathbb{E}^3 \to \mathbb{E}^3$ by $h_2(x_1, x_2, x_3) = (x_1, x_2, x_3 + \phi(x_1, x_2))$.

The homeomorphism $h = h_1^{-1}h_2h_1$ satisfies the conclusion of Lemma 1.

**Lemma 2.** Suppose $S$ is a polyhedral 2-sphere in $\mathbb{E}^3$ and $K$ is a polyhedral disk in $\mathbb{E}^3$ such that $\text{Bd}(K) \cap S = 0$. Then for each $\varepsilon > 0$ there is a piecewise linear homeomorphism $h$ of $\mathbb{E}^3$ onto itself which moves no point as much as $\varepsilon$ and such that each component of $K \cap h(S)$ is a simple closed curve in $\text{Int}(K)$.

The proof of Lemma 2 is a simple variation of the proof of Lemma 1.

**Lemma 3.** Suppose $X$ is a compact continuum in $\mathbb{E}^n (n \geq 2)$ which separates $\mathbb{E}^n$, $U$ is a component of $\mathbb{E}^n - X$, and $A$ is a closed set which lies in $U$. Then there
is a positive number \( \gamma \) such that if \( f \) is any map of \( X \) into \( U \) which moves no point as much as \( \gamma \), then \( f(X) \) separates \( A \) from \( X \) in \( E^n \).

**Proof.** Let \( S \) be the unit \((n-1)\)-sphere in \( E^n \) with center at the origin. For any point \( p \in E^n \) let \( \pi_p : E^n - \{p\} \to S \) be the map given by \( \pi_p(x) = (x - p)/\|x - p\| \).

It is a well-known fact that a compact set \( Y \) in \( E^n \) separates two points \( p \) and \( q \) in \( E^n \) if and only if the maps \( \pi_p|_Y \) and \( \pi_q|_Y \) are not homotopic [7, p. 97].

We suppose without loss of generality that \( A \) is connected.

Take \( p \in A \) and let \( q \) be a point of some component of \( E^n - X \) other than \( U \). Let \( \gamma \) be a positive number less than \( d(X, A \cup \{q\}) \).

Now suppose \( f \) is a map of \( X \) into \( U \) which moves no point as much as \( \gamma \). Note that there may be no such map, in which case the lemma is vacuously satisfied. We first show that \( f(X) \) separates \( p \) from \( q \) in \( E^n \). Suppose not; then the maps \( \pi_p|f(X) \) and \( \pi_q|f(X) \) are homotopic. That is, there is a homotopy \( \phi : f(X) \times I \to S \) such that \( \phi(y, 0) = \pi_p(y) \) and \( \phi(y, 1) = \pi_q(y) \). We define a homotopy \( \psi : X \times I \to S \) by:

\[
\psi(x, t) = \begin{cases} 
\pi_p((1 - 3t)x + 3tf(x)), & 0 \leq t \leq 1/3; \\
\phi(f(x), 3t - 1), & 1/3 \leq t \leq 2/3, \\
\pi_q((3t - 2)x + (3 - 3t)f(x)), & 2/3 \leq t \leq 1.
\end{cases}
\]

Now \( \psi \) is a homotopy connecting \( \pi_p|X \) and \( \pi_q|X \) contrary to the fact that \( X \) separates \( p \) from \( q \) in \( E^n \). Thus \( f(X) \) must separate \( p \) from \( q \) in \( E^n \). Since \( A \) is connected and \( f(X) \cap A = 0, f(X) \) separates \( A \) from \( q \). Finally, since \( f(X) \subset U \) and \( X \) is connected, \( q \) and \( X \) must lie in the same component of \( E^n - f(X) \). Thus \( f(X) \) separates \( A \) from \( X \) in \( E^n \).

Before proceeding with the next lemma, we give a definition of the Brouwer degree of a map. If \( M^n \) and \( N^n \) are closed, connected, orientable, \( n \)-manifolds then \( H_n(M^n) = \mathbb{Z} \) and \( H_n(N^n) = \mathbb{Z} \). If \( f \) is a map of \( M^n \) into \( N^n \) then \( f \) induces a homomorphism \( f^* : H_n(M^n) \to H_n(N^n) \). The integer \( f^*(1) \) is called the Brouwer degree of \( f \). If \( M^n = N^n = S^n \), then \( f \) is essential if and only if the degree of \( f \) is different from 0 [5, p. 304].

**Lemma 4.** Suppose \( L \) is a straight line in \( E^3 \), \( D \) is a disk, and \( f \) is a map of \( \text{Bd}(D \times I) = [\text{Bd}(D) \times I] \cup [D \times \text{Bd}(I)] \) into \( E^3 \) such that for every \( t \in I \), \( f|[(\text{Bd}(D) \times t) \cup [D \times \text{Bd}(I)] \) into \( E^3 - L \). Suppose further that \( r \) is a point of \( L \) such that \( f(D \times 0) \cap L_t = f(D \times 1) \cap L_0 = 0 \), where \( L_0 \) and \( L_1 \) are the closures of the components of \( L - r \). Then \( f \) is an essential map of the 2-sphere \( \text{Bd}(D \times I) \) into \( E^3 - r \).

**Proof.** Let \( R \) be the unit 2-sphere with center at \( r \). Let \( q_0 \) and \( q_1 \) be the points \( L_0 \cap R \) and \( L_1 \cap R \), respectively. Let \( \phi \) be the radial retraction of \( E^3 - r \) onto \( R \) (that is, \( \phi(x) = r + (x - r)/\|x - r\| \)).
We note that for any \( t \in I \), \( \phi f \mid (Bd(D) \times t) \) is homotopic to \( \phi f \mid (Bd(D) \times t) \) in \( E^3 - L \). In particular, \( \phi f \mid (Bd(D) \times t) \) is not homotopic to a constant in \( R - (q_0 \cup q_1) \). Furthermore, since \( \phi^{-1}(q_0) \subset L_0 \) and \( f(D \times 1) \cap L_0 = 0 \), it follows that \( q_0 \notin \phi f(D \times 1) \). Similarly, \( q_1 \notin \phi f(D \times 0) \).

We will show that \( \phi f \) is an essential map of \( Bd(D \times I) \) onto \( R \). We let \( B_0 \) and \( B_1 \) be the hemispheres of \( R \) whose poles are \( q_0 \) and \( q_1 \) respectively. We can find a map \( g : Bd(D \times I) \to R \) such that

\[
g((Bd(D) \times [0, 1/2]) \cup D \times 0) = B_0,
\]

\[
g((Bd(D) \times [1/2, 1]) \cup D \times 1) = B_1,
\]

\( g \) is homotopic to \( \phi f \) on \( R \), and

\[
g \mid (Bd(D) \times 1/2) \text{ is an essential map of } Bd(D) \times 1/2 \text{ onto the equator, } B_0 \cap B_1, \text{ of } R.
\]

We can obtain \( g \) as follows. Let \( C_0 \) and \( C_1 \) be mutually exclusive disks on \( R \) such that \( q_1 \in \text{Int}(C_1) \subset C_1 \subset \text{Int}(B_1) \), \( C_0 \cap \phi f(Bd(D) \times [1/2, 1] \cup D \times 1) = 0 \), and \( C_1 \cap \phi f(Bd(D) \times [0, 1/2] \cup D \times 0) = 0 \). There is a map \( \psi : R \to R \) which expands \( C_i \) onto \( B_i \) and shrinks \( R - (C_0 \cup C_1) \) onto \( B_0 \cap B_1 \). We can choose \( \psi \) so that it is homotopic to the identity by a homotopy which at each stage is fixed on \( q_0 \cup q_1 \) and takes \( R - (q_0 \cup q_1) \) onto itself. The map \( g = \psi \phi f \) will have the desired properties. Now \( g \) takes the hemispheres \( (Bd(D) \times [0, 1/2]) \cup (D \times 0) \) and \( (Bd(D) \times [1/2, 1]) \cup (D \times 1) \) of \( Bd(D \times I) \), respectively, onto the hemispheres \( B_0 \) and \( B_1 \) of \( R \). Hence by [5, p. 304] it follows that the Brouwer degree of \( g \) is the same as the Brouwer degree of \( g \mid (Bd(D) \times 1/2) \). Since \( g \mid (Bd(D) \times 1/2) \) is an essential map of \( Bd(D) \times 1/2 \) onto \( B_0 \cap B_1 \), \( g \), and therefore \( \phi f \), is an essential map of \( Bd(D \times I) \) onto \( R \).

Finally, \( f \) is not homotopic to a constant in \( E^3 - r \); for if \( H : Bd(D \times I) \times I \to E^3 - r \) were a homotopy connecting \( f \) and a constant map, then \( \phi f : Bd(D \times I) \times I \to R \) would be a homotopy connecting \( \phi f \) and a constant map contrary to the fact that \( \phi f \) is essential. This completes the proof of Lemma 4.

3. Conditions under which a surface is tame. It is a well-known fact that a tamely embedded surface can be deformed into either of its complementary domains (in fact, it will have a cartesian product neighborhood). The purpose of this section is to show that the converse is also true. We state this result as follows.

**Theorem 1.** Suppose \( M \) is a closed, connected 2-manifold in \( E^3 \) with the property that for each component \( U \) of \( E^3 - M \) there is a homotopy \( h : M \times I \to \bar{U} \) such that \( h(x, 0) = x \) and \( h(x, t) \in U \) for \( t > 0 \). Then \( M \) is tamely embedded in \( E^3 \).

**Proof.** Bing has shown [3] that \( M \) is tame if \( E^3 - M \) is 1-ULC. Actually, the proof given in [3] does not use the full strength of this hypothesis but only the seemingly weaker condition:
CONDITION A. For each component $U$ of $E^3 - M$ and each $\varepsilon > 0$, there is a number $\delta > 0$ such that each polyhedral unknotted simple closed curve in $U$ of diameter less than $\delta$ can be shrunk to a point in $U$ on a set of diameter less than $\varepsilon$.

We will prove Theorem 1 by showing that Condition A is satisfied.

Suppose that $U$ is a component of $E^3 - M$ and $h : M \times I \to \hat{U}$ is the homotopy promised by the hypothesis of Theorem 1. Let $\varepsilon > 0$ be given. There is a number $\eta > 0$ such that any subset of $M$ of diameter less than $\eta$ lies in a disk in $M$ of diameter less than $\varepsilon/2$. Take $\delta > 0$ so that any polyhedral unknotted simple closed curve in $U$ of diameter less than $\delta$ bounds a polyhedral disk in $E^3$ of diameter less than $\eta/3$. We will show that $\delta$ satisfies the requirements of Condition A.

Suppose $J$ is a polyhedral unknotted simple closed curve in $U$ of diameter less than $\delta$. Then $J$ bounds a polyhedral disk $K$ in $E^3$ of diameter less than $\eta/3$. We can choose $t_0$ small enough that for $0 < t < t_0, h_t(M)$ separates $M$ from $J$ in $E^3$. The existence of $t_0$ is justified by Lemma 3. We may, in addition, suppose that for $0 < t < t_0, d(x, h_t(x)) < \eta/3$; hence the set $\bigcup_{0 \leq t \leq t_0} h_t^{-1}(h_t(M) \cap K)$ has diameter less than $\eta$ and thus lies in a disk $E$ in $M$ of diameter less than $\varepsilon/2$. Note that for $0 < t < t_0, h_t(E)$ separates $J$ from $K \cap M$ on $K$. Let $F$ be a disk on $M$ such that $E = \text{Int}(F)$ and diameter $(F) < \varepsilon/2$.

It follows from [4] that $M$ can be pierced by a tame arc $A$ at a point $x_0$ of $\text{Int}(E)$. We denote $A$ by $p x_0 q$ where $p \in U$ and $q \in E^3 - \hat{U}$. There is a homeomorphism $k$ of $E^3$ onto itself which takes $A$ onto a straight line interval $A'$. Let $L'$ be the straight line obtained by extending $A'$, and let $L = k^{-1}(L)$. Now $L$ may intersect $E$ at points other than $x_0$; however there is a subdisk $D$ of $E$ which contains $x_0$ in its interior and such that $D \cap L = x_0$. In particular, $\text{Bd}(D)$ cannot be shrunk to a point in $E^3 - L$. We choose $t_1$ $(0 < t_1 < t_0)$ with the following properties:

(i) $h(\text{Bd}(F) \times [0, t_1]) \cap h(E \times [0, t_1]) = 0$;
(ii) $h(D \times [0, t_1]) \cap L - A = 0$;
(iii) $h((F - D) \times [0, t_1]) \cap A = 0$;
(iv) for $0 \leq t \leq t_1, h_t(\text{Bd}(D))$ is not homotopic to a constant in $E^3 - L$.

Note that (iv) is actually a consequence of (ii), (iii), and the fact that $\text{Bd}(D)$ cannot be shrunk to a point in $E^3 - L$.

There is a point $p_1$ of $A \cap U$ such that the subarc $p p_1$ of $A$ contains $h_{t_1}(D) \cap L$. The existence of $p_1$ is justified by (ii). Figure 1 illustrates the construction we are making.

By Lemma 3 we choose $t_2$ $(0 < t_2 < t_1)$ so that $h_{t_2}(M)$ separates $M$ from $h_{t_1}(M) \cup p p_1$ in $E^3$. There is a point $p_2$ of $A \cap U$ such that the subarc $p p_2$ of $A$ contains $(h_{t_1}(D) \cup h_{t_2}(D)) \cap L$. Choose $t_3$ $(0 < t_3 < t_2)$ so that $h_{t_1}(M)$ separates $M$ from $h_{t_2}(M) \cup p p_2$ in $E^3$.

Now let $f$ be the map $h | \text{Bd}(D \times [t_3, t_1])$, where $\text{Bd}(D \times [t_3, t_1]) = \text{Bd}(D) \times [t_3, t_1] \cup D \times t_3 \cup D \times t_1$. Let $r$ be a point of $h_{t_2}(D) \cap L$. Note that
$h_t(D) \cap L$ is nonempty since $h_t \mid \partial D$ is not homotopic to a constant in $E^3 - L$.

Let $L_0$ and $L_1$ be the closures of the components of $L - r$ containing $p$ and $q$ respectively.

Now $f(D \times t_1) \cap L = h_{t_1}(D) \cap L \subseteq p_1 \cap L_0 = r_1$, and $f(D \times t_3) \cap L = h_{t_3}(D) \cap L \subseteq p_2 x_0 \subseteq L_1 - r$; so $f(D \times t_1) \cap L_1 = 0$ and $f(D \times t_3) \cap L_0 = 0$.

This, together with (iv), gives us the hypothesis of Lemma 4. Although $L$ is not a straight line, it is taken onto the straight line $L'$ by the space homeomorphism $k$; thus Lemma 4 applies here and we obtain the result that $f$ is not homotopic to a constant in $E^3 - r$.

Let $g$ be the map $h \mid \partial D = h[Bd(F) \times [t_3, t_1]]$. We wish to show that $g$ is not homotopic to a constant in $E^3 - r$. To see this, let $\theta': D \times I \to F$ be an isotopy with the properties that:
\[ \theta'(x,0) = x; \]
\[ \theta'(x,s) \in F - D \text{ for } x \in \text{Bd}(D) \text{ and } s \in I; \text{ and} \]
\[ \theta'_s \text{ is a homeomorphism of } D \text{ onto } F. \]

Now \( \theta' \) induces an isotopy \( \theta: \text{Bd}(D \times [t_3,t_1]) \times I \to F \times [t_3,t_1] \) given by
\[ \theta((x,t),s) = (\theta'(x,s),t); \ (x,t) \in \text{Bd}(D \times [t_3,t_1]), \ s \in I. \]

We define a homotopy \( H: \text{Bd}(D \times [t_3,t_1]) \times I \to E^3 \) by \( H(y,s) = h(\theta(y,s)) \); \( y \in \text{Bd}(D \times [t_3,t_1]), s \in I \). Note that \( H(y,0) = f(y) \) and \( H(y,1) = g(\theta_1(y)) \); furthermore \( \theta(\text{Bd}(D \times [t_3,t_1]) \times I) \subset (F - D) \times [t_3,t_1] \cup F \times t_3 \cup F \times t_1 \), while, by (iii), \( h((F - D) \times [t_3,t_1] \cup F \times t_3 \cup F \times t_1) = 0 \). Hence \( H \) maps into \( E^3 - r \). Now suppose \( g \) is homotopic to a constant in \( E^3 - r \); that is, there is a homotopy \( \phi: \text{Bd}(F \times [t_3,t_1]) \times I \to E^3 - r \) such that \( \phi(y,0) = g(y) \) and \( \phi_1 \) is a constant map. We define a homotopy \( \psi: \text{Bd}(D \times [t_3,t_1]) \times I \to E^3 - r \) by
\[ \psi(y,s) = \begin{cases} 
H(y,2s), & 0 \leq s \leq 1/2, \\
\phi(\theta_1(y), 2s-1), & 1/2 \leq s \leq 1.
\end{cases} \]

Now \( \psi(y,0) = f(y) \) and \( \psi_1 \) is a constant map. This gives the contradiction that \( f \) is homotopic to a constant in \( E^3 - r \), and thus shows that \( g \) is not homotopic to a constant in \( E^3 - r \).

Now \( r \in h_{t_2}(E) \) and by (i) \( g(\text{Bd}(F \times [t_3,t_1])) \cap h_{t_2}(E) = 0 \); thus it follows that \( g \) is not homotopic to a constant in \( E^3 - h_{t_2}(E) \). In particular, \( g \) is not homotopic to a constant in \( U - h_{t_2}(E) \). Thus \( g \) is an essential map of the 2-sphere \( \text{Bd}(F \times [t_3,t_1]) \) into the 3-manifold \( U - h_{t_2}(E) \). It follows from the sphere theorem [9] that there is a polyhedral 2-sphere \( S \) in \( U - h_{t_2}(E) \) which cannot be shrunk to a point in \( U - h_{t_2}(E) \). It follows from the proof of the sphere theorem as given by Papakyriakopoulos that \( S \) may be chosen to lie in any preassigned neighborhood of \( g(\text{Bd}(F \times [t_3,t_1])) \). Now diameter \( (F) < \varepsilon/2 \) and for \( t \leq t_1 \), \( d(x,h_t(x)) < \eta/3 \). Hence we may suppose that diameter \( (S) < 5\varepsilon/6 \), that \( S \cap J = 0 \), and, in light of Lemma 2, that each component of \( S \cap K \) is a simple closed curve.

Since \( \text{Bd}(U) = M \) is connected and \( S \cap M = 0 \), \( M \) lies in a component of \( E^3 - S \). We may suppose that \( \varepsilon \) was chosen smaller than the diameter of \( M \) so that \( M \) must lie in the unbounded component of \( E^3 - S \). Thus the bounded component of \( E^3 - S \) lies in \( U \). Since \( S \) cannot be shrunk to a point in \( U - h_{t_2}(E) \), and since \( h_{t_2}(E) \) is connected, it follows that \( h_{t_2}(E) \) lies in the bounded component of \( E^3 - S \). Then \( S \) must separate \( J \) from \( M \cap K \) on \( K \); for if not there is an arc \( B \) from \( J \) to \( M \) in \( K - S \). In this case \( B \) must lie in the unbounded component of \( E^3 - S \) since \( M \) is in this component. But then \( B \) misses \( h_{t_2}(E) \) contrary to the fact that \( h_{t_2}(E) \) separates \( J \) from \( M \cap K \) on \( K \). Figure 2 illustrates the situation that we have obtained.
Let $K'$ be the closure of the component of $K - S$ which contains $J$. Then $K' \subset U$ and each component of $K' \cap S$ is a simple closed curve. Each of these simple closed curves can be shrunk to a point in $S$; hence $J$ can be shrunk to a point in $K' \cup S \subset U$. Finally, diameter $(K' \cup S) < \eta/3 + 5\varepsilon/6 < \varepsilon$. Thus we have shown that $J$ can be shrunk to a point in $U$ on a set of diameter less than $\varepsilon$; hence Condition A is satisfied and the proof of Theorem 1 is complete.

**Question.** Is the full hypothesis of Theorem 1 required? For example, if $M$ is a closed, connected 2-manifold in $E^3$ such that for each component $U$ of $E^3 - M$ and each positive number $\varepsilon$ there is a map of $M$ into $U$ that moves no point as much as $\varepsilon$, is $M$ necessarily tame? It can be shown under these conditions that the complement on $M$ is "nice." So if, for example, $M$ is a 2-sphere, then the components of $E^3 - M$ are an open 3-cell and an open 3-cell minus a point, respectively. However, it is known (see [6, Example 3.2] for example) that this is not enough to insure that $M$ is tame.

All example of wild surfaces known to the author have the property that there is a component $U$ of $E^3 - M$, a number $\delta > 0$, a sequence $\{J_i\}$ of simple closed curves on $M$ with $\lim(\text{diameter}(J_i)) = 0$, and a sequence $\{\eta_i\}$ of positive numbers such that any simple closed curve in $U$ which is homeomorphically within $\eta_i$ of $J_i$ cannot be shrunk to a point in $U$ on a set of diameter less than $\delta$. A surface satisfying the hypothesis of the above question cannot have this property.
4. The image of a tame surface under a relative homeomorphism.

**Theorem 2.** Suppose $M$ is a closed, connected 2-manifold which is tamely embedded in $E^3$ and $f$ is a map of $E^3$ onto itself such that $f|_M$ is a homeomorphism and $f(E^3 - M) = E^3 - f(M)$. Then $f(M)$ is tamely embedded in $E^3$.

**Proof.** Let $M' = f(M)$ and $U'$ be a component of $E^3 - M'$. Let $U$ and $V$ be the components of $E^3 - M$. By assumption neither $f(U)$ nor $f(V)$ intersect $M'$. Since each of $f(U)$ and $f(V)$ is connected they must each lie in some component of $E^3 - M'$. Since $f$ maps onto $E^3$, these sets must be the components of $E^3 - M'$. We suppose that the notation is chosen so that $f(U) = U'$.

Since $M$ is tame, it has a cartesian product neighborhood [8, Lemma 1]. That is, there is a homeomorphism $g: M \times [-1,1] \rightarrow E^3$ such that $g(x,0) = x$ for each $x \in M$. We suppose that $g(M \times (0,1]) \subset U$. We define a homotopy $h: M' \times I \rightarrow \bar{U}'$ by $h(y,t) = f(g(f^{-1}(y),t))$, $y \in M'$, $t \in I$. Then $h(y,0) = y$ for every $y \in M'$ and for $t > 0$, $h(y,t) \in U'$. Thus the hypothesis of Theorem 1 is satisfied, and we conclude that $M'$ is tame.

**Corollary.** The same result holds if we replace the hypothesis that $f$ be an onto map by the assumption that $f(U)$ and $f(V)$ lie in different components of $E^3 - M'$, where $U$ and $V$ are the components of $E^3 - M$.

The proof is the same as that of Theorem 2.

Theorems 1 and 2 were stated for $E^3$, although they obviously remain valid for the 3-sphere $S^3$. The following theorem is stated in terms of $S^3$ since it has a more symmetric form in this setting.

It is a well-known fact that if $M$ and $M'$ are tamely embedded 2-spheres in $S^3$, then any homeomorphism of $M$ onto $M'$ can be extended to a homeomorphism of $S^3$ onto itself. The corresponding statement is clearly false for surfaces of positive genus. The following theorem shows that by placing some restrictions both as to the placement of $M$ in $S^3$ and as to the nature of the homeomorphism of $M$ onto $M'$ that this result can be generalized to closed, connected surfaces of any genus.

**Theorem 3.** Suppose $M$ is a closed, connected, tame 2-manifold in $S^3$ such that the closure of each component of $S^3 - M$ is a cube with handles. If $f$ is a map of $S^3$ onto itself such that $f|_M$ is a homeomorphism and $f(S^3 - M) = S^3 - f(M)$, then $f|_M$ can be extended to a homeomorphism of $S^3$ onto itself.

**Proof.** It follows by Theorem 2 that $f(M)$ is tame. We assume without loss of generality that $M$ and $M' = f(M)$ are polyhedra.

Let $U$ be a component of $S^3 - M$. As was noted in the proof of Theorem 2, $f(U)$ is a component of $S^3 - M'$. We denote this component by $U'$. Now $\bar{U}$ is a polyhedral cube with handles whose boundary is $M$. We will obtain an extension.
The map $F$ given by $F(x) = F_1(x), x \in U$, and $F(x) = F_2(x), x \in V'$, will be the desired homeomorphism of $S^3$ onto itself. Thus the proof of Theorem 3 will be completed by the following theorem.

**Theorem 4.** Suppose $C$ and $C'$ are polyhedral 3-manifolds with boundary in $S^3$ such that $C$ is a cube with handles and such that there is a map $f$ of $C$ onto $C'$ which takes $\partial C$ homeomorphically onto $\partial C'$. Then $C$ and $C'$ are homeomorphic; in particular, $f|\partial C$ can be extended to a homeomorphism of $C$ onto $C'$.

**Proof.** Let $C$ be of genus $n$. Then $C$ is the union of a polyhedral cube $C_0$ and $n$ mutually exclusive polyhedral cubes $C_1, C_2, \ldots, C_n$ such that $C \cap C_0$ consists of two mutually exclusive polyhedral disks, $D_{i1}$ and $D_{i2}$, on the boundary of each. Note that

$$
\partial C = \partial C_0 \cup \cdots \cup \partial C_n - (\text{Int}(D_{11}) \cup \text{Int}(D_{12}) \cup \cdots \cup \text{Int}(D_{n2})),
$$

and that $D_{ij} \cap \partial C = \text{Int}(D_{ij})$. We break the proof into several steps.

**Step 1.** First we note a well-known property of $n$-cells. If $K$ and $K'$ are two $n$-cells and $f$ is a homeomorphism of $\partial K$ onto $\partial K'$, then $f$ can be extended to a homeomorphism of $K$ onto $K'$.

**Step 2.** It follows from [8, Lemma 1] that there is a piecewise linear homeomorphism $g: \partial C' \times I \to C'$ such that $g(x,0) = x$ for every $x \in \partial C'$. To simplify notation we will suppose that $\partial C' \times I$ is embedded in $C'$ and we will denote the point $g(x,t)$ simply by $(x,t)$.

Since $g$ is piecewise linear, $P \times I$ will be a polyhedron in $C'$ whenever $P$ is a polyhedron in $\partial C'$. We define a homeomorphism $\phi$ of $C'$ into $C'$ by

$$
\phi(y) = \begin{cases} 
(x, 1/2 + 1/2t), & y = (x, t) \in \partial C' \times I, \\
(0, y), & y \in C' - \partial C' \times I.
\end{cases}
$$

The effect of $\phi$ is to shrink $C'$ onto a polyhedral manifold contained in $\text{Int}(C')$.

**Step 3.** Let $J_{ij} = \text{Int}(D_{ij})$ and $J'_{ij} = f(J_{ij})$. We show that there is no loss of generality in assuming that each $J'_{ij}$ is a polygon. Let $A$ be an annular neighborhood of $J'_{11}$ on $\partial C'$ which does not intersect any other $J'_{ij}$ and such that $J'_{11}$ circles $A$ exactly once. There is a polygonal simple closed curve $K_{11}$ in $\text{Int}(A)$ which circles $A$ exactly once and which contains at least two points $p_1$ and $p_2$ in common with $J'_{11}$. Let $L_1$ and $L_2$ be the components of $\partial A$. Let $s_1$ and $s_2$ be two points of $L_1$, and let $t_1$ and $t_2$ be two points of $L_2$. Now $A - J'_{11}$ has two components and the closure of each is an annulus. Thus there is an arc $\alpha_1$ from $s_1$ to $t_1$ such that $\text{Int}(\alpha_1) \subset \text{Int}(A)$ and $\alpha_1 \cap J'_{11} = p_1$. Similarly there is an arc $\alpha_2$ from $s_2$ to $t_2$ such that $\text{Int}(\alpha_2) \subset \text{Int}(A) - \alpha_1$ and $\alpha_2 \cap J'_{11} = p_2$. In the same
fashion there is a polygonal arc $\beta_1$ from $s_1$ to $t_1$ such that $\text{Int}(\beta_1) \subseteq \text{Int}(A)$ and $\beta_1 \cap K_{11} = p_1$. By constructing $\beta_1$ so as to circle $A$ an appropriate number of times near $\text{Bd}(A)$, we can insure that the simple closed curve

$$\beta_1 \cup [(s_1 \cup t_1) \times I] \cup [x_1 \times 1]$$

can be shrunk to a point in $A \times I$. There is a polygonal arc $\beta_2$ from $s_2$ to $t_2$ such that $\text{Int}(\beta_2) \subseteq \text{Int}(A - \beta_1)$ and such that $\beta_2 \cap K_{11} = p_2$. Now $A - (x_1 \cup x_2)$ has two components. Let $E_1$ and $E_2$ denote the closures of these components. Similarly, let $F_1$ and $F_2$ denote the closures of the two components of $A - (\beta_1 \cup \beta_2)$. We assume the notation is chosen so that $E_i \cap \text{Bd}(A) = F_i \cap \text{Bd}(A)$.

There is a homeomorphism $h_1$ of $\text{Bd}(A) \cup J_1' \cup x_1 \cup x_2$ onto $\text{Bd}(A) \cup K_{11} \cup \beta_1 \cup \beta_2$ which is the identity on $\text{Bd}(A)$ and which takes $J_1' \cap E_i$ onto $K_{11} \cap F_i$ ($i = 1, 2$). Now $A - (J_1' \cup x_1 \cup x_2)$ has exactly four components and the closure of each is a disk. If $G$ is one of these components, then $G$ lies in some $E_i$ and $h_1(\text{Bd}(G))$ bounds a disk in the corresponding $F_i$. We can apply Step 1 to each such component to extend $h_1$ to a homeomorphism $h_2$ of $A$ onto itself which is the identity on $\text{Bd}(A)$ and which takes $J_1'$ onto $K_{11}$. We can extend $h_2$ by the identity on $(\text{Bd}(A) \times I) \cup (A \times 1)$ to obtain a homeomorphism of $\text{Bd}(A \times I)$ onto itself. It follows from Dehn’s lemma [9] that the polygonal simple closed curve $\beta_1 \cup [(s_1 \cup t_1) \times I] \cup [x_1 \times 1]$ bounds a polyhedral disk $X_1$ whose interior lies in $\text{Int}(A \times I)$. Similarly, $\beta_2 \cup [(s_2 \cup t_2) \times I] \cup [x_2 \times 1]$ bounds a polyhedral disk $X_2$ whose interior lies in $\text{Int}(A \times I) - X_1$. We can use Step 1 to extend $h_2$ to map $x_1 \times I$ homeomorphically onto $X_1$. Again applying Step 1 to each component of $A \times I - (x_1 \cup x_2) \times I$ we obtain a homeomorphism $h_3$ of $A \times I$ onto itself such that $h_3 \mid \text{Bd}(A) = h_2$. Extending by the identity outside $A \times I$, we obtain a homeomorphism of $C'$ onto itself. By repeating this process for each $J_1'i$ we obtain a homeomorphism $h$ of $C'$ onto itself with the property that each $h(J_1'i)$ is a polygon. Now $hf$ satisfies the hypothesis of Theorem 4 and $hf(J_1'i)$ is a polygon. If there is a homeomorphism $F$ of $C$ onto $C'$ which extends $hf \mid \text{Bd}(C)$, then $h^{-1}F$ is a homeomorphism of $C$ onto $C'$ which extends $f \mid \text{Bd}(C)$. This justifies our assumption; we assume without any change in notation that each $J_1'i$ is a polygon.

**Step 4.** Now $f$ takes $D_{ij}$ onto a singular disk in $C'$ whose boundary is the polyhedral singular disk $J_1'i$. $f \mid D_{ij}$ can be replaced by a piecewise linear map $f_{ij}$ which agrees with $f$ on $J_{ij}$. Let $E_{ij} = J_1'i \times [0, 1/2] \cup \phi_{ij}(D_{ij})$, where $\phi$ is the homeomorphism described in Step 2. Since $f_{ij}$ and $\phi$ are piecewise linear, $E_{ij}$ is a polyhedral singular disk in $C'$ whose boundary is $J_1'i$. Furthermore $E_{ij}$ has no singularities near its boundary; that is, there is a neighborhood of $J_{ij}$, namely $C' - \phi(C')$, which intersects $E_{ij}$ in the annulus $J_{ij} \times [0, 1/2]$. Thus the hypotheses of Dehn’s lemma [9] are satisfied. Hence there is a nonsingular polyhedral disk $F_{ij}$ in $C'$ whose boundary is $J_1'i$. Since $F_{ij}$ can be chosen to lie in
Step 5. Now the $F_{ij}$'s may intersect one another. We wish to replace each $F_{ij}$ by a disk having the same boundary such that the resulting disks are mutually exclusive.

Let $D'_{11} = F_{11}$. We apply Lemma 1 to $F_{12}$ with $\eta(x) = d(x, \text{Bd}(C'))$ and obtain a piecewise linear homeomorphism $h$ of $S^3$ onto itself such that $h(F_{12}) \cap \text{Bd}(C') = \text{Bd}(h(F_{12})) = J'_{12}$, and such that each component of $D'_{11} \cap h(F_{12})$ is a simple closed curve in $\text{Int}(D'_{11})$. Let $J$ be an interior (with respect to $D'_{11}$) simple closed curve in $D'_{11} \cap h(F_{12})$. That is, $J$ bounds a disk $D$ in $D'_{11}$ whose interior contains no points of $h(F_{12})$. Let $E$ be the disk in $h(F_{12})$ bounded by $J$. Since $D'_{11}$ has a cartesian product neighborhood, the disk $(h(F_{12}) - E) \cup D$ can be pushed to one side of $D'_{11}$ in a neighborhood of $D$ to obtain a new disk whose intersection with $D'_{11}$ has fewer components than does $D'_{11} \cap h(F_{12})$. We can continue this process to obtain a polyhedral disk $F'_{12}$ which does not intersect $D'_{11}$. $F'_{12}$ is chosen so that $F'_{12} \cap \text{Bd}(C') = \text{Bd}(F'_{12}) = J_{12}$. We repeat the above process to replace each $F_{ij}$ ($i \geq 2$) by a polyhedral disk $F'_{ij}$ which does not intersect $D'_{11}$.

Let $D'_{12} = F'_{12}$. We repeat the above process to replace each $F'_{ij}$ ($i \geq 2$) by a polyhedral disk $F''_{ij}$ which does not intersect $D'_{12}$. Since $F_{ij} \cap D'_{11} = 0$, $D'_{11} \cap D'_{12} = 0$, and $F''_{ij}$ can be chosen to lie in $F''_{ij}$ plus any neighborhood of $D'_{12}$, $F''_{ij}$ may be chosen so as to miss both $D'_{11}$ and $D'_{12}$. We let $D'_{21} = F'_{21}$. We can continue in this manner to obtain a collection $\{D'_{ij}\}$ of mutually exclusive polyhedral disks in $C'$ such that $D'_{ij} \cap \text{Bd}(C') = \text{Bd}(D'_{ij}) = J'_{ij}$.

Step 6. We have now divided $C'$ by the $D'_{ij}$'s just as $C$ is divided by the $D_{ij}$'s. Let $S'_0 = f(\text{Bd}(C_0) - \bigcup D_{ij}) \cup \bigcup D'_{ij}$, and for $i \geq 1$ let

$$S'_i = f(\text{Bd}(C_i) - \{D_{ij} \cup D'_{ij}\}) \cup D'_{ij} \cup D'_{ij}.$$  

Now $S'_i (0 \leq i \leq n)$ is a polyhedral 2-sphere. By Alexander's sphere theorem [1], $S'_i$ bounds two 3-cells in $S^3$. Since $S'_i \cap \text{Bd}(C') \neq 0$, exactly one of these, which we denotes by $C'_i$, lies in $C'$; moreover $C' = \bigcup C'_i$, and $\text{Int}(C'_i) \cap \text{Int}(C'_j) = 0$ for $i \neq j$. By Step 1 we can extend $f \vert J_{ij}$ to a homeomorphism $g_{ij}$ taking $D_{ij}$ onto $D'_{ij}$. This gives us a homeomorphism $G$ of $\bigcup \text{Bd}(C'_i)$ onto $\bigcup \text{Bd}(C'_i)$ given by:

$$G(x) = \begin{cases} f(x), & x \in \text{Bd}(C), \\ g_{ij}(x), & x \in D_{ij}. \end{cases}$$

Again by Step 1, $G \vert \text{Bd}(C_i)$ can be extended to a homeomorphism $G_i$ of $C_i$ onto $C'_i$. The map $F: C \to C'$ given by $F(x) = G_i(x), x \in C_i$, is a homeomorphism of $C$ onto $C'$ with the property that $F \vert \text{Bd}(C) = f \vert \text{Bd}(C)$. This completes the proof of Theorem 4.

5. Examples. In this section we construct examples to show that the hypothesis in Theorem 3 that the closure of each component of $S^3 - M$ be a cube
with handles is indeed necessary. This may be given the following interpretation. While Theorem 3 states that it is impossible to take an "unknotted" surface $M$ onto a "knotted" one by a map of $S^3$ which is a homeomorphism relative to $M$, it is possible to take a "knotted" surface onto an "unknotted" one. The existence of examples to support this statement is provided by the following theorem.

**Theorem 5.** Let $M$ be a tame torus (genus 1) in $S^3$ and let $M'$ be a tame unknotted torus in $S^3$ (that is, the closure of each component of $S^3 - M'$ is a cube with one handle). Then there is a map $f$ of $S^3$ onto itself such that $f|_M$ is a homeomorphism of $M$ onto $M'$ and such that $f(S^3 - M) = S^3 - M'$.

**Proof.** We assume without loss of generality that $M$ and $M'$ are polyhedra. Let $U$ and $V$ be the components of $S^3 - M$. It follows from [1] that the closure of at least one of these components, say $U$, is a solid torus (cube with one handle).

We may choose simple closed curves $a$ and $b$ on $M$ which generate $H_1(M)$ and such that $a$ bounds a disk in $U$. We consider the following segment of the Mayer-Vietoris sequence of the proper triad $(M; U, V)$:

$$H_1(S^3) \cong H_1(U) \oplus H_1(V) \cong H_1(M) \cong H_2(S^3).$$

Since $H_1(S^3) = H_2(S^3) = 0$, $\psi$ is an isomorphism of $H_1(M)$ onto $H_1(U) \oplus H_1(V)$. Each of $H_1(U)$ and $H_1(V)$ is infinite cyclic. Let these groups be generated by $a$ and $b$, respectively. For any $\gamma \in H_1(M)$, $\psi(\gamma) = (i^*(\gamma), -j^*(\gamma))$, where $i$ and $j$ are the injection maps of $M$ into $U$ and $V$, respectively. Since $i^*(a) = 0$, we may put $\psi(a) = (0, rb)$ and $\psi(b) = (pa, qb)$, where $p$, $q$, and $r$ are integers. Since $\psi$ is an isomorphism, it follows that $p = r = 1$. Let $\gamma$ be a simple closed curve on $M$ which is homologous to $-qa + b$. Then $\psi(\gamma) = (a, -qb + qb) = (a, 0)$; so $j^*(\gamma) = 0$. Thus we have found a simple closed curve $\gamma$ on $M$ which is not homologous to 0 on $M$ (and, hence does not separate $M$), but which is homologous to 0 in $V$. Hence $\gamma$ bounds an orientable, polyhedral 2-manifold $K$ which lies, except for its boundary, in $V$.

Let $U'$ and $V'$ be the components of $S^3 - M'$. We may choose simple closed curves $a'$ and $b'$ on $M'$ which generate $H_1(M')$ and so that $a'$ bounds a disk in $U'$, and $b'$ bounds a disk in $V'$.

There is a homeomorphism $f$ of $\bar{U}$ onto $\bar{U}'$ such that $f(\gamma)$ is homologous to $\beta'$ on $M'$. One may obtain $f$ as follows. Let $h$ be any homeomorphism of $\bar{U}$ onto $\bar{U}'$ such that $h(a) = a'$. Such homeomorphisms exist since $a$ and $a'$ bound disks in $\bar{U}$ and $\bar{U}'$, respectively. Then $h(\beta)$ is homologous on $M'$ to a simple closed curve of the form $n a' + b'$. We suppose the orientation on $M'$ is chosen so that $h(\beta)$ is homologous to $n a' + b'$. Since $\gamma$ is represented by $-qa + b$, $h(\gamma)$ is homologous to $(n - q)a' + b'$ on $M'$. Let $g$ be a homeomorphism of $\bar{U}'$ onto itself obtained by cutting $\bar{U}'$ along the disk bounded by $a'$, rotating one of the free ends through $-(n - q)$ rotations, and then rejoining along this disk. Then $f = gh$ will be the
desired homeomorphism. We may assume that $\alpha, \beta, \alpha', \text{ and } \beta'$ are polyhedra, and that $f$ is piecewise linear.

Since $f(\gamma)$ is homologous to $\beta'$ on $M'$, $f(\gamma)$ bounds a polyhedral disk $K'$ which lies, except for its boundary, in $V'$. Now by [8, Lemma 1] there is a piecewise linear homeomorphism $k$ of $K \times I$ into $\mathcal{P}$ such that $k(x, 0) = x$, $k(\partial(K) \times I) \subset M$, and $k(\text{Int}(K) \times I) \subset V$. Let $A$ be the annulus $k(\gamma \times I)$ on $M$, and let $A' = f(A)$. Let $\gamma_1 = k(\gamma \times 1)$, and let $K_1 = k(K \times 1)$. Figure 3 illustrates this construction in a particular instance.

Now $f(\gamma_1)$ bounds a polyhedral disk $K_1'$ which lies except for its boundary in $V'$. By applying the technique of Step 5 of the proof of Theorem 4, we may assume that $K_1' \cap K' = 0$. We can apply the Tietze extension theorem to extend $f\big|_{\gamma}$ to a map of $K$ onto $K'$. Similarly we can extend $f\big|_{\gamma_1}$ to a map of $K_1$ onto $K_1'$. We thus have a map of $\partial(k(K \times I))$ onto $K' \cup K_1' \cup A'$. Now $K' \cup K_1' \cup A'$ is a polyhedral 2-sphere which bounds a 3-cell $C$ in $\mathcal{P}'$. Again by the Tietze
extension theorem, we may extend to get a map, which we continue to denote by $f$, which takes $k(K \times I)$ onto $C'$. By the same process we may further extend $f$ to take $V - k(K \times I)$ onto the 3-cell $\tilde{V}' - \tilde{C}'$. We thus have a map $f$ of $S^3$ onto itself which takes $M$ homeomorphically onto $M'$, which takes $U$ onto $U'$ and which takes $V$ into $\tilde{V}'$. Since both $M$ and $M'$ have cartesian product neighborhoods we can alter the above process in such a way that will insure that $f$ takes $V$ onto $V'$. Once this is done the resulting map will satisfy the conclusion of Theorem 5.

REFERENCES

2. R. H. Bing, *Conditions under which a surface in $E^3$ is tame*, Fund. Math. 47 (1959), 105-139.

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