A CLASS OF NILSTABLE ALGEBRAS

BY
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1. Introduction. In what follows we shall consider strictly power-associative algebras over a field of characteristic different from two and three that satisfy the identity

\[(1) x(xa) + (ax)x = 2(xa)x,\]

whose equivalent linearized form is

\[(2) x(ya) + y(xa) + (ay)x + (ax)y = 2(ya)x + 2(xa)y.\]

Rosier has shown [10] that every semisimple algebra of this class has an identity and is the direct sum of simple algebras. Moreover, the simple algebras of degree greater than two are Jordan or quasiassociative. Hereafter let \(A\) be a central simple degree two algebra of this class. Then \(A\) has an identity, \(1 = e + f\), that is, the sum of two orthogonal primitive idempotents, \(e\) and \(f\), every scalar extension \(A_K\) of \(A\) is simple, and \(e\) and \(f\) are primitive in every \(A_K\).

It is known [2; 3] that \(A\) can be decomposed relative to \(e\) into a vector space direct sum \(A = A(1) + A(\frac{1}{2}) + A(0)\), where \(e \cdot x = \frac{1}{2}(ex + xe) = \lambda x\) for all \(x\) in \(A(\lambda), \lambda = 1, \frac{1}{2}, 0\). Moreover \(A(1) = eF + N_1\) and \(A(0) = fF + N_0\) are vector space decompositions in which \(N_1^+\) and \(N_0^+\) are nilsubalgebras of the algebra \(A^+\). \((A^+\) is the same linear space as \(A\) with the multiplication \(x \cdot y = \frac{1}{2}(xy + yx)\).) \(A(1)^+\) and \(A(0)^+\) are orthogonal subalgebras of \(A^+, e\) is a two-sided identity in \(A(1)\) and a two-sided annihilator of \(A(0),\)

\[xy = yx = 0, \text{ for all } x \text{ in } A(1) \text{ and } y \text{ in } A(0),\]

\[A(1) \cdot A(\frac{1}{2}) \subset A(\frac{1}{2}) + A(0),\]

\[A(0) \cdot A(\frac{1}{2}) \subset A(\frac{1}{2}) + A(1),\]

and, for each \(x\) and \(y\) in \(A(\frac{1}{2}),\) there is an \(\alpha\) in \(F\) and \(n\) in \(N_1 + N_0\) for which \(x \cdot y = \alpha 1 + n\).

Without further information about the multiplication of subspaces little can be predicted even for commutative power-associative algebras. L. A. Kokoris has

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(T) This paper contains the results of the author's Ph. D. thesis submitted at Illinois Institute of Technology.
shown [5] that simple commutative power-associative algebras of degree two exist that are not Jordan. On the other hand let us define \( A \) to be nilstable(2) in case \( A_u(\lambda) A_u(\tau) \) and \( A_u(\lambda) A_u(\tau) = A_u(\lambda + \tau) + N_u(1 - \lambda) \), \( \lambda, \tau = 0,1 \), where \( u \) is any idempotent in \( A \), \( A_u(\lambda) \), \( \lambda = 0,1, \) are the subspaces in the decomposition of \( A \) relative to \( u \), and \( N_u(1 - \lambda) \) is the nilsubspace of \( A_u(1 - \lambda) \), \( \lambda = 0,1 \). Then nilstable simple commutative power-associative algebras were shown by Kokoris [8] to be Jordan. In fact, the added assumption of nilstability has produced the solution of the degree two flexible algebras [9]. We shall therefore assume hereafter that \( A \) is nilstable.

Lemmas 2.4 and 2.6 will show that \( A \) is nilstable if and only if \( A^+ \) is nilstable, and Theorem 6 in [6] states that \( A^+ \) is nilstable whenever \( F \) has characteristic zero, so that we shall also have described all algebras of degree two and characteristic zero. Actually, by these same lemmas it suffices to assume that \( A_u(\lambda) A_u(\tau) \) and \( A_u(\lambda) A_u(\tau) \) are contained in \( A_u(\lambda + \tau) + N_u(0) + N_u(1) \) for every idempotent \( u \) and \( \lambda, \tau = 0,1 \).

The principal result will be that \( A \) is a noncommutative Jordan algebra and so possesses a known [9] multiplication table. We shall obtain this result by showing that \( A^+ \) is Jordan. Then, by considering a certain trace-like function on \( A \), we shall show that \( A^+ \) is simple and that \( A \) satisfies the flexible law \( x(yx) = (xy)x \).

Being Jordan admissible and flexible \( A \) is known [12] to satisfy the Jordan identity \( x^2(yx) = (x^2y)x \). In case \( A \) is not nilstable the algebras constructed by Kokoris provide examples of algebras of our class that do not satisfy the Jordan identity even though they are nilstable with respect to at least one idempotent. Finally, as a consequence of (1) and the flexible law, \( A \) satisfies \( x(xy) + (yx)x = x(yx) + (xy)x \). Moreover, we can write each \( x \) in \( A \) as \( x = \alpha_1 e + x^{1/2} + \alpha_0 f \) with \( x^{1/2} = \beta_1 \) and then define \( t(x) = \alpha_0 + \alpha_1 \) and \( n(x) = \alpha_0 \alpha_1 - \beta_1 \). Then \( A \) is a quadratic algebra and the results of [13] apply.

2. Construction of an ideal in \( A \). Let \( A \) be decomposed relative to the idempotent \( e \). We shall show that \( N_1 + N_0 = 0 \) by showing that each \( N_x \) is an ideal of \( A(\lambda) \) and then constructing a proper ideal of \( A \) containing \( N_1 + N_0 \). The first few propositions below repeat results found in [10].

**Lemma 2.1.** If \( x \) is in \( A(\lambda) \), then \( ex \) is in \( A(\lambda) \).

In case \( A \) is commutative \( ex = \frac{1}{2} x \). Here let \( ex = a_1 + (\frac{1}{2} x + x^*) + a_0 \) with \( a_x \) in \( A(\lambda) \) and \( x^* \) in \( A(\frac{1}{2}) \), then \( xe = -a_1 + (\frac{1}{2} x - x^*) - a_0 \), since \( ex + xe = x \).

Similarly, write \( ex^* = b_1 + (\frac{1}{2} x^* + x^{**}) + b_0 \). Now (1), with \( x \) and \( a \) replaced by \( e \) and \( x \), respectively, becomes \( ex^* = a_1 - a_0 - x^* + 3x^*e \) so that \( ex^* = \frac{1}{2} a_1 + \frac{1}{2} x^* - \frac{3}{4} a_0 \). Consequently \( x^{**} = 0 \). Apply (1) again by replacing \( a \) and \( x \) by \( x^* \) and \( e \) respectively and obtain \( 0 = ex^{**} = (1/16) a_1 + (1/16) a_0 \). By equating corresponding \( A(\lambda) \) components, \( a_1 = a_0 = 0 \). Thus, for all \( x \) in \( A(\frac{1}{2}) \), we can write

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(2) The results of this paper for the stable case were announced by Kosier in Abstract 61T-296, Notices Amer, Math. Soc. 8 (1961), 618.
\[ e x = \frac{1}{2} x + x^*, \]
\[ e x = \frac{1}{2} x - x^*, \]
\[ e x^* = x^* e = \frac{1}{2} x^*. \]

Consequently \((e x)e = \frac{1}{2} x = e(xe)\) so that we have proved

**Lemma 2.2.** Every \(x\) in \(A(\frac{1}{2})\) is of the form \(e y\) and \(z e\) for some \(y\) and \(z\) in \(A(\frac{1}{2})\).

**Lemma 2.3.** If \(e x = a x\) for some \(x\) in \(A(\frac{1}{2})\) and \(a \neq \frac{1}{2}\) in \(F\), then \(x = 0\).

For now (1), with \(a\) and \(x\) replaced by \(x\) and \(e\), respectively, becomes \((2a - 1)^2 x = 0\), so that \(x = 0\).

**Theorem 2.1.** The subspaces \(A(1)\) and \(A(0)\) are subalgebras of \(A\).

Let \(x\) and \(y\) be in \(A(1)\) and write \(x y\) as the sum of its \(A(\lambda)\) components, \(x y = a_1 + a_{1/2} + a_0\), so that \(y x = b_1 - a_{1/2} - a_0\). Then (2), with \(x\) in the asymmetric position and \(y\) and \(e\) in the others, becomes \((a_{1/2} + 4a^*) + 2a_0 = 0\), so that \(a_0 = 0\) and \(e a_{1/2} = \frac{1}{4} a_{1/2}\), implying \(a_{1/2} = 0\). Similarly, for \(x\) and \(y\) in \(A(0)\), call \(x y = a_1 + a_{1/2} + a_0\) and \(y x = -a_1 - a_{1/2} + b_0\). Again giving \(x\) the distinguished position in (2) produces \(a_1 = a_{1/2} = 0\).

Continuing in this vein, let \(x\) be in \(A(1)\) and \(y\) in \(A(\frac{1}{2})\) and write \(x y = a_1 + a_{1/2} + a_0\) and \(y x = -a_1 + b_{1/2} + b_0\). At the same time call \(x y^* = r_1 + r_{1/2} + r_0\) and \(y^* x = -r_1 + s_{1/2} + s_0\).

**Lemma 2.4.** For each \(x\) in \(A(1)\) and \(y\) in \(A(\frac{1}{2})\), \(x y = a_{1/2} + a_0\), \(y x = (a_{1/2} - 4a^*) + a_0\), and \(x y^* = y^* x = a^*\).

Replacing \(x, y,\) and \(a\) in (2) by \(e, y,\) and \(x,\) respectively, and using (3) results in

\[ \frac{1}{2}(b_{1/2} - a_{1/2}) + (b_0 - a_0) = a^* - 3b^*, \]

so that, by equating corresponding components,

\[ a_0 = b_0, \]

and

\[ b_{1/2} - a_{1/2} = 2a^* - 6b^*. \]

Multiplying (4) on the left by \(e\) produces

\[ b_{1/2} - a_{1/2} = 4a^* - 8b^*. \]

From (6) and (7) it is apparent that

\[ a^* = b^*. \]

Applying (2) to \(x, y,\) and \(e\) again, this time with \(y\) in the asymmetric position, and using (5) and (8) produces
\[ a_1 + 2a^* = 3y^*x - xy^*, \]
while replacing \( a \) with \( e \) results in
\[ a_1 + \frac{1}{2}(a_{1/2} - b_{1/2}) = 3y^*x - xy^*, \]
so that
\[ a_1 = 0 \text{ and } b_{1/2} = a_{1/2} - 4a^*. \]

Making the same computations with \( x, y^*, \) and \( e \) will therefore result in
\[ r_1 = 0, \quad r_0 = s_0, \quad \text{and } s_{1/2} = r_{1/2} - 4r^*, \]
that is, \( xy^* = r_{1/2} + r_0 \) and \( y^*x = (r_{1/2} - 4r^*) + r_0. \)

However, if we consider (2) in the light of these results and with \( y^* \) in the select position we find
\[ xy^* - y^*x = 4r^* = 0, \]
so that
\[ xy^* = y^*x = r_{1/2} + r_0. \]

But then (9) with the use of (10) and (11) becomes
\[ 2a^* = 3y^*x - xy^* = 2(r_{1/2} + r_0), \]
and consequently, by equating corresponding components, \( r_{1/2} = a^* \) and \( r_0 = 0. \)

**Lemma 2.5.** For every \( x \) in \( A(1) \) and \( y \) in \( A(\frac{1}{2}) \), \( (xy)_{1/2} = (yx)_{1/2} = (x \cdot y)_{1/2} \)
where the subscript \( \frac{1}{2} \) indicates the \( A(\frac{1}{2}) \) component.

By Lemma 2.4 we have \( (xy)_{1/2} = a_{1/2} \) and \( (yx)_{1/2} = a_{1/2} - 4a^* \). Therefore \( (xy)_{1/2} = a^* \), and \( e(yx)_{1/2} = \frac{1}{2}a_{1/2} + a^* - 2a^* = \frac{1}{2}(yx)_{1/2} + a^* \) so that \( (yx)_{1/2} = a^*. \)

Analogous methods will produce the corresponding results for products of zero-space elements by half-space elements. Calling \( xy = a_1 + a_{1/2} + a_0 \) and \( yx = b_1 + b_{1/2} - a_0 \) for \( x \) in \( A(0) \) and \( y \) in \( A(\frac{1}{2}) \), we obtain

**Lemma 2.6.** For each \( x \) in \( A(0) \) and \( y \) in \( A(\frac{1}{2}) \), \( xy = a_1 + a_{1/2}, \)
\[ yx = a_1 + (a_{1/2} + 4a^*), \]
\[ xy^* = y^*x = a^*, \]
and \( (xy)_{1/2} = (yx)_{1/2} = (x \cdot y)_{1/2}. \]

At this point we can conclude that \( A \) is \( e \)-nilstable if and only if \( A^+ \) is, since
Lemmas 2.4 and 2.6 show that the \( A(0) + A(1) \) components of \( xy \) and \( yx \) are the same whenever \( y \) is in \( A(\frac{1}{2}) \) and \( x \) is in \( A(1) + A(0) \). Now let \( x \) and \( y \) lie in \( A(\frac{1}{2}) \)
and set \( xy = a_1 + a_{1/2} + a_0 \) and \( yx = b_1 - a_{1/2} + b_0. \)

**Lemma 2.7.** For each \( x \) and \( y \) in \( A(\frac{1}{2}) \), \( xy^* = \frac{1}{2}(b_1 - a_1) - a^* + \frac{1}{2}(a_0 - b_0), \)
\[ y^*x = \frac{1}{2}(b_1 - a_1) + a^* + \frac{1}{2}(a_0 - b_0), \]
\[ xy^* = y^*x = 3y^*x - xy^*. \]

Substituting \( x, y, \) and \( e \) in (2) with \( y \) in the place of \( a \), we obtain
\[ \frac{1}{3}(b_1 - a_1) + 4a^* + \frac{1}{2}(a_0 - b_0) = 3y^*x - xy^*. \]

Repeating the substitution, this time with \( e \) in the place of \( a \), gives
\[ yx^* - 3x^*y = 3y^*x - xy^*. \]

Now call \( xy^* = r_1 + r_{1/2} + r_0 \) and \( y^*x = s_1 - r_{1/2} + s_0. \) Then, corresponding to (12), we obtain
\[ \frac{1}{2}(r_1 - s_1) + 4r^* + \frac{1}{2}(r_0 - s_0) = 3y^*x - xy^* = 0 \]
so that \( r_1 = s_1, r^* = 0, \) and \( r_0 = s_0. \) Then the right side of (12) becomes
\[ 3(r_1 - r_{1/2} + r_0) - (r_1 + r_{1/2} + r_0) \]
so that, by equating corresponding components in (12),
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Let $r_1 = \frac{1}{4}(b_1 - a_1)$, $r_{1/2} = -a^*$, and $r_0 = \frac{1}{4}(a_0 - b_0)$. Again, corresponding to (13) is $y^*x^* - 3x^*y^* = 3y^*x - xy^* = 0$, that is, for all $x$ and $y$ in $A(\lambda)$, $3x^*y^* = y^*x^*$. But then, by the symmetry of the assumptions for $x$ and $y$, $3y^*x^* = x^*y^*$, and so $x^*y^* = y^*x^* = 0$. If we continue by calling $y^*x^* = c_1 + c_{1/2} + c_0$ so that, by what has been shown so far, $x^*y = c_1 - c_{1/2} + c_0$ with $c^* = 0$, then, by (13), $c_1 = \frac{1}{2}(a_1 - b_1) = -r_1$, $c_{1/2} = a^* = -r_{1/2}$, and $c_0 = \frac{1}{2}(b_0 - a_0) = -r_0$. As a corollary to this lemma we might observe that $xx^* = -xx^* = 0$ for all $x$ in $A(\lambda)$.

To obtain another corollary to Lemma 2.7 write $xy = (a_1e + n_1) + a_{1/2} + (a_0f + m_0)$ and $yx = (\beta_1e + m_1) - a_{1/2} + (\beta_0f + m_0)$ with $a_i$ and $\beta_i$ in $F$ and $n_i$ and $m_i$ in $N_i$, for $i = 1, 0$ then by Lemma 2.7

\[ xy^* = \frac{1}{4}[(\beta_1 - a_1)e + (m_1 - n_1)] - a^* + \frac{1}{4}[(a_0 - \beta_0)f + (n_0 - m_0)] \]

\[ y^*x = \frac{1}{4}[(\beta_1 - a_1)e + (m_1 - n_1)] + a^* + \frac{1}{4}[(a_0 - \beta_0)f + (n_0 - m_0)]. \]

But $x \cdot y = \gamma + n$ and $x \cdot y^* = \delta 1 + m$ for some $\gamma$ and $\delta$ in $F$ and $n$ and $m$ in $N_1 + N_0$. Consequently $\gamma 1 = \frac{1}{2}(a_1 + \beta_1)e + \frac{1}{2}(a_0 + \beta_0)f$ and $\delta 1 = \frac{1}{2}(\beta_1 - a_1)e + \frac{1}{2}(a_0 - \beta_0)f$, that is, $a_1 + \beta_1 = a_0 + \beta_0$ and $-a_1 + \beta_1 = a_0 - \beta_0$, and so $\beta_1 = a_0$ and $\beta_0 = a_1$, producing the following.

**Corollary.** For each $x$ and $y$ in $A(\frac{1}{2})$, $xy = (a_1e + n_1) + a_{1/2} + (a_0f + m_0)$, $yx = (a_0e + m_1) - a_{1/2} + (\beta_1f + m_0)$.

By a method similar to that used in [11] we can now show that each $N_\lambda$ is an ideal of $A(\lambda)$.

**Lemma 2.8.** For every $x$ in $A(1)$ and $y$ in $A(\frac{1}{2})$, $(xy)^{1/2} = 2[(e(x \cdot y)]^{1/2} = 2(x \cdot ey)^{1/2}$ and $(yx)^{1/2} = 2[(x \cdot ye)^{1/2} = 2(x \cdot ey)^{1/2}$. For every $x$ in $A(0)$ and $y$ in $A(\frac{1}{2})$, $(xy)^{1/2} = 2[(x \cdot ye)^{1/2} = 2(x \cdot ye)^{1/2}$ and $(yx)^{1/2} = 2[(x \cdot ye)^{1/2} = 2(x \cdot ye)^{1/2}$.

The first statement is a consequence of Lemma 2.4. Using the notation of the lemma we have $2[e(x \cdot y)]^{1/2} = 2[e(a_{1/2} - 2a^* + a_0)]^{1/2} = 2(4a_{1/2} + a_1 - 2a^*) = (xy)^{1/2}$. In the same way $2(x \cdot ey)^{1/2} = 2[(x \cdot y_1)^{1/2} + 2(x \cdot y_0)^{1/2} = (a_1 + a_0 + a_0 + a_0)^{1/2} = (xy)^{1/2}$.

**Theorem 2.2.** The nilsubspace $N_\lambda$ is an ideal of $A(\lambda)$ for $\lambda = 1, 0$.

Suppose that $N_\lambda$ is not a subalgebra. Then there are elements $x$ and $y$ of $N_\lambda$ for which $xy = e + n$, so that $yx = -e + m$, with $m$ and $n$ in $N_1$. Let $a$ be any
element of $A(\frac{1}{2})$ and write, by Lemma 2.4, $xa = b_{1/2} + b_0$ and $ax = (b_{1/2} - 4b^*) + b_0$. Now (2) with $x$ and $a$ interchanged becomes

\[(14) \quad a + 4a^* = 2ma - am - na + b_{1/2}v - yb_{1/2} + 4yb^* - 8b^*y.\]

The left side of (14) is an element in $A(\frac{1}{2})$ while each term on the right is of the form $zw$ or $wz$ with $z$ in $N_1$ and $w$ in $A(\frac{1}{2})$. Indeed, since $zw$ and $wz$ are in $A(\frac{1}{2}) + A(0)$ by Lemma 2.4, their zero-space components vanish, and we need only consider their half-space components. But these can all be written, by Lemma 2.8, in the form $z
ot w$ or $(z \cdot ew)_{1/2}$. That is to say, $a + 4a^*$ lies in $[N_1 \cdot A(\frac{1}{2})]_{1/2}$. But $a + 4a^* = 4e(ea)$, and every element in $A(\frac{1}{2})$ is of the form $ec$ for some $c$ in $A(\frac{1}{2})$. Consequently,

\[(15) \quad A(\frac{1}{2}) = [N_1 \cdot A(\frac{1}{2})]_{1/2}.\]

To state (15) differently, define, for each $x$ in $A(1)$, a linear transformation, $S_x$, in $A(\frac{1}{2})$ by the formula $S_x(w) = (x \cdot w)_{1/2}$ for all $w$ in $A(\frac{1}{2})$. It has been shown that $S_x$ is a nilpotent transformation whenever $x$ is a nilpotent element [3] and that the associative algebra generated by a set of transformations, $S_x$, each determined by a nilpotent $x$, is a nilpotent algebra [1]. Calling this enveloping algebra, $S_{N_1}$, we can rewrite (15) as $A(\frac{1}{2}) = A(\frac{1}{2})S_{N_1}$, and therefore $A(\frac{1}{2}) = A(\frac{1}{2})S_{N_1} = \cdots = A(\frac{1}{2})S_{N_1} = \cdots$ for all positive integers $n$. Since $S_{N_1}^k = 0$ for some $k$, $A(\frac{1}{2}) = 0$. But then $A$ is the algebra direct sum $A(1) \oplus A(0)$, contrary to the simplicity of $A$. Hence $N_1$ is a subalgebra and therefore an ideal of $A_1$.

In the same way $N_0$ is shown to be a subalgebra. Here we use the fact that for every $x$ in $A(0)$ there is a linear transformation, $T_x$, in $A(\frac{1}{2})$ given by $T_x(w) = (x \cdot w)_{1/2}$ for all $w$ in $A(\frac{1}{2})$. Again, $T_x$ is nilpotent whenever $x$ is in $N_0$, the enveloping algebra $T_{N_0}$ generated by all $T_x$ with $x$ in $N_0$ is nilpotent, and $A(\frac{1}{2}) = A(\frac{1}{2})T_{N_0}$, producing the same sort of contradiction.

Now call $N = N_0 + N_1$, $B = A(\frac{1}{2}) + N$, and $I = \{x \in A : Ax + xA \subset B\}$. Then, by nilstability and Theorem 2.2, $N \subset I$. In fact $N = I \cap [A(1) + A(0)]$, since $e^2 = e$ is not in $B$, so that we may write $I = I_{1/2} + N \subset B$. We shall show that $I$ is an ideal(3). To do this we shall show separately that $AN + NA$ and $AI_{1/2} + I_{1/2}A$ are contained in $I$. First observe that $[A(1) + A(0)]N + N[A(1) + A(0)] \subset N \subset I$. Next, call $*A = [NA(\frac{1}{2})]_{1/2}$ and $*A = [A(\frac{1}{2})N]_{1/2}$ so that

$$A(\frac{1}{2})N + NA(\frac{1}{2}) \subset A* + *A + N.$$ 

We shall see that $A* + *A \subset I$. We begin with methods similar to those used in [9].

**Lemma 2.9.** If $x \cdot y$ and $x \cdot ey$ are in $N$ for some $x$ and $y$ in $A(\frac{1}{2})$ then $xy$ and $yx$ are in $B$.

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(3) The author is grateful to the referee for suggesting $I$ as an ideal easier to establish than that in the original.
Using the notation and results of Lemma 2.7 we have $x \cdot y = \frac{1}{2}(a_1 + b_1) + \frac{1}{2}(a_0 + b_0)$ in $N$ and $x \cdot ey = \frac{1}{2}x \cdot y + x \cdot y^* = \frac{1}{2}b_1 + \frac{1}{2}a_0$ in $N$. Hence $b_1$ is in $N_1$ and $a_0$ is in $N_0$ so $a_1$ and $b_0$ are in $N_1$ and $N_0$, respectively, and $xy$ and $yx$ are in $B$.

**Lemma 2.10.** The subspace $B$ is not an ideal in $A^+$. 

For, $N(A(1) + A(0)) = [A(1) + A(0)]N \subseteq N \subseteq B$ since $N_2$ is an ideal of $A(\lambda)$, and $A(\frac{1}{2})(A(1) + A(0))$ and $[A(1) + A(0)]A(\frac{1}{2}) \subseteq B$ by nilstability so that, were $B$ an ideal in $A^+$, $A(\frac{1}{2}) \cdot A(\frac{1}{2})$ would be contained in $B$, indeed in $N$. But then $x \cdot y$ and consequently $x \cdot ey$ is in $N$ for all $x, y$ in $A(\frac{1}{2})$ so that $A(\frac{1}{2})A(\frac{1}{2}) \subseteq B$ and $B$ would be an ideal in $A$. By simplicity of $A$, and since $e$ is not in $B$, $A(\frac{1}{2}) \subseteq B = 0$, and $A = A(1) \oplus A(0)$ contrary to simplicity.

As a result of Lemma 2.10 we can prove, as Kokoris did in Lemma 7 of [7], that $A(\frac{1}{2})$ has a non-nilpotent element. In fact, all the results of that paper concluding with the statement that the set $C = A(\frac{1}{2}) \cdot N + N$ is an ideal in $A^+$ are now at our disposal.

**Lemma 2.11.** If $x$ is in $A^+ + A^*$ and $y$ is in $A(\frac{1}{2})$ then $xy$ and $yx$ are in $B$.

Every such $x$ is a sum of elements of the form $(nz)_{1/2}$ or $(zn)_{1/2}$ for some $n$ in $N$ and $z$ in $A(\frac{1}{2})$. For any such $n$ and $z$ and $y$ in $A(\frac{1}{2})$, $y \cdot (n \cdot z)$ is in $C$ since $C$ is an ideal in $A^+$ and $n \cdot z$ is in $A(\frac{1}{2}) \cdot N \subseteq C$. Indeed, $y \cdot (n \cdot z)_1$ and $y \cdot (n \cdot z)_0$ are in $C$ since $(n \cdot z)_1$ and $(n \cdot z)_0$ are in $N$ by nilstability, while $y \cdot (n \cdot z)_{1/2}$ is in $C$ and in $A(1) + A(0)$, and therefore in $N$. That is, $y \cdot (n \cdot z)_{1/2}$ is in $N$ for every $y$ in $A(\frac{1}{2})$, hence $ey \cdot (n \cdot z)_{1/2}$ is in $N$ for every $y$ in $A(\frac{1}{2})$, and, by Lemma 2.9, $y(n \cdot z)_{1/2}$ and $(n \cdot z)_{1/2}y$ are in $B$. In case $n$ is in $N_1$, $(nz)_{1/2} = 2(n \cdot ez)_{1/2} = (n \cdot z)_{1/2} + 2(n \cdot z^*)_{1/2}$ and $(zn)_{1/2} = 2(n \cdot ze)_{1/2} - 2(n \cdot z^*)_{1/2}$ by Lemma 2.8. In case $n$ is in $N_0$, $(nz)_{1/2} = (n \cdot z)_{1/2} = (n \cdot z)_{1/2} + 2(n \cdot z^*)_{1/2}$ and $(zn)_{1/2} = (n \cdot z)_{1/2} = (n \cdot z^*)_{1/2}$ by the same lemma. So in either case $y(zn)_{1/2}$, $y(nz)_{1/2}$, $(zn)_{1/2}y$, and $(nz)_{1/2}y$ will be in $B$ provided $y(n \cdot z^*)_{1/2}$ and $(n \cdot z^*)_{1/2}y$ are in $B$. But, in fact, for any $z$ in $A(\frac{1}{2})$, $z^*$ is in $A(\frac{1}{2})$ and so $y \cdot (n \cdot z^*)$ is in $C$ and, as in the preceding, $y(n \cdot z^*)_{1/2}$ and $(n \cdot z^*)_{1/2}y$ are in $B$.

We now remark that, for all $x$ in $A^+ + A^*$, $Nx + xN \subseteq B$ by nilstability, $ex$ and $xe$ and consequently $fx$ and $xf$ are in $B$ by Lemma 2.1, and therefore, by Lemma 2.11, $A^* + A^* \subseteq C$. So we have shown that $AN + NA \subseteq C$, and to show that $AI_{1/2} + I_{1/2}A \subseteq I$ now requires only showing that $eI_{1/2}$ (and consequently $I_{1/2}e, fI_{1/2}$, and $I_{1/2}f$) and $A(\frac{1}{2})I_{1/2} + I_{1/2}A(\frac{1}{2})$ are contained in $I$. We begin with the former.

**Lemma 2.12.** If $x$ is in $I_{1/2}$ then $ex$ is in $I$.

For all $y$ in $A(\frac{1}{2})$, we have, by Lemma 2.7, $x^*y = -y^*x$ is in $B$ and $yx^* = -yx^*$ is in $B$. Further, $x^*e = ex^* = x^*f = f x^* = \frac{1}{2}x^*$ is in $B$. Finally, for all $n$ in $N$,
Lemma 2.13. If $x$ is in $I_{1/2}$ and $y$ is in $A(\frac{1}{2})$ then $xy$ and $yx$ are in $I$.

We have that $(xy)_1, (yx)_0, (yx)_1,$ and $(yx)_0$ are in $N$ since $x$ is in $I_{1/2}$, and

$$(xy)_{1/2} = -(xy)_{1/2}$$

so we need only examine $(xy)_{1/2}$. But, for all $n$ in $N$, $(xy)_{1/2}$ and $n(xy)_{1/2}$ are in $A^* + A^* + N \subseteq B$. Also $(xy)_{1/2}e$ and $e(xy)_{1/2}$ are in $A(\frac{1}{2}) \subseteq B$, so we need only examine products $(xy)_{1/2}z$ and $(xy)_{1/2}z$ with $z$ in $A(\frac{1}{2})$. Replace $a$ and $y$ in $(2)$ by $y$ and $z$, respectively, and obtain $z(xy) + (yx)z - 2(xy)z = 2(zx) - (zy)x - x(zy)$. The right side here is a member of $B$ since $x$ is in $I$.

On the left, the terms $z(xy)_1, z(xy)_0, y, z$, respectively, and obtain $z(xy) + (yx)z - 2(xy)z = 2(zx) - (zy)x - x(zy)$. The right side here is a member of $B$ since $x$ is in $I$.

Thus $x*, n x*$ are in $A^* + A^* + N \subseteq B$. Thus $x*$, and consequently $ex = \frac{1}{2}x + x^*$, are in $I$.

Theorem 2.3. The set $I$ is an ideal and $N \subseteq I \subseteq B$.

3. Classification of $A$.

Theorem 3.1. $A$ is Jordan admissible.

Since $A$ is simple and $e$ is not in $I$, $N \subseteq I = 0$ and $A = eA + A(\frac{1}{2}) + fF$. From this fact follows the argument on p. 331 of [4] by which it is shown that $A^+$ is Jordan.

Now let $\delta(x) = \alpha_x + \alpha_0$ for each $x = x_1, x_1 + x_0$ in $A$. Then $\delta$ is a linear functional. We shall show that $\delta$ satisfies, for all $x, y, z$ in $A$,

(a) $\delta(xy) = \delta(yx)$,

(b) $\delta[(xy)z] = \delta[x(yz)]$, and

(c) $\delta(x) = 0$, whenever $x$ is nilpotent.

Then it is known that the set $N_\delta$ of all $x$ in $A$ for which $\delta(xy) = 0$ for every $y$ in $A$ is an ideal of $A$ containing the nilradical. Now $\delta(ee) = \delta(e) = 1$ so, as before, $N_\delta = 0$.

In fact we shall show that $\delta$ satisfies the three properties in the attached algebra $A^+$; hence $N_\delta^+$ is an ideal of $A^+$ containing the nilradical of $A^+$. But $N_\delta^+$ and $N_\delta$ are the same subspace since $\delta(x \cdot y) = \frac{1}{2}\delta(xy + yx) = \frac{1}{2}[\delta(xy) + \delta(yx)] = \delta(xy)$. Hence $A^+$ is semisimple and consequently is either simple or else the direct sum.
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A(1) ⊕ A(0). The latter contradicts the simplicity of A, so that, pending proofs
of (a), (b), and (c), we have the following.

THEOREM 3.2. A is J-simple.

To prove (a) write
\[ x = x_1e + x_{1/2} + \alpha_0f \] and
\[ y = \beta_1e + y_{1/2} + \beta_0f \] and, using the
corollary to Lemma 2.7.,
\[ x_{1/2}y_{1/2} = \gamma_1e + a_{1/2} + \gamma_0f \] and
\[ y_{1/2}x_{1/2} = \gamma_0e - a_{1/2} + \gamma_1f. \]
Then
\[ xy = (\alpha_1\beta_1 + \gamma_1)e + (xy)_{1/2} + (a_0\beta_0 + \gamma_0)f \] and
\[ yx = (\alpha_1\beta_1 + \gamma_0)e + (yx)_{1/2} + (a_0\beta_0 + \gamma_1)f \] so that
\[ \delta(xy) = \alpha_1\beta_1 + \gamma_1 + \alpha_0\beta_0 + \gamma_0 = \delta(yx). \]
Of course
\[ \delta(x \cdot y) = \delta(y \cdot x) \] since \( x \cdot y = y \cdot x. \)

To verify (b) observe first of all that
\[ \delta[(xy)z - x(yz)] = \delta[(x \cdot y) \cdot z - x \cdot (y \cdot z)] \]
so that \( \delta \) satisfies (b) in \( A \) if and only if it does in \( A^+ \). By (2) and the linearity of \( \delta \),
\[ \delta[x(yz)] + \delta[y(xz)] + \delta[(zy)x] + \delta[(xz)y] = 2\delta[(xz)y] + 2\delta[(yz)x], \]
which, by application of (a) to its right member, becomes
\[ \delta[(xz)y] = 0. \]
Applying (a) again produces
\[ \delta[(yz)x] - \delta[(xy)z] + \delta[(xy)] + \delta[(yz)] = 0 \]
or
\[ \delta[x(yz) - (xz)y] = \delta[(yx)x - y(xz)]. \]
Consequently
\[ 4\delta[x \cdot (y \cdot z) - (x \cdot y) \cdot z] = \delta[(x \cdot y) \cdot z - x \cdot (y \cdot z)]. \]

Now \( \delta \) can be shown to satisfy (b) in \( A^+ \) by a direct computation. Let
\[ x = x_1e + x_{1/2} + a_0f, \quad y = \beta_1e + y_{1/2} + a_0f, \] and
\[ z = \gamma_1e + z_{1/2} + a_0f. \] Then
\[ x \cdot (y \cdot z) = \alpha_1\beta_1\gamma_1e + \alpha_1(y_{1/2} \cdot z_{1/2})_1 + \frac{1}{2}(\beta_1 + \beta_0)(x_{1/2} \cdot z_{1/2})_1 + \frac{1}{2}(\gamma_1 + \gamma_0)(x_{1/2} \cdot y_{1/2})_1 + \frac{1}{2}(\gamma_0)(x_{1/2} \cdot z_{1/2})_0 \] and
\[ (x \cdot y)z = \alpha_1\beta_0\gamma_0f + \alpha_0(y_{1/2} \cdot z_{1/2})_0 + \frac{1}{2}(\beta_1 + \beta_0)(x_{1/2} \cdot z_{1/2})_0 + \frac{1}{2}(\gamma_1 + \gamma_0)(x_{1/2} \cdot y_{1/2})_0. \] If we call \( (x_{1/2} \cdot y_{1/2})_1 = \phi e \), then, by the corollary to
Lemma 2.7.,
\[ (x_{1/2} \cdot y_{1/2})_1 = \phi e. \] Similarly, let
\[ (x_{1/2} \cdot z_{1/2})_1 = \psi e \] and
\[ (y_{1/2} \cdot z_{1/2})_1 = \omega e. \] Then
\[ \delta[x \cdot (y \cdot z)] = \alpha_1\beta_1\gamma_1 + \frac{1}{2}(\beta_1 + \beta_0)(x_{1/2} \cdot \gamma_1) + \frac{1}{2}(\gamma_1 + \gamma_0)(x_{1/2} \cdot \gamma_1) + \frac{1}{2}(\gamma_0)(x_{1/2} \cdot \gamma_1) + \frac{1}{2}(\beta_1 + \beta_0)(x_{1/2} \cdot \gamma_1) + \frac{1}{2}(\gamma_1 + \gamma_0)(x_{1/2} \cdot \gamma_1) + \frac{1}{2}(\gamma_0)(x_{1/2} \cdot \gamma_1). \] In the same way we find the value of
\[ \delta[(x \cdot y) \cdot z] \] and so verify (b) in \( A^+ \).

Property (c) can be obtained by observing that \( A \) has the subspace decomposition
\[ A = 1F + uF + A(\notin) \] where \( u = e - f \) and \( u^2 = 1 \) and \( u \cdot x = 0 \) for all \( x \) in \( A(\notin) \). Let
\[ x = x_1e + x_{1/2} + a_0f, \] with \( x_{1/2}^2 = \gamma 1 \), be any element in \( A \), then
\[ \delta(x) = 2x \] and
\[ x^2 = (x^2 + \beta^2 + \gamma)1 + 2axu + 2ax_{1/2} \] so that
\[ 2x^2 - 2xx + (x^2 - \beta^2 - \gamma)1 = 0. \]

Whenever \( x \) is nilpotent, the minimal polynomial for \( x \) is of the form \( x^k \) for some
\( k > 1 \) and divides the left member of (17), so that \( k = 2 \) and \( 2x = \delta(x) = 0 \). By
the agreement of powers in \( A \) and \( A^+ \) \( \delta \) satisfies (c) in \( A^+ \) as well.

Moreover, it is shown in [10] that \( A \) satisfies the flexible law by showing that,
as a consequence of (2) and properties (a) and (b) of \( \delta \),
\[ \delta([(xy)x - x(yx)]z) = 0 \] for all \( x, y, \) and \( z \) in \( A \).
Thus \( (xy)x - x(yx) \) is in \( N \) and so vanishes for all \( x \) and \( y \) in \( A \).
Being flexible and \( J \)-admissible, \( A \) is known \([12]\) to satisfy the Jordan identity. We can summarize the results of this section in the following principal result.

**Theorem 3.3.** If \( A \) is a nilstable central simple strictly power-associative algebra of degree two satisfying (1) over a field of characteristic \( \neq 2, 3 \) then \( A \) is a noncommutative Jordan algebra that is \( J \)-simple.

The multiplication table for every such algebra has been described \([9]\). On the other hand, when \( F \) has a characteristic different from zero and the assumption of nilstability is dropped, examples of commutative (hence satisfying(1)), power-associative, central simple, degree two algebras are known \([5]\) that fail to satisfy the Jordan identity because they are not \( u \)-stable with respect to every idempotent \( u \) they possess.

**Bibliography**


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