SEQUENTIALLY 1-ULC TORI

BY

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1. Introduction. A closed set \( X \) in Euclidean 3-space \( E^3 \) is called tame if there exists a homeomorphism \( h \) of \( E^3 \) onto itself such that \( h(X) \) is a polyhedron. A set which is not tame is called wild. In this paper, we investigate conditions which determine tameness of an arc in \( E^3 \). Examples of wild arcs in \( E^3 \) are abundant; see, for example, [3; 8]. Also abundant are conditions implying tameness of an arc; see [7; 10].

Consider the following conditions placed on an arc \( \mathcal{A} \) in \( E^3 \):

1. \( \mathcal{A} \) lies on a 2-sphere \( S \) in \( E^3 \).
2. \( \mathcal{A} \) lies on a simple closed curve \( J \) in \( E^3 \) which is the intersection of a nested sequence of (two-dimensional) tori plus their interiors.

This paper was motivated by a belief that (1) and (2) implied that \( \mathcal{A} \) is tame. This turns out not to be the case; the wild arc constructed in [1] is a counterexample. With this in mind, we make the following definition. A sequence \( \{M_1, M_2, \ldots\} \) of 2-manifolds in \( E^3 \) is sequentially 1-ULC if, given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) and integer \( N \) such that: Whenever \( n > N \), and \( \alpha \) is a simple closed curve on \( M_n \) of diameter less than \( \delta \) which bounds a disk on \( M_n \), then \( \alpha \) bounds a disk of diameter less than \( \varepsilon \) on \( M_n \).

We now add another condition.

3. The sequence of tori of condition 2 is sequentially 1-ULC.

Our primary result is that these three conditions imply tameness of the arc \( \mathcal{A} \). This theorem yields as a corollary an answer to a question raised by Bing in [3]: No subarc of the “Bing sling” [3] lies on a disk.

A simple closed curve \( J \) is said to pierce a disk \( D \) if \( J \) links \( \text{Bd} \ D \) (boundary of \( D \)) and \( J \cap D \) is a single point. As the “Bing sling” is the only example in the literature of a simple closed curve that pierces no disk, one is now led to a natural question. Can a different simple closed curve \( \mathcal{K} \) be constructed where \( \mathcal{K} \) pierces no disk, yet lies on a disk? In §3, we show the existence of such a simple closed curve \( \mathcal{K} \). That \( \mathcal{K} \) lies on a disk will be immediate from its construction. To show that \( \mathcal{K} \) pierces no disk, we will use the following. Define \( P_{\mathcal{K}} \) to be the set of points of an arc \( \mathcal{A} \) at which \( \mathcal{A} \) pierces a disk. We set up an alternate condition to (3) given above.

3'. \( P_{\mathcal{K}} \) is dense in \( \mathcal{A} \).

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Conditions (1), (2), and (3') are also shown to imply tameness. This result is then used to establish that $\mathcal{A}$ pierces no disk.

2. No subarc of the "Bing sling" lies on a disk.

**Theorem 1.** If $\mathcal{A}$ is an arc in $E^3$ such that

1. $\mathcal{A}$ lies on a 2-sphere $S$ in $E^3$;
2. $\mathcal{A}$ lies on a simple closed curve $J$ in $E^3$ which is the intersection of a decreasing sequence of tori plus their interiors;
3. The sequence of tori of (2) is sequentially 1-ULC;

then $\mathcal{A}$ is tame.

**Proof.** We assume without loss of generality that the 2-sphere $S$ is locally polyhedral mod $\mathcal{A}$ [4]. We will use Theorem 6 of [5] to establish that $S$ is locally tame at all non-endpoints of the arc $\mathcal{A}$. Toward this goal, we prove the following.

**Assertion.** Given a non-endpoint $p$ of $\mathcal{A}$, and $\varepsilon > 0$, there exists a $\delta > 0$ such that: if $\beta$ is a simple closed curve lying in a $\delta$-neighborhood of $p$, $\beta \cap S = \emptyset$, and $\beta$ bounds a disk $B$ in $E^3$, then $\beta$ bounds a disk $B'$ in $E^3 - S$ such that $B'$ lies in an $\varepsilon$-neighborhood of $p$. This Assertion is a bit weaker than the statement that $E^3 - S$ is locally simply connected at $p$, which is the hypothesis of Theorem 6 of [5]. However, the Assertion is sufficiently strong so that the proof of Theorem 6 of [5] still remains valid, showing that $S$ is locally tame at $p$. We now prove the Assertion in six steps, numbered for convenience.

1. There exists an integer $N_1$ and a positive number $\gamma$ such that if $\alpha$ is any simple closed curve on $T_n, n > N_1$, and if $\alpha$ lies in a neighborhood of $p$ of radius $\gamma$ (which we abbreviate $\partial\gamma(p)$), then either $\alpha \cap S \neq \emptyset$, or $\alpha$ bounds a disk on $T_n$.

Step 1 is devoted to a justification of this statement.

The arc $\mathcal{A}$ is now extended to form a simple closed curve $K, \mathcal{A} \subset K \subset S$. We assume $K \cap J = A$, i.e., $K \cup J$ is a $\theta$-curve. Call the end points of $\mathcal{A}$ $a$ and $b$, and call the sequence of tori given by our hypothesis $\{T_1, T_2, T_3, \ldots\}$; these may be taken to be polyhedral [4] and in general position. Let $P$ be a plane missing $p$ and separating $a$ from $b$ in $E^3$, with $P$ in general position with respect to $\{T_1, T_2, \ldots\}$. For fixed $n$, $T_n \cap P$ is a collection of simple closed curves $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$. Let $L_i$ be the disk on $P$ bounded by $\lambda_i$. There exists an integer $N_1$ such that if $n$ were chosen above to be larger than $N_1$, then $L_i$ would not intersect both $\mathcal{A}$ and $J - \mathcal{A}$, for $i = 1, 2, \ldots, k$. This fact can be used to show that at least one of the $\lambda_i$'s links $J$ (in the sense of §9 of [4]). Thus, if $n > N_1$, at least one $\lambda_i$ will link $J$. Note that such a $\lambda_i$ could not bound a disk on $T_n$, but must lie on $T_n$ in a nontrivial way.

There exists a positive number $\gamma$ such that if a simple closed curve $\alpha$ lies in $\partial\gamma(p)$, then $\alpha$ does not intersect the plane $P$, and $\alpha$ does not link the simple closed curve $(J \cup K) - \text{Int} \mathcal{A}$. Now let us suppose that $\alpha$ lies on $T_n$, for $n > N_1$, and $\alpha$ does not bound a disk on $T_n$. Then $\alpha$ and the $\lambda_i$ of the preceding paragraph can be joined by an annulus on $T_n$; since $\lambda_i$ links $J, \alpha$ must also link $J$. Since $\alpha$ does not
link \((J \cup K) - \text{Int}\mathcal{A}\), it follows from Theorem 9 of [4] that \(x\) links \(K\). Since \(K\) lies on the 2-sphere \(S\), it follows that \(x \cap S \neq \emptyset\), which completes step 1.

(2) We assume that diameter \(\mathcal{A} > \varepsilon/3\). There exists a \(\delta_1 > 0\) such that any \(\delta_1\)-simple closed curve (a \(\delta_1\)-set is a set of diameter less than \(\delta_1\)) on \(S\) bounds an \(\varepsilon/3\)-disk on \(S\). In particular \(\delta_1 < \varepsilon/3\), of course. We use the sequential 1-ULC hypothesis to select a \(\delta_2 > 0\) and integer \(N_2\) such that any \(\delta_2\)-simple closed curve on \(T_n, n > N_2\), which bounds a disk on \(T_n\) bounds a \(\delta_1/3\)-disk on \(T_n\); in particular \(\delta_2 < \delta_1/3\).

(3) We now select a disk \(U\) on \(S\) containing \(\mathcal{A}\) on its interior, “thin” enough so \(U\) has the following property: If \(W\) is any open set containing \(\mathcal{A}\), and \(X\) is an open set containing \(S\), then there exists a homeomorphism \(H\) of \(E^3\) onto itself such that \(H(S) = S, H = \text{identity on } E^3 - X, H(U) \subset W,\) and \(H\) moves no point of \(E^3\) more than the minimum of the two numbers \(\delta_2/3\) and \(\gamma/2\).

The existence of such a disk \(U\) follows from the fact that \(S\) is locally tame, mod \(\mathcal{A}\). To see this, note that if we had asked in the preceding paragraph that \(H\) be defined only on the 2-sphere \(S\), then it is clear how to select \(U\). In fact, in this case, \(H\) could be defined to be the identity on a small disk \(D_w\), with \(\mathcal{A} \subset \text{Int}\, D_w \subset D_w \subset \text{Int}\, U\). On the set \(S - D_w\), where \(H\) is not the identity, \(H\) is isotopic to the identity. Furthermore, this set is tame, since it misses \(\mathcal{A}\); hence it is bicollared in \(E^3\). We now extend \(H\) to the bicollar in the obvious way, so that \(H = \text{identity, except on this bicollar. By choosing the bicollar to lie in } X, \text{ we find } H\) satisfies all required properties.

(4) We now select the \(\delta > 0\) required in the Assertion, by requiring that \(\delta < \delta_2/6, \delta < \gamma/2\) and \(\mathcal{O}_\delta(p) \cap [S - U] = \emptyset\). We now prove the Assertion. Let \(\beta\) be a simple closed curve in \(\mathcal{O}_\delta(p)\) such that \(\beta\) bounds a disk \(B\), and \(\beta \cap S = \emptyset\). We may assume that \(B\) lies in \(\mathcal{O}_\delta(p)\) simply by pushing it there without moving \(\beta\).

(5) In this step, we show that \(\beta\) bounds a disk \(B'\) of diameter less than \(\delta_1\), and such that \(B' \cap \mathcal{A} = \emptyset\). Let \(m\) be an integer, \(m > N_1, m > N_2\), so that \([T_m \cup \text{Int}\, T_m] \cap \beta = \emptyset\). Let \(\text{Int}\, T_m\) be the open set \(W\) of step 3, and let \(X\) of step 3 be sufficiently small so that \(X \cap \beta = \emptyset\). Step 3 guarantees the existence of a homeomorphism \(H\), and the disk \(H(B)\) has certain nice properties: Firstly, its diameter is less than diameter \(B + \delta_2/3 + \delta_2/3 < \delta_2\). Secondly, \(\beta\) bounds \(H(B)\), by choice of \(X\). Most important, \(S \cap T_m \cap H(B) = \emptyset\). This follows since \(B \cap S \subset U\) so \(H(B) \cap H(S) \subset H(U)\), but since \(H(S) = S\), we have \(H(B) \cap S \subset H(U)\), and since \(H(U) \subset \text{Int}\, T_m\), we have \(H(B) \cap S \subset \emptyset\).

We assume without loss of generality that \(H(B)\) is polyhedral on its interior and in general position with respect to \(T_m\). Thus, \(H(B) \cap T_m\) is a collection of simple closed curves \(x_1, x_2, \ldots, x_r\). Since \(H(B) \cap T_m \cap S = \emptyset\), each \(x_i\) does not intersect \(S\). By step 1, each \(x_i\) bounds a disk on \(T_m\). Since \(H(B)\) has diameter less than \(\delta_2\), and \(x_i \subset H(B)\), it follows that each \(x_i\) bounds a \(\delta_1/3\)-disk on \(T_m\), by step 2. The usual disk replacement process (see step 6 for details) is now performed.
on the disk $H(B)$, yielding a disk $B'$, of diameter less than diameter $H(B) + \delta_1 / 3 + \delta_1 / 3 < \delta_1$. Furthermore $B' \cap \text{Int } T_m = \emptyset$, so $B' \cap \mathcal{A} = \emptyset$.

(6) The disk $B'$ is placed in general position with respect to $S$, and the usual disk replacement process used to modify $B'$ into a new disk $B''$. That is, $B' \cap S$ is a collection of simple closed curves $l_1, l_2, \ldots, l_t$. Note that $l_i \cap \mathcal{A} = \emptyset$, $i = 1, 2, \ldots, t$. An “innermost” $l_i$ on $S$ is selected, and the disk it bounds on $B'$ is replaced by the $\varepsilon/3$-disk it bounds on $S$. (See step 2 for why we have an $\varepsilon/3$-disk.) This new disk on $B'$ is pushed slightly to one side of $S$; this can be done because the new disk cannot contain $\mathcal{A}$ on its interior, as diameter $\mathcal{A} > \varepsilon/3$. Thus, this new disk is polyhedral.

This process is continued with another $l_i$, until all intersection is eliminated, yielding $B''$. We have $B'' \cap S = \emptyset$, $\beta$ is the boundary of $B''$, and diameter $B'' < \text{diameter } B' + \varepsilon/3 + \varepsilon/3 < \varepsilon$. This establishes the Assertion.

It remains to show that $\mathcal{A}$ is tame at its end points $a$ and $b$. Now that we know that $\mathcal{A}$ is locally tame mod $a \cup b$, it is easy to construct arbitrarily small 2-spheres around $a$ (or $b$) out of the tori $\{T_i\}$, such that each 2-sphere intersects $\mathcal{A}$ in exactly one point. Thus $\mathcal{A}$ will be tame at its end points by satisfying Properties P and Q of [10]. We omit details of this construction as they are tedious, and similar to the proof of Theorem 2. Indeed, all that we really need to establish Corollary 1 is that $\mathcal{A}$ is locally tame on its interior.

**Corollary. 1.** No subarc of the “Bing sling” [3] lies on a disk.

**Proof.** If some subarc does lie on a disk, then a smaller subarc lies on a 2-sphere $S$, by §5 of [5]. We observe that the “Bing sling” satisfies Properties 2 and 3 of Theorem 1, with the necessary tori being provided by its very construction. Thus, this small subarc is tame, by Theorem 1, which is a contradiction to the fact that it pierces no disk.

3. The simple closed curve $\mathcal{A}$ which pierces no disk, yet lies on a disk. Using a technique developed by Bing [2], one can construct a 2-sphere $\mathcal{S}$ in $E^3$ whose wild points from a wild, cellular arc in $E^3$, which we call $\xi$. For an exact description, see [1]. The arc $\xi$ can be completed to a simple closed curve $Z$ on $\mathcal{S}$, and the same argument which shows that $\xi$ is cellular (see [9]) will establish that $Z$ is the intersection of a decreasing sequence of tori plus their interiors (note: these tori cannot be sequentially 1-ULC, by Theorem 1).

If any non-endpoint $x$ of $\xi$ had the property that $\xi$ pierced a disk at $x$, then one could use the symmetry given by the construction of $\xi$ to show that $\xi$ pierces a disk at a dense subset of itself, i.e., $P_x$ is dense in $\xi$. This, however, gives us a contradiction on account of the following.

**Theorem 2.** If $\mathcal{A}$ is an arc in $E^3$ such that

(1) $\mathcal{A}$ lies on a 2-sphere $S$ in $E^3$;
(2) $\mathcal{A}$ lies on a simple closed curve $J$ in $E^3$ which is the intersection of a nested sequence of tori plus their interiors;

(3) $P_\mathcal{A}$ is dense in $\mathcal{A}$;

then $\mathcal{A}$ is tame.

Proof. Let $x$ be any non-endpoint of $\mathcal{A}$. Given $\varepsilon > 0$, we will show that there exists a 2-sphere $S$ of diameter less than $\varepsilon$, such that $x \in \text{Int } S$, and $S \cap J$ is a set of two points. This will establish that $\mathcal{A}$ is locally tame at all non-endpoints [10]. The endpoints of $\mathcal{A}$ can then be taken care of by the same method, with only slight changes necessary in the construction of $S$.

The 2-sphere $S$ will be constructed as shown in the figure. That is, an annulus of the torus $T$ will connect two disks $D_1$ and $D_2$ which are pierced by $J$ on opposite sides of $x$. The 2-sphere $S$ will consist of the annulus plus one subdisk of each $D_i$, $i = 1, 2$. Of course, it must be justified that there is a torus and two disks which intersect as nicely as shown in the figure. This is done in eight steps.

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In this step we select the torus $T$ of the figure. Select a ray $R$ starting at $x$, such that $R$ and $[uxv \cup D_1 \cup D_2]$ are disjoint. $R$ may be taken to be locally polyhedral mod $x$. There is a positive number $\eta$, such that if $A$ is an arc in $E^3$ of diameter less than $\eta$, and if $A$ intersects both $uxv$ and $uyv$, then $A$ intersects $D_1 \cup D_2$. This follows from the fact that $J$ pierces the disks $D_1$ and $D_2$ at $u$ and $v$, respectively. We also assume that $\eta < \text{dist}(R, uyv)$ and $\eta < \rho$.

Let $D'_0$ be an $\eta/8$-disk with $u \in \text{Int } D'_0 \subset \text{Int } D_0$; let $D'_1$ be similarly situated in $D_2$. We choose $\gamma$ sufficiently small so that a $\gamma$-neighborhood of $J$ intersects $D_i$ only in a subset of $D'_i$, $i = 1, 2$. We have $\gamma < \eta/8$, of course. The torus $T$ is now selected from our sequence so $T$ lies in this $\gamma$-neighborhood of $J$. By applying [4], we may assume that $T$ is polyhedral, that $D_1$ is locally polyhedral mod $u$, that $D_2$ is locally polyhedral mod $v$; furthermore, we assume that $T$, $D_1$, $D_2$ and $R$ are in general position.

At the present time, $T$ may intersect $D_1$ and $D_2$ very differently from the way indicated in the figure. We now simplify this intersection.

Let us examine a simple closed curve $L$ of $T \cap [D_1 \cup D_2]$. $L$ may be classified thus:

1. $L$ bounds a disk on $T$;
2. $L$ does not bound a disk on $T$.

$L$ may also be classified in a different way. Assume for convenience that $L \subset D_1$.

1'. $L$ bounds a subdisk $E_1$ of $D_1$ which does not contain $u$.
2'. The subdisk $E_1$ of $D_1$ bounded by $L$ does contain $u$.

We show that $L$ is of Type 1 if and only if $L$ is of Type 1', that is, these classifications are really the same.

If $L$ is of Type 1, then $L$ does not link $J$. Thus, $L$ is also of Type 1'. If $L$ is of Type 1', then using techniques of Theorem 1 of [6], one can show that $L$ bounds a disk which does not intersect $J$, and whose interior does not intersect $T$. If $L$ were of Type 2, then by cutting $T$ along $L$ and inserting two copies of this disk, one could construct a 2-sphere in contradiction to step 1. Thus, if $L$ is of Type 1', then $L$ must also be of Type 1.

All $L$ of Type 1 are now removed. That is, we suppose that $L$ is an “innermost” simple closed curve of Type 1 in $D_1$. The subdisk $E_1$ of $D_1$ bounded by $L$ will not contain any simple closed curves of Type 2. This is obvious from the equivalence of Types 1 and 2 with Types 1' and 2'. Thus, $T$ may be altered by removing the disk bounded by $L$ on $T$, replacing it by $E_1$, then pushing to one side slightly. This process is repeated until all Type 1 simple closed curves have been removed, forming a new torus $T'$.

We now show that $J \subset \text{Int } T'$. The first stage of the alteration of $T$ consisted of interchanging two disks. This will change $\text{Int } T$ only by adding to or subtracting from it the 3-cell bounded by these two disks. $J$ cannot lie in this 3-cell, by step 1. The same line of reasoning is continued during each alteration in the construction of $T'$, showing that $J \subset \text{Int } T'$.
(6) If \( t \) is a point of \( T' - D_1 - D_2 \), then \( t \) can be joined to \( J \) by an arc \( A(t) \) which is disjoint from \( D_1 \cap D_2 \), and which is of diameter less than \( \eta/4 \). To see this we examine two cases: either \( t \) lies on \( T \), or \( t \) lies very close \( D_1' \) or \( D_2' \). The latter case is clear from the choice of \( D_1' \) and \( D_2' \); in fact, the arc will have diameter less than \( \eta/8 \). In the former case, we begin by joining \( t \) to \( J \) with an \( \eta/8 \) arc, which may intersect \( D_1' \) (or \( D_2' \)). If it does, we modify it by bending it just before it hits \( D_1' \) so it instead runs down the side of \( D_1' \) to \( J \). This bent arc will have diameter less than \( \eta/8 + \eta/8 = \eta/4 \), as desired.

(7) We now look at the components \( C_1, C_2, \ldots, C_m \) of \( T' - D_1 - D_2 \). If \( t \) and \( t' \) are both points of the same component, say \( C_1 \), then \( A(t) \) and \( A(t') \) both intersect the same component of \( J - u - v \). Otherwise, let \( \overline{tt'} \) be an arc of \( C_1 \) joining \( t \) and \( t' \). Let \( s \) and \( s' \) be two points of this arc such that \( A(s) \) and \( A(s') \) intersect different components of \( J - u - v \), and such that the subarc \( ss' \) of \( \overline{tt'} \) has diameter less than \( \eta/2 \). The path \( [A(s) \cup ss' \cup A(s')] \) has diameter less than \( \eta \), contradicting the definition of \( \eta \). Thus, each \( C_i \) lies in an \( \eta/4 \)-neighborhood of either \( \overline{uxv} \) or \( \overline{uyv} \).

(8) The desired 2-sphere \( \mathcal{S} \) may now be selected. Since the ray \( R \) hits \( T' \) an odd number of times, and since \( R \cap [D_1 \cup D_2] = \emptyset \), \( R \) hits some component \( C_N \) of \( T' - D_1 - D_2 \) an odd number of times. \( C_N \) cannot lie in an \( \eta/4 \)-neighborhood of \( \overline{uyv} \), since \( C_N \cap R \neq \emptyset \), and \( \eta < \text{dist}(R, \overline{uyv}) \). Thus, \( C_N \) lies in an \( \eta/4 \)-neighborhood of \( \overline{uxv} \).

\( C_N \) cannot be all of \( T' \), as
\[
\text{diam}(C_N) < \text{diam}(\overline{uxv}) + \eta/4 + \eta/4 < \varepsilon,
\]
whereas \( \text{Int} \ T' \) contains \( J \), a set of diameter larger than \( \varepsilon \), so \( T' \) has diameter larger than \( \varepsilon \). Thus, \( C_N \) will be an annulus of \( T' \), with two boundary simple closed curves of Type 2', by steps 4 and 5. Furthermore, both of these simple closed curves do not lie on \( D_1 \). If they did, the annulus lying between them on \( D_1 \) could be added to \( C_N \) to produce a torus \( T'' \) disjoint from \( J \). Since \( R \cap T'' \) would contain an odd number of points, \( J \) would lie in \( \text{Int} \ T'' \). Thus diameter \( (T'') > \varepsilon \). But diameter \( (T'') < \text{diameter} (\overline{uxv} \cup D_1 \cup D_2) + \eta/4 + \eta/4 < \varepsilon \) which gives us a contradiction. By similar reasoning, both of these simple closed curves do not lie on \( D_2 \).

Let \( \mathcal{S} \) be the 2-sphere composed of \( C_N \) plus the subdisk of \( D_1 \) bounded by a boundary simple closed curve of \( C_{N_1} \), plus the subdisk of \( D_2 \) bounded by the other boundary simple closed curve of \( C_N \). Then,
\[
\text{diam}(\mathcal{S}) < \text{diam}(\overline{uxv} \cup D_1 \cup D_2) + \eta/4 + \eta/4 < \varepsilon,
\]
and \( \mathcal{S} \cap J \) will be just the two points \( u \) and \( v \). That \( x \) lies in \( \text{Int} \ \mathcal{S} \) follows from the fact that \( \mathcal{S} \cap R \) consists of an odd number of points. This completes the proof of Theorem 2.
It is a simple matter to construct a simple closed curve $\mathcal{K}$ such that $\mathcal{K}$ looks locally just like the arc $\xi$, and with $\mathcal{K}$ lying on a 2-sphere in $E^3$. To do this the construction of [1] is simply performed with eyebolts hooking in a circular fashion at each stage. Thus, $\mathcal{K}$ lies on a 2-sphere in $E^3$, yet pierces no disk in $E^3$.

**Question.** Is $\mathcal{K}$ homogeneously embedded in $E^3$? Precisely, given points $p$ and $q$ in $\mathcal{K}$, is there a homeomorphism $h$ of $E^3$ onto itself such that $h(\mathcal{K}) = \mathcal{K}$ and $h(p) = q$?

**References**


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