DISJOINT OPEN SUBSETS OF $\beta X \setminus X$

BY

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1. **Introduction.** It is well known that if X denotes a countably infinite discrete set, then $\beta X \setminus X$ admits an uncountable collection of pairwise disjoint nonempty open subsets(2). In this paper we offer a number of generalizations of this theorem. We show for example (Corollary 3.2) that this statement about $\beta X \setminus X$ holds if X is any locally compact nonpseudocompact space. Our main result, Theorem 3.3, asserts that for an arbitrary completely regular space X and an arbitrary cardinal number $\mathfrak{m}$, the space $\beta X \setminus X$ admits a collection of $\mathfrak{m}$ pairwise disjoint nonempty open subsets if and only if X admits a collection $\mathscr{U}$ of $\mathfrak{m}$ cozero-sets with the following properties: (1) Each element of $\mathscr{U}$ contains a noncompact zero-set; (2) If U_a and U_b are distinct elements of $\mathscr{U}$, then $U_a \cap U_b$ has compact closure in X. This theorem is new and interesting only as it applies to infinite cardinal numbers. Those spaces X for which $\beta X \setminus X$ admits no pair of disjoint nonempty open subsets, for example, have been studied extensively. Building on the results of Doss [2] and Gál [4], Gillman and Jerison have listed thirteen characterizations of these spaces in 6J and 15R of [5].

The general results of §3 are coupled with a theorem of Tarski to answer in §4 the following question about pairs $(\mathfrak{m}, \mathfrak{n})$ of cardinal numbers: If D is the discrete space with $\mathfrak{n}$ points, is it true that $\beta D \setminus D$ admits a collection of $\mathfrak{m}$ pairwise disjoint nonempty open subsets? The answer (for infinite $\mathfrak{n}$) is "yes" if and only if $\mathfrak{m} \leqq \mathfrak{n}^{\aleph_0}$; there is, then, for a given cardinal number $\mathfrak{m}$, a nonpseudocompact space X for which $\beta X \setminus X$ admits a collection of $\mathfrak{m}$ pairwise disjoint nonempty open subsets. In §5 we construct, for each cardinal number $\mathfrak{m}$, a compact space Y which has $\mathfrak{m}$ pairwise disjoint nonempty open subsets, but for which the equation $Y = \beta X \setminus X$ has a solution only for pseudocompact X.

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2. **Definitions and references to the literature.** When X is a topological space, we denote by $C(X)$ the set of real-valued continuous functions on X, and by $C^*(X)$ the set of bounded real-valued continuous functions on X. A zero-set of X is a set of the form $f^{-1}\{0\}$, where $f \in C(X)$. The complement of a zero-set of X is called a cozero-set of X.

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(2) This fact can be derived from material in any one of the papers [7; 9; 10]. As far as we can determine, it is stated explicitly for the first time in 6 Q.2 of [5].

Throughout this paper we shall be concerned with the Stone-Čech compactification βX of the completely regular space X. The space βX is characterized among the compactifications of X (to within a homeomorphism keeping X pointwise fixed) by the fact that each element of $C^*(X)$ is the restriction to X of an element of $C(\beta X)$.

For a detailed development and study of βX, we refer the reader to [5]. We shall need the following two facts, which are Theorems 6.9 (d) and 6.5 (IV) respectively in [5].

2.1 LEMMA. *X is open in βX if and only if X is locally compact.*

2.2 LEMMA. *If Z_1 and Z_2 are zero-sets of X, then* $\mathrm{cl}_{\beta X}(Z_1 \cap Z_2) = \mathrm{cl}_{\beta X}Z_1 \cap \mathrm{cl}_{\beta X}Z_2$.

For our work in §4 with discrete spaces, we need Tarski's generalization (see [13], especially Theorem 14) of the following theorem, proved in [12] by Sierpiński: The set of integers admits a collection of $\mathfrak{c}$ infinite subsets, each pair with finite intersection. Because Tarski's result covers a situation much more general than the case which concerns us, we are able to offer a proof which is, at least superficially, simpler than his. The ideas involved, however, are all Tarski's.

2.3 THEOREM (TARSKI). *Let $\mathfrak{n}$ and $\mathfrak{m}$ be cardinal numbers, with $\mathfrak{n} \geqq \aleph_0$, and let D be a set with $\mathfrak{n}$ points. Then the following assertions are equivalent:*

(i) *D admits a collection $\mathscr{E}$ of infinite subsets, each pair with finite intersection, with* card $\mathscr{E} = \mathfrak{m}$;

(ii) $\mathfrak{m} \leqq \mathfrak{n}^{\aleph_0}$.

Proof. (i) $\Rightarrow$ (ii). With each E in $\mathscr{E}$ we associate a countably infinite subset S_E of E. The map $E \to S_E$ is one-to-one from $\mathscr{E}$ into the set of countably infinite subsets of D, and (ii) follows.

(ii) $\Rightarrow$ (i). Let F be the collection of all finite sequences drawn without repetition from D and let $\mathscr{E}$ be the set of all (countably infinite) sequences drawn without repetition from D. Then card $F = \mathfrak{n}$ and card $\mathscr{E} = \mathfrak{n}^{\aleph_0}$. For each e in $\mathscr{E}$, let $S_e = \{f \mid f \in F$ and f is an initial segment of $e\}$. Then card $S_e = \aleph_0$ for each e in $\mathscr{E}$, and distinct sets of the form S_e have only finitely many elements of F in common. Thus the identity $F = \bigcup_{e \in \mathscr{E}} S_e$ expresses a set of cardinality $\mathfrak{n}$—F, to be sure, not D—in the desired form.

2.4 REMARK. In the light of Tarski's theorem it is worthwhile to recall that exceeding any cardinal number there is a cardinal number $\mathfrak{m}$ for which $\mathfrak{m} = \mathfrak{m}^{\aleph_0}$ and (by König's theorem) a cardinal number $\mathfrak{n}$ for which $\mathfrak{n} < \mathfrak{n}^{\aleph_0}$. A characterization of those infinite cardinal numbers $\mathfrak{n}$ for which $\mathfrak{n} < \mathfrak{n}^{\aleph_0}$ does not particularly concern us here; the interested reader is referred to [1], especially 36.1.2.

2.5 Definition. Let X be a topological space and let $\mathfrak{m}$ be a cardinal number. We say that X has $d(\mathfrak{m})$ provided that X admits a collection $\mathscr{U}$ of pairwise disjoint nonempty open subsets for which card $\mathscr{U} = \mathfrak{m}$.

2.6 Discussion. Šanin in [11], and Erdös and Tarski in [3], have constructed, for a (hypothetical) inaccessible uncountable limit cardinal $\mathfrak{m}$, a topological space which has $d(\mathfrak{n})$ for each $\mathfrak{n} < \mathfrak{m}$ but which does not have $d(\mathfrak{m})$. But from Theorem 1 of [3] it follows that if $\mathfrak{m}$ is an accessible limit cardinal, or if $\mathfrak{m} = \aleph_0$, and if X is a topological space with $d(\mathfrak{n})$ whenever $\mathfrak{n} < \mathfrak{m}$, then X has $d(\mathfrak{m})$.

3. **The main theorems.** We recall that a collection $\mathscr{T}$ of subsets of a topological space X is said to be locally finite (in X) if each point in X admits a neighborhood intersecting at most finitely many elements of $\mathscr{T}$. A subset U of a topological space X is said to be relatively compact (in X) if $\mathrm{cl}_X U$ is compact.

3.1 Theorem. *Suppose that the completely regular space X admits a locally finite collection $\mathscr{T}$ of nonempty relatively compact open subsets, with* card $\mathscr{T}$ $= \mathfrak{m} \geqq \aleph_0$. *Then $\beta X \setminus X$ has $d(\mathfrak{m}^{\aleph_0})$.*

Proof. This first paragraph is devoted to showing that the sets in $\mathscr{T}$ may be shrunk to pairwise disjoint compact sets with nonvanishing interiors. Let ϕ be a function from $\mathscr{T}$ to X for which $\phi T \in T$ whenever $T \in \mathscr{T}$. We set $A = \{\phi T \mid T \in \mathscr{T}\}$, and for each a in A we choose T_a in $\mathscr{T}$ with $\phi(T_a) = a$. The map $a \to T_a$ is one-to-one from A onto a subset of $\mathscr{T}$, and the local finiteness of $\mathscr{T}$ guarantees that card $A = \mathfrak{m}$. Using again the local finiteness of $\mathscr{T}$ (and the complete regularity of X) we can find for each a in A compact sets U_a and Y_a for which simultaneously

$$a \in \mathrm{int}\ U_a \subset U_a \subset \mathrm{int}\ Y_a \subset Y_a \subset T_a$$

and $Y_a \cap A = \{a\}$. Now define

$$V_a = U_a \setminus \bigcup_{b \in A;\, b \neq a} \mathrm{int}\ Y_b.$$

Evidently V_a is a compact neighborhood of the point a. If a and b are distinct points in A and $x \in V_a$, then $x \notin \mathrm{int}\ Y_b$, so that $x \notin V_b$. Thus, with

$$\mathscr{V} = \{V_a \mid a \in A\},$$

we have: $\mathscr{V}$ is a locally finite collection of pairwise disjoint compact subsets of X, each with nonempty interior, for which card $\mathscr{V} = \mathfrak{m}$.

We now use the implication (ii) $\Rightarrow$ (i) of 2.3 to find a collection $\mathscr{E}$ of infinite subsets of A, each pair with finite intersection, for which card $\mathscr{E} = \mathfrak{m}^{\aleph_0}$. For each E in $\mathscr{E}$ we define

$$W_E = \beta X \setminus \mathrm{cl}_{\beta X}\Big(X \setminus \bigcup_{a \in E} V_a\Big).$$

The sets W_E being open in βX, we may complete the proof by showing

(i) each of the sets W_E meets $\beta X \setminus X$; and

(ii) the sets $W_E \cap (\beta X \setminus X)$ are pairwise disjoint.

We begin with (ii). Choosing distinct elements E_1 and E_2 of $\mathscr{E}$, we consider the relation

$$(*) \qquad X = \Big(X \setminus \bigcup_{a \in E_1} V_a\Big) \cup \Big(X \setminus \bigcup_{a \in E_2} V_a\Big) \cup \Big(\bigcup_{a \in E_1} V_a \cap \bigcup_{a \in E_2} V_a\Big).$$

The last of these sets, which for convenience we shall call K, is simply $\bigcup_{a \in E_1 \cap E_2} V_a$, and consequently is a compact subset of X. It now follows from (*), taking closures in βX, that

$$\beta X = \mathrm{cl}\Big(X \setminus \bigcup_{a \in E_1} V_a\Big) \cup \mathrm{cl}\Big(X \setminus \bigcup_{a \in E_2} V_a\Big) \cup K.$$

Complementation in βX yields the relation

$$W_{E_1} \cap W_{E_2} \subset K \subset X,$$

and (ii) follows.

To prove (i), we begin by invoking complete regularity of X to associate with each a in A a continuous function f_a on X to $[0, 1]$ for which $f_a(a) = 1$ and $f_a(x) = 0$ whenever $x \notin V_a$. Next for each E in $\mathscr{E}$ we set $f_E = \sum_{a \in E} f_a$. Local finiteness of $\mathscr{V}$ guarantees that the function f_E is continuous. Let

$$Z_E = \{x \in X \mid f_E(x) = 1\}$$

and notice that $E \subset Z_E \subset X$. Now E is an infinite, closed subset of X with no accumulation point in X; hence Z_E is not compact. Since Z_E is closed in X, then, there is a point p_E in $\beta X \setminus X$ for which $p_E \in \mathrm{cl}_{\beta X} Z_E$. Denoting by g_E the continuous extension to βX of f_E, we see that $g_E(p_E) = 1$. Since f_E vanishes on the set $X \setminus \bigcup_{a \in E} V_a$, g_E vanishes on the closure (in βX) of this set. Hence $p_E \in W_E$ and the proof is complete.

The converse to 3.1 is invalid, as Corollary 5.5 shows. The property of X equivalent to the property $d(\mathfrak{m})$ for $\beta X \setminus X$ is given in 3.3. Obviously the conclusion of 3.1 holds if one replaces the requirement that the sets in $\mathscr{T}$ be relatively compact with the hypothesis that X be locally compact; but Example 3.4 shows that the hypothesis cannot be dropped altogether.

The topological space X is said to be pseudocompact if each real-valued continuous function on X is bounded, i.e., if $C(X) = C^*(X)$. It is easy to show (see p. 370 of [6], for example) that a completely regular space is pseudocompact if and only if it admits no infinite locally finite collection of nonempty open subsets. These remarks, together with the observation that $\aleph_0^{\aleph_0} = \mathfrak{c}$, yield the following corollary to 3.1.

3.2 Corollary. *Let the completely regular space X be locally compact but not pseudocompact. Then $\beta X \setminus X$ has $d(\mathfrak{c})$.*

3.3 THEOREM. *Let X be a completely regular space and let* $\mathfrak{m}$ *be a cardinal number. Then the following assertions are equivalent*:

(i) $\beta X \setminus X$ *has* $d(\mathfrak{m})$;

(ii) *the space X admits a collection* $\{U_a\}_{a \in A}$ *of cozero-sets, with* card $A = \mathfrak{m}$, *for which*

(1) *each U_a contains a noncompact zero-set*;

(2) $U_a \cap U_b$ *is relatively compact in X whenever* $a \neq b$.

Proof. (i) $\Rightarrow$ (ii). There is a collection $\mathscr{V} = \{V_a\}_{a \in A}$ of pairwise disjoint nonempty open subsets of $\beta X \setminus X$, where card $A = \mathfrak{m}$. For each a in A we choose p_a in V_a and an open subset T_a of βX for which

$$T_a \cap (\beta X \setminus X) = V_a.$$

We next choose a continuous function f_a mapping βX into $[0, 1]$, with $f_a(p_a) = 1$ and $f_a(x) = 0$ whenever $x \notin T_a$, and we define

$$U_a = \{x \in X \mid f_a(x) > 1/2\};$$

$$Z_a = \{x \in X \mid f_a(x) \geqq 1/3\}.$$

To verify (1) it will suffice to show $p_a \in \mathrm{cl}_{\beta X} Z_a$. For this we need only show $p_a \notin \mathrm{cl}_{\beta X}(X \setminus Z_a)$ which is obvious since $f_a \leqq 1/3$ on $\mathrm{cl}_{\beta X}(X \setminus Z_a)$ and $f_a(p_a) = 1$. Now suppose that (2) fails; then there is (for a certain pair $\{a, b\}$ of indices in A) a point p in $\beta X \setminus X$ for which

$$p \in \mathrm{cl}_{\beta X}\mathrm{cl}_X(U_a \cap U_b) = \mathrm{cl}_{\beta X}(U_a \cap U_b).$$

Then we have $f_a(p) \geqq 1/2$ and $f_b(p) \geqq 1/2$. It follows that

$$p \in T_a \cap T_b \cap (\beta X \setminus X) = V_a \cap V_b = \varnothing,$$

a contradiction completing the proof that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Let Z_a be a noncompact zero-set of X, with $Z_a \subset U_a$. Using 2.2 and Urysohn's theorem we choose a continuous function f_a on βX to $[0, 1]$ for which $f_a \equiv 0$ on $\mathrm{cl}_{\beta X}(X \setminus U_a)$ and $f_a \equiv 1$ on $\mathrm{cl}_{\beta X} Z_a$. Let

$$V_a = \{p \in \beta X \mid f_a(p) > 0\}.$$

To complete the proof we need only show that each of the sets $V_a \cap (\beta X \setminus X)$ is nonempty and that $V_a \cap V_b \cap (\beta X \setminus X) = \varnothing$ whenever $a \neq b$. For the first of these statements we notice $\varnothing \neq \mathrm{cl}_{\beta X} Z_a \cap (\beta X \setminus X) \subset V_a \cap (\beta X \setminus X)$. For the second, suppose that $p \in V_a \cap V_b$ for some distinct a, b in A. Then $(f_a f_b)(p) > 0$, whence

$$p \notin \mathrm{cl}_{\beta X}(X \setminus U_a) \cup \mathrm{cl}_{\beta X}(X \setminus U_b).$$

Hence $p \in \mathrm{cl}_{\beta X}(U_a \cap U_b)$, so that $p \in X$ by hypothesis (ii)(2). Thus $V_a \cap V_b \cap (\beta X \setminus X) = \varnothing$.

3.4 EXAMPLE. Let Q be the space of rational numbers. Then $\beta Q \setminus Q$ does not have $d(\mathfrak{c})$.

Proof. Use (i) $\Rightarrow$ (ii) of 3.3, noticing that Q does not have $d(\mathfrak{c})$ and that the empty set is the only relatively compact open subset of Q.

Each of the next two results is an easy consequence of 3.2 and 3.3.

3.5 COROLLARY. *Let the completely regular space X be locally compact and not pseudocompact. Then X admits a collection of $\mathfrak{c}$ open subsets none of which is relatively compact but whose pairwise intersections are relatively compact.*

3.6 COROLLARY. *For a cardinal number $\mathfrak{m}$, the following assertions are equivalent:*

(i) *$\beta R \setminus R$ has $d(\mathfrak{m})$;*

(ii) $\mathfrak{m} \leqq \mathfrak{c}$.

4. A theorem about $\beta D \setminus D$ for discrete D.

4.1 THEOREM. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be cardinal numbers, with $\mathfrak{n} \geqq \aleph_0$, and let D be the discrete space for which* card $D = \mathfrak{n}$. *Then $\beta D \setminus D$ has $d(\mathfrak{m})$ if and only if $\mathfrak{m} \leqq \mathfrak{n}^{\aleph_0}$.*

Proof. Each subset of D is both a zero-set and a cozero-set of D, and the compact subsets of D are the finite subsets of D. From Theorem 3.3, then, we have: $\beta D \setminus D$ has $d(\mathfrak{m})$ if and only if D admits a collection of $\mathfrak{m}$ infinite subsets whose pairwise intersections are finite. Now 2.3 yields the result.

4.2 COROLLARY. *Let D be a discrete space and let $\mathfrak{m}$ be an infinite cardinal number. Then $\beta D \setminus D$ has $d(\mathfrak{m})$ if and only if $\beta D \setminus D$ has $d(\mathfrak{m}^{\aleph_0})$.*

Proof. Our assertion is vacuously true if card D is finite, and otherwise the result follows from 4.1 and the fact that $\mathfrak{m}^{\aleph_0} \leqq \mathfrak{n}^{\aleph_0}$ whenever $\mathfrak{m} \leqq \mathfrak{n}^{\aleph_0}$.

4.3 COROLLARY. *Let D be a discrete space and suppose that $\beta D \setminus D$ has $d(\mathfrak{m})$ for some infinite cardinal number $\mathfrak{m}$. Then $\beta D \setminus D$ has $d(\mathfrak{c})$.*

Proof. We have $\mathfrak{c} \leqq \mathfrak{m}^{\aleph_0}$.

We call attention to the following special case of 4.1, which is surprising when contrasted with the result given in the first sentence of this paper.

4.4 THEOREM. *Let D be the discrete space for which* card $D = \mathfrak{c}$, *and suppose that $\beta D \setminus D$ has $d(\mathfrak{m})$. Then $\mathfrak{m} \leqq \mathfrak{c}$.*

5. **Converse problems.** In the preceding sections we considered a given topological space, usually not pseudocompact, and we obtained results concerning those cardinal numbers $\mathfrak{m}$ for which $\beta X \setminus X$ has $d(\mathfrak{m})$. Here we consider briefly other aspects of the same situation. Theorems 5.4 and 5.6 constitute a short step toward a solution to the following question, whose solution is not known to us:

For what compact spaces Y is there a nonpseudocompact space X for which $Y = \beta X \setminus X$?

5.2 Definition. Let x be a point in the topological space X. Then x is a P-point in X if each G_δ in X that contains x is a neighborhood of x.

5.3 Theorem. *Suppose that the completely regular space X is not pseudocompact. Then $\beta X \setminus X$ contains at least $\mathfrak{c}$ non-P-points.*

Proof. It follows from 1.21 of [5] that $\beta X \setminus X$ contains a copy of $\beta N \setminus N$ It is easy to check that any non-P-point in this copy of $\beta N \setminus N$ is a non-P-point in $\beta X \setminus X$. We need only show, then, that $\beta N \setminus N$ contains at least $\mathfrak{c}$ non-P-points. From 4K.1 and 6S.6 of [5], it follows that the set of non-P-points in $\beta N \setminus N$ is dense in $\beta N \setminus N$. Our 3.2 (applied to N) now yields the result.

5.4 Theorem. *Let $\mathfrak{m}$ be a cardinal number. Then there is a locally compact space X with the following properties:*

(i) *$\beta X \setminus X$ has $d(\mathfrak{m})$;*

(ii) *$\beta X \setminus X$ does not have $d(\mathfrak{n})$ if $\mathfrak{n} > \mathfrak{m}$;*

(iii) *if X' is any topological space for which $\beta X' \setminus X' = \beta X \setminus X$, then X' is pseudocompact.*

Proof. We let Y denote either the discrete space with $\mathfrak{m}$ points or the one-point compactification of this space, according as $\mathfrak{m}$ is finite or infinite. We choose for X any space for which Y is (homeomorphic with) $\beta X \setminus X$. (If Z is any space for which card $(\beta Z \setminus Z) = 1$, then from Theorems 1 and 3 of [6] it follows that we may use $X = Y \times Z$. A similar construction for X is given in 9K of [5].) Clearly (i) and (ii) are satisfied, and X is locally compact by 2.1. Since $\beta X \setminus X$ contains at most one non-P-point, (iii) follows from 5.3.

The next result, promised earlier, is the example showing that the converse to 3.1 fails. The proof uses (iii) of 5.4 and the Glicksberg pseudocompactness criterion invoked in the proof of 3.2.

5.5 Corollary. *For each cardinal number $\mathfrak{m}$ there is a compact space Y, with $d(\mathfrak{m})$, with the following property: If $\mathscr{T}$ is a locally finite collection of nonempty open subsets of a topological space X for which $Y = \beta X \setminus X$, then $\mathscr{T}$ is finite.*

A topological space is said to be separable if it admits a countable dense subset.

5.6 Theorem. *For each a in A let Y_a be a compact separable space, and let $Y = \prod_{a \in A} Y_a$. If $Y = \beta X \setminus X$, then X is pseudocompact.*

Proof. It is easy to deduce from [8] the fact that no product of separable spaces can admit an uncountable collection of pairwise disjoint nonempty open subsets. (This fact follows from Marczewski's Theorem 3.3, once it is observed that any separable space has the property (K).) Hence the space Y does not have $d(\mathfrak{c})$. The result then follows from 3.2, 2.1 and the fact that Y is compact.

References

1. H. Bachmann, *Transfinite Zahlen*, Springer, Heidelberg, 1955.

2. R. Doss, *On uniform spaces with a unique structure*, Amer. J. Math. **71** (1949), 19–23.

3. P. Erdös and A. Tarski, *On families of mutually exclusive sets*, Ann. of Math. (2) **44** (1943), 315–329.

4. I. S. Gál, *Uniformizable spaces with a unique structure*, Pacific J. Math. **9** (1959), 1053–1060.

5. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N.J., 1960.

6. I. Glicksberg, *Stone-Čech compactifications of products*, Trans. Amer. Math. Soc. **90** (1959), 369–382.

7. M. Katětov, *On nearly discrete spaces*, Časopis Pěst. Mat. Fys. **75** (1950), 69–78.

8. E. Marczewski, *Séparabilité et multiplication cartésienne des espaces topologiques*, Fund. Math. **34** (1947), 127–143.

9. B. Pospíšil, *On bicompact spaces*, Publ. Fac. Sci. Univ. Masaryk No. 270 (1939), 16 pp.

10. W. Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409–419, 633.

11. N. A. Šanin, *On the product of topological spaces*, Dokl. Akad. Nauk SSSR **53** (1946), 591–593.

12. W. Sierpiński, *Sur une décomposition d'ensembles*, Monatsh. Math. Phys. **35** (1928), 239–243.

13. A. Tarski, *Sur la décomposition des ensembles en sousensembles presque disjoints*, Fund. Math. **12** (1928), 188–205.

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