ON THE HOPF-TODA INVARIANT

BY

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0. Introduction. One line of attack on the problem of computing the unstable homotopy groups of spheres is to attempt to construct the elements whose reduced product filtration exceeds 1. The classical Hopf construction associates with a map $S^m \times S^{t-1} \to S^n$ of type (α, β) an element of $\pi_{m+t}(S^{n+1})$. The indeterminacy of the construction is the suspension subgroup in the sense that the elements associated with any two maps of the same type differ by a suspension. However, it was proved by I. M. James [8] that the filtration of the element obtained does not exceed 2. In [3] I defined a generalization of the Hopf construction which can yield elements of arbitrarily large filtration. Thus, for example, let ∂ denote the homotopy boundary homomorphism and let $\alpha \in \pi_m(S^n_{r-2})$, $\beta \in \pi_t(S^n_{r-1}, S^n_{r-2})$ be elements[1] such that the Whitehead product $[\alpha, \partial\beta]$ is trivial. Then there exists a map $F: S^m \times S^{t-1} \to S_{r-2}$ of type $(\alpha, \partial\beta)$. Applying the construction we obtain an element $c(F) \in \pi_{m+t}(S^{n+1})$ with indeterminacy equal to the subgroup of filtration $r-2$. As always the problem is to prove nontriviality. To this end we study a homomorphism

$$(0.1) \qquad H: \pi_i(S^n_{r-1}, S^n_{r-2}) \to \pi_{i+1}(S^{rn})$$

equivalent to a special case of the "relative Hopf homomorphism" of H. Toda [14]. If $r = 2$ then H reduces to the generalized Hopf invariant H^* of P. J. Hilton [5]. Our main theorem may be stated as follows.

Let n be even and let $r = p^s$ be a power of a prime $p \geq 2$. Let $h_{r-1}: S^n_{r-1}, S^n_{r-2} \to S^{(r-1)n}$, $*$ be a map of degree 1, let E denote suspension and suppose that $(h_{r-1})_* \beta = E\beta''$. Let

$$\tau = E^{(r-1)n-1}(\phi i\alpha) \circ E^{m+1}\beta'' \in \pi_{m+t}(S^{rn}),$$

where $\phi: \pi_m(S^n_\infty) \to \pi_{m+1}(S^{n+1})$ denotes the James canonical isomorphism [6], and i is induced by the injection $S^n_{r-2} \to S^n_\infty$. Let $\{c(F)\}$ denote the coset of $c(F)$ in $\pi_{m+t}(S^{n+1})/\phi i\pi_{m+t-1}(S^n_{r-2})$. Then we have:

THEOREM 0.2. (a) *If* (i) τ *is of order* p^w, (ii) $\tau = p^k E^2\theta$, *where* $\theta \in \pi_{m+t-2}(S^{rn-2})$ *and* $E^3\theta$ *generates* $\pi_{m+t+2}(S^{rn+1}; p)$, *then* $\{c(F)\}$ *is of order* p^v, *where* $v \geq \min(s-k, w)$.

Received by the editors October 22, 1962.

[1] S^n_r denotes the rn-skeleton of the reduced product complex S^n_∞ of the n-sphere.

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(b)   *If $\tau = p^k \iota_{r_n}$ then $\{c(F)\}$ is of order p^v, where $v \geqq s - k$.*

For example, using the formula 4.4 due to Toda it is easy to prove that $[\iota_2, [\iota_2]^{p-1}] = 0 \in \pi_{2p-2}(S_{p-2}^2)$. Applying 0.2(b) we recover the nonzero elements of $\pi_{2p}(S^3)$. Our main application of 0.2 is however as follows. Let $\alpha_t \in \pi_{2t(p-1)+2}(S^3)$ $(t \geqq 1)$ be the elements of order p described by Toda in [16]. Then we prove for all $m + t \leqq p$ the existence of maps of type

$$(\iota_{2m}, [\iota_{2m}]^{p-1} \circ E^{2m(p-1)-4} \alpha_t)$$

whose Hopf construction elements are nontrivial. Using the computations due to Toda of the stable groups we are able to compute the orders of the p-components of the unstable groups of the q-stem for $q < 2p(p-1)$. Let p be an odd prime and let $[r, n]$ denote the p-primary component of $\pi_r(S^n)$. We obtain[2]

THEOREM 0.3.
(a) $[2t(p-1) + 2m - s, 2m + 1] = 0 \quad (m \geqq 1, \ 1 \leqq t \leqq p, \ 2 \leqq s \leqq 2p-3)$;
(b)   (i) $[2t(p-1) + 2m - 1, 2m + 1] = Z_p \quad (1 \leqq t \leqq p-1, \ 1 \leqq m \leqq t-1)$;
     (ii) $[2t(p-1) + 2m - 1, 2m + 1] = 0 (1 \leqq t \leqq p-1, m \geqq t)$;
     (iii) $[2p(p-1) + 1, 3] = Z_p$;
     (iv) $[2p(p-1) + 2m - 1, 2m + 1] = Z_{p^2}$ or $Z_p + Z_p \quad (2 \leqq m \leqq p-1)$
                                        $= Z_p \quad (m \geqq p)$;
(c)   (i) $[2t(p-1) + 2m, 2m + 1] = Z_p \quad (1 \leqq t \leqq p-1, \ m \geqq 1)$;
     (ii) $[2p(p-1) + 2m, 2m + 1] = Z_{p^2} \quad (m \geqq 2)$;
     (iii) $[2p(p-1) + 2, 3] = Z_p$.

1. **The relative Hopf invariant.**   Let $K = K_0 \cup e^r$ be a finite cell complex which is the result of attaching an r-cell e^r to the subcomplex K_0. Let $\theta: K \to K \vee S^r$ be a map which shrinks an $(r-1)$-sphere contained in e^r, is such that the composition of θ with the projection on to S^r is a map of degree one and such that the composition of θ with the projection on to K is homotopic to the identity map. Then we recall that the relative Hopf invariant due to Toda [14] is a homomorphism H' such that the diagram

$$
\begin{array}{ccccc}
 & & \xrightarrow{\hspace{3cm} Q' \hspace{3cm}} & & \downarrow \\
\pi_q(K \vee S^r, K) & \xleftarrow{\ j\ } & \pi_q(K \vee S^r) & \xleftarrow{\ \partial\ } & \pi_{q+1}(K \times S^r, K \vee S^r) \\
\uparrow{\theta_*} & & & & \downarrow{\chi} \\
\pi_q(K, K_0) & & \xrightarrow{\hspace{2.5cm} H' \hspace{2.5cm}} & & \pi_{q+1}(K \# S_r)
\end{array}
$$

is commutative where j is an injection, ∂ is the homotopy boundary, χ is induced by the collapsed product identification and Q' is a left inverse of $j\partial$.

Let V^n and S^n denote the n-element and n-sphere as represented in [7]. We shall assume that orientations for V^n, S^n and products have been chosen as in [7]. Then the homotopy boundary homomorphism $\partial : \pi_n(X, A) \to \pi_{n-1}(A)$ is defined accordingly. Let S denote the reduced suspension function in the category of based spaces and based maps. Let ι_1 be the preferred generator of $\pi_1(S^1)$ so that if $\gamma \in \pi_n(X)$ then

$$E\gamma = \gamma \# \iota_1 \in \pi_{n+1}(X \# S^1) = \pi_{n+1}(SX).$$

If $\alpha \in \pi_{t-1}(S^{q-1})$ and $\alpha' \in \pi_t(V^q, S^{q-1})$ are elements such that $\alpha = \partial\alpha'$ then, again following James, we shall denote by $\alpha^* : \pi_{q-1}(X) \to \pi_{t-1}(X)$ and $\alpha^{**} : \pi_q(X, A) \to \pi_t(X, A)$ the operators determined by composition with α and α'. We recall that

(1.1) $\alpha^{**} = (-1)^{q+t}(E\alpha)^* : \pi_q(X, *) \to \pi_t(X, *)$

It is well known [17] that α^* and α^{**} are homomorphisms if α is a suspension element. The following lemma is easily proved using 1.1 and the definition of H'.

LEMMA 1.2. $H'\alpha^{**} = (E^2\alpha)^* H'$.

Let $h : K, K_0 \to S^r, *$ be a map of degree one on e^r. Applying an argument essentially due to Toda we obtain the following theorem in which the square brackets refer to the Blakers-Massey generalized Whitehead product defined in [1]. Let $i : \pi_n(K_0) \to \pi_n(K)$ denote the injection.

THEOREM 1.3. If $\alpha \in \pi_m(K_0)$ and $\beta \in \pi_t(K, K_0)$ then

$$H'[\alpha, \beta] = (i\alpha) \# h_* \beta \in \pi_{m+t}(K \# S^r).$$

Proof. Let $\bigstar$ denote the product defined in [1; 5]. Then we have $\chi(i\alpha \bigstar h_*\beta) = i\alpha \# h_*\beta$. In view of the commutativity of the diagram it is thus sufficient to prove that $j\partial(i\alpha \bigstar h_*\beta) = \theta_*[\alpha, \beta]$. This however follows from [1, 5.7, 3.10] and the definition of θ.

Now suppose that α and β are such that $[\alpha, \partial\beta] = 0 \in \pi_{m+t-2}(K_0)$. Then there is a map $F : S^m \times S^{t-1} \to K_0$ of type $(\alpha, \partial\beta)$. Let

(1.4) $F' : (S^m \times S^{t-1}) \cup (* \times V^t) \to K$

be an extension of F such that $F'(*, x) = g(x)$ $(x \in V^t)$ where $g : V^t, S^{t-1} \to K, K_0$ represents β. Let ψ be a characteristic map for the $(m + t)$-cell of $S^m \times V^t$ and let

(1.5) $\gamma = \{F'\psi \mid S^{m+t-1}\} \in \pi_{m+t-1}(K).$

Then we have the theorem:

THEOREM 1.6. $j\gamma = [\alpha, \beta] \in \pi_{m+t-1}(K, K_0).$

Proof. Let E_+^{t-1} and E_-^{t-1} be the 'Northern' and 'Southern' hemispheres of S^{t-1}. We can certainly replace F by a homotopic map with the property that

(1.7) $F(x, y) = F(x, *)$ if $(x, y) \in S^m \times E_+^{t-1}$.

Then the injection $V^t \to (S^m \times S^{t-1}) \cup (* \times V^t)$ followed by F' is a map $(V^t, S^{t-1}, E_+^{t-1}) \to (K, K_0, *)$. Let $\psi_m : (V^m, S^{m-1}) \to (S^m, *)$ be a map of degree one and let $(V^m \times V^t)^{\cdot} = (S^{m-1} \times V^t) \cup (V^m \times S^{t-1})$. Then $\phi : (V^m \times V^t)^{\cdot} \to K$ represents γ where

(1.8) $\phi = F'((\psi_m \times 1) \,|\, (V^m \times V^t)^{\cdot})$.

However, if $A = (S^{m-1} \times V^t) \cup (V^m \times E_+^{t-1})$ then it follows from the definition [1, 3.1] that $\phi \,|\, A$ represents $[\alpha, \beta]$. $V^m \times E_-^{t-1}$ is the face of $V^m \times V^t$ complementary to A and $\phi(V^m \times E_-^{t-1}) \subseteq K_0$. Since A and $(V^m \times V^t)^{\cdot}$ are required to be oriented coherently, 1.6 follows from [10, 22.1].

2. **Reduced products.** Let A be a countable CW-complex with but one 0-cell at the base-point $*$. The reduced product complex A_∞ [6] is a space whose points are equivalence classes of finite sequences of points of A. Let $A_n (n \geq 0)$ denote the subcomplex determined by the sequences with n terms or fewer, A_0 being identified with $*$ and A_1 with A. Let Ω be the space of based loops on SA, let $\sigma : S\Omega \to SA$ be the map such that $\sigma(x, t) = x(t)$ and let $\phi_1 : A_\infty \to \Omega$ be a *canonical* map in the sense of [6]. Then if $\psi = \sigma S(\phi_1)$ we recall that

(2.1) $\phi = \psi_* E : \pi_q(A_\infty) \to \pi_{q+1}(SA)$

is the *canonical isomorphism* of James. We shall also denote by ψ the maps from $S(A_n)$ to $S(A)$ which agree with ψ.

Let $\alpha \in \pi_m(A_n)$, $\beta' \in \pi_{t-1}(A_n)$ be elements whose Whitehead product $[\alpha, \beta'] \in \pi_{m+t-2}(A_n)$ vanishes. Then there is a map $F : S^m \times S^{t-1} \to A_n$ of type (α, β'). The generalization of the Hopf construction described in [3] associates with F the element

(2.2) $c(F) = (\psi S(F)\eta)_* \iota \in \pi_{m+t}(SA)$,

where ι denotes the orientation class of $S(S^m \# S^{t-1})$ and η is a homotopy right inverse of the suspension of the collapsed product identification $\chi : S^m \times S^{t-1} \to S^m \# S^{t-1}$.

$\pi_q(A_\infty)$ is filtered by the subgroups $i\pi_q(A_n)$ and the *reduced product filtration* in $\pi_{q+1}(SA)$ is that induced by ϕ. The following lemma (on which we shall not rely) is a consequence of the definition 2.2.

LEMMA 2.3.   *If $\sigma \in \pi_{m+t}(SA)$ then there exists a map $G : S^m \times S^{t-1} \to A_n$ of type (α, β') such that $c(G) = \sigma$ if and only if the filtration of $\sigma - c(F)$ does not exceed n.*

Now let $A = S^n$, $K = S_{r-1}^n$, $K_0 = S_{r-2}^n$. Then H' as defined in 1 is a homomorphism

$$H' : \pi_q(S^n_{r-1}, S^n_{r-2}) \to \pi_{q+1}(S^n_{r-1} \# S^{(r-1)n}).$$

Let

(2.4) $$H = \lambda_* H' : \pi_q(S^n_{r-1}, S^n_{r-2}) \to \pi_{q+1}(S^{rn}),$$

where λ is the map defined by the diagram

$$
\begin{array}{ccc}
S^n_{r-1} \# S^{(r-1)n} = S^n_{r-1} \# S' \# S^{(r-1)n-1} = S(S^n_{r-1}) \# S^{(r-1)n-1} \\
\lambda \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \psi \# 1 \\
S^{rn} =\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!=\!\!= S(S^n) \# S^{(r-1)n-1}
\end{array}
$$

in which the equalities denote orientation preserving homeomorphisms and $\psi \# 1$ the collapsed product map. Let $\alpha \in \pi_m(S^n_{r-2})$, $\beta \in \pi_t(S^n_{r-1}, S^n_{r-2})$ then we have:

THEOREM 2.5.   *If* $h_* \beta = E\beta''$, *where* $\beta'' \in \pi_{t-1}(S^{(r-1)n-1})$ *then*

$$H[\alpha, \beta] = (-1)^k E^{(r-1)n-1} (\phi i \alpha) \circ E^{m+1} \beta'',$$

where $k = n(m+1)(r-1) + mt - 1$.

Proof.   Applying 1.3, 2.1 and finally [7, 3.5] we obtain

$$
\begin{aligned}
H[\alpha, \beta] &= \lambda_*(i\alpha \# \beta'' \# \iota_1) \\
&= (-1)^{t-1} \lambda_*(i\alpha \# \iota_1 \# \beta'') \\
&= (-1)^{t-1} (\phi i \alpha) \# \beta'' \\
&= (-1)^k E^{(r-1)n-1} (\phi i \alpha) \circ E^{m+1} \beta''.
\end{aligned}
$$

Now suppose that $F : S^m \times S^{t-1} \to S^n_{r-2}$ is a map of type $(\alpha, \partial \beta)$ and let $\gamma \in \pi_{m+t-1}(S^n_{r-1})$ be defined by 1.5. Then we assert that

(2.6) $$\phi i \gamma = \pm c(F).$$

A proof of a special case can be found in [4, p. 248–9][3]. The argument is quite general, however, and in the present situation yields 2.6.

Let $h_r : S^n_r, S^n_{r-1} \to S^{rn}, *$ be a map of degree 1. We shall also denote by h_r any map and any homomorphism induced by any map which agrees with a cellular approximation of the combinatorial extension $h_r : S^n_\infty \to S^{rn}_\infty$ of h_r. (See [6, p. 173].) Let p be a prime and let $\mathscr{C}_p$ denote the class of all finite Abelian groups of order prime to p. We shall require the following result of [2].

LEMMA 2.7.   *If* n *is even and* $r = p^q$ *then there are* $\mathscr{C}_p$-*isomorphisms*

$$h_r : \pi_i(S^n_\infty, S^n_{r-1}) \to \pi_i(S^{rn}_\infty) \qquad (i > 0).$$

---

[3] There is a misprint in [4, p. 249, line 5]: $(k'')_* \iota$ generates the summand $\pi_{2p}(S^{2p})$.

3. The E-H-Q sequence. Let n be even, let r be a power of a prime $p \geqq 2$. Then, in view of 2.7, $\phi h_r : \pi_q(S^n_\infty, S^n_{r-1}) \to \pi_{q+1}(S^{rn+1})$ certainly defines an isomorphism of the p-components. Let μ_t denote the characteristic class of the tn-cell of S^n_r and also its image under injection in $\pi_{tn-1}(S^n_\infty, S^n_{t-1})$. Let $k : \pi_{q+1}(S^{rn+1}) \to \pi_q(S^n_\infty, S^n_{r-1})$ be the homomorphism such that

$$ka = (\phi h_r)^{-1} a \quad \text{if the order of } \alpha \text{ is a power of } p,$$

(3.1)
$$= 0 \quad\quad \text{if } \alpha \text{ is of finite order prime to } p,$$

$$= \mu_r \quad\quad \text{if } \alpha = \iota_{rn+1}.$$

Then since $\pi_{q+1}(S^{rn+1})$ is finite unless $q = rn$, k is a $\mathscr{C}_p$-isomorphism. Let $\mathscr{C}$ be a class of Abelian groups. We recall that a sequence $G_1 \xrightarrow{\phi_2} G_2 \xrightarrow{\phi_2} G_3$ is $\mathscr{C}$-exact at G_2 in the sense of Spanier [12] if $\phi_2 \phi_1 = 0$ and if the homology group at G_2 is in $\mathscr{C}$. Let $H_r = \phi h_r j \phi^{-1}$, $Q_r = \partial k$ and let $E_r = \phi i$. It follows easily that the sequence

(3.2)
$$\to \pi_q(S^n_{r-1}) \xrightarrow{E_r} \pi_{q+1}(S^{n+1}) \xrightarrow{H_r} \pi_{q+1}(S^{rn+1}) \xrightarrow{Q_r} \pi_{q-1}(S^n_{r-1}) \to$$

is $\mathscr{C}_p$-exact.

Let $\alpha \in \pi_{q-1}(S^{rn-1})$ be of infinite order or else of order a power of p. We require the lemma:

LEMMA 3.3. $Q_r E^2 \alpha = (-1)^q [\iota_n]^r \circ \alpha.$

Proof. In the following diagram, i' is an injection, θ a map of degree 1 and $P = ([\iota_n]^r)_*$.

$$
\begin{array}{ccccccc}
\pi_{q-1}(S^n_{r-1})1 & \xleftarrow{\partial} & \pi_q(S^n_\infty, S^n_{r-1}) & \xrightarrow{h_r} & \pi_q(S^{rn}_\infty) & \text{---------} & \\
\Big\uparrow P & & \Big\uparrow \mu_* & & \Big\uparrow i' & & \Big\downarrow \phi \\
\pi_{q-1}(S^{rn-1}) & \xleftarrow{\partial} & \pi_q(V^{rn}, S^{rn-1}) & \xrightarrow{\theta_*} & \pi_q(S^{rn}) & \xrightarrow{E} & \pi_{q+1}(S^{rn+1}). \\
\Big| & & & (-1)^q E & & & \\
\end{array}
$$

The lower rectangle is commutative, for let $\beta' \in \pi_q(V^{rn}, S^{rn-1})$ and let $\beta = \partial \beta'$. Then $\theta_* \beta' = \beta^{**} \iota_{rn} = (-1)^q (E\beta)^* \iota_{rn} = (-1)^q E\beta$. The other portions of the diagram are also commutative and the lemma follows from the definition of Q_r.

Let Δ be defined by the commutative diagram

(3.4)
$$
\begin{array}{ccc}
\pi_{q-1}(S^n_{r-1}) & \xrightarrow{j} & \pi_{q-1}(S^n_{r-1}, S^n_{r-2}) \\
\Big\uparrow Q_r & & \Big\downarrow H \\
\pi_{q+1}(S^{rn+1}) & \xrightarrow{\Delta} & \pi_q(S^{rn}).
\end{array}
$$

By [14, 5.9″] we have $j[\iota_n]^r = r[\iota_n, \mu_{r-1}]$ and hence applying 3.3, 1.2 and 2.5 we obtain:

LEMMA 3.5.   If $\gamma \in \pi_{q-2}(S^{rn-2})$ is of infinite order or else of order a power of p then $\Delta E^3 \gamma = -rE^2 \gamma$.

Proof of Theorem 0.2. Let i' denote the injection $\pi_{m+t-1}(S^n_{r-2}) \to \pi_{m+t-1}(S^n_{r-1})$. Then by the exactness of the homotopy sequence of the pair (S^n_{r-1}, S^n_{r-2}) we have

$$\text{kernel } j = i' \pi_{m+t-1}(S^n_{r-2}).$$

Moreover, by the $\mathscr{C}_p$-exactness of the E-H-Q sequence we have $p^v \gamma \in \text{kernel } E_r$ if and only if $p^v \gamma \in Q_r \pi_{m+t+1}(S^{rn+1})$. Since, by 2.6, $c(F) = \pm E_r \gamma$, it follows that to prove 0.2(a) we have to show that

$$(3.6) \qquad p^v \gamma \notin i' \pi_{m+t-1}(S^n_{r-2}) + Q_r \pi_{m+t+1}(S^{rn+1}) \quad (v < \min(s-k, w)).$$

Now combining 1.6 and 2.5 we obtain

$$(3.7) \qquad Hj\gamma = \pm E^{(r-1)n-1}(\phi i\alpha) \circ E^{m+1}\beta'' = \pm \tau$$

and if the conditions of 0.2(a) are satisfied then 3.5 implies that

$$\Delta E^3 \theta = -rE^2 \theta = -p^s E^2 \theta$$

and hence that

$$(3.8) \qquad p^v \tau = p^{k+v} E^2 \theta \notin \Delta \pi_{m+t+1}(S^{rn+1}) \quad (v < \min(w, s-k)).$$

3.6 follows from 3.7 and 3.8. If $\tau = p^k \iota_{rn}$ then $\Delta \pi_{rn+1}(S^{rn+1})$ is generated by $p^s \iota_{rn}$ and the assertion of 3.6 holds for $v < s - k$. This completes the proof.

Let $\beta'' \in \pi_{t-1}(S^{(r-1)n-1})$. For our main application of 0.2(a) we need the lemma:

LEMMA 3.9.   (a) $\partial \beta''^{**} \mu_{r-1} = [\iota_n]^{r-1} \circ \beta''$, (b) $(h_{r-1})_* \beta''^{**} \mu_{r-1} = (-1)^t E\beta''$. Since $\partial \mu_{r-1} = [\iota_n]^{r-1}$, 3.9(a) follows from the definition of β''^{**}. But $(h_{r-1})_* \mu_{r-1} = \iota_{(r-1)n}$ and hence we obtain 3.9(b) on applying 1.1.

4. The double suspension. Let p be an odd prime. The following lemma is due to J. C. Moore [11].

LEMMA 4.1. $E^2 : [q, 2m-1] \to [q+2, 2m+1]$ is   an   isomorphism   if $q < 2mp - 3$ and an epimorphism if $q = 2mp - 3$.

In the case $q = 2mp - 3$ the kernel of E^2 is Z_p or 0 according as the mod p Hopf invariant is trivial or not. (See the exact sequence on p. 147 of [15, II].) It follows from the result of Liulevicius [9] that the former is the case for $m > 1$. Let i_{p-1} denote the homomorphism induced by the injection $S^{2m} \to S^{2m}_{p-1}$. To obtain further information Toda [14] developed a technique for studying the homomorphisms E_p and

$$i_{p-1}E : [q + 2m - 1, 2m - 1] \to \pi_{q+2m}(S^{2m}_{p-1}; p)$$

whose composition is E^2. As we shall see this is particularly effective in the range $2m(p-1) \leqq q < 2m(p^2-1) - 4$.

Let $n \geqq 4$ be even and let ΩS^n_{p-1} be the space of loops on S^n_{p-1} based at $*$. S^{n-1} is canonically imbedded in ΩS^n_{p-1} and 7.8(ii) of [14] defines an isomorphism

$$\bar{H}_p : \pi_{q-1}(\Omega S^n_{p-1}, \ S^{n-1}; p) \to [q, pn-1].$$

Let $\Omega : \pi_q(S^n_{p-1}) \to \pi_{q-1}(\Omega S^n_{p-1})$ be the natural isomorphism. Then if $m > 1$ we have a sequence

$$\xrightarrow{J} [q + 2m + 1, 2mp - 1] \xrightarrow{\chi} [q + 2m - 1, 2m - 1] \xrightarrow{i_{p-1}E} \pi_{q+2m}(S^2_{p-1}; p) \xrightarrow{J},$$

where $\chi = \partial(\bar{H}_p)^{-1}$ and $J = \bar{H}_p j \Omega$, which is exact in the range $q > 2m(p-1)$ since the corresponding groups in the homotopy sequence of $(\Omega S^{2m}_{p-1}, S^{2m-1})$ are finite if $q > 2m(p-1)$. Suppose that $m > 1$ and $2m(p-1) < q < 2m(p^2-1) - 4$. Then applying 4.1 we obtain the commutative diagram

(4.2)

$$
\begin{array}{c}
G(w+2) \\
\downarrow \chi \\
[q+2m-1, 2m-1] \longrightarrow \\
\downarrow i_{p-1}E \qquad\qquad \downarrow E^2 \\
G(w+1) \xrightarrow{Q_p} \pi_{q+2m}(S^{2m}_{p-1}; p) \xrightarrow{E} [q+2m+1, 2m+1] \xrightarrow{H_p} G(w) \\
\downarrow J \\
G(w+1)
\end{array}
$$

where $w = q - 2m(p-1)$, and in which $G(q)$ denotes the p-component of the stable group of the q-stem. We remark that the horizontal sequence of 4.2 applies also in the case $m = 1$.

We shall assume the following information concerning the stable groups extracted from [15, 4.15].

LEMMA 4.3.

$$G(2t(p-1) - s) = 0 \quad (1 \leqq t \leqq p, \ 3 \leqq s \leqq 2p-2);$$

$$G(2t(p-1) - 2) = 0 \quad (1 \leqq t \leqq p-1); \qquad G(2p(p-1) - 2) = Z_p;$$

$$G(2t(p-1) - 1) = Z_p \ (1 \leqq t \leqq p-1); \qquad G(2p(p-1) - 1) = Z_{p^2}.$$

Proof of Theorem 0.3. We are concerned with the p-components of the groups of the q-stem (odd-dimensional spheres only) for $q < 2p(p-1)$. Thus we already know the stable p-component and using the result [14, 4.2]

(4.4) $\pi_{q+2}(S_{p-1}^2; p) = G(q - 2p + 3)$ $(1 \leqq q < 2p^2 - 5)$

we have a clue (via the E-H-Q sequence) to $[q + 3, 3]$. When these extremes are trivial and the groups $G(w)$ and $G(w + 1)$ of diagram 4.2 are trivial $(m \geqq 1)$, all double suspensions are epimorphisms and hence isomorphisms and all p-components are trivial. That such considerations yield 0.3(a) we leave to the reader. The idea of the remainder of the proof is as follows. Using 0.3(a) we prove trivial many Whitehead products. When the conditions of 0.2(a) are satisfied, corresponding Hopf construction elements appear in the cokernels of E^2. Thus for certain stems we are able to obtain a lower bound for the product of the orders of the successive cokernels of E^2. However, using diagram 4.2 we obtain an upper bound and when the bounds are equal all can be settled.

Consider the Whitehead products

(4.5) $\xi_{m,t} = \left[\iota_{2m}, [\iota_{2m}]^{p+1} \circ E^{2m(p-1)-4}\alpha_t\right] \in \pi_z(S_{p-2}^{2m})$,

where $z = 2(m + t)(p - 1) + 2m - 3$. We first prove:

LEMMA 4.6. $\xi_{m,t} = 0$ if $m + t \leqq p$.

We have, by $[14, 4.2]$ and 0.3(a), that

$$\pi_{2(t+1)(p-1)-1}(S_{p-2}^2; p) \approx G(2t(p-1))   (t \leqq p-1)$$

and hence 4.3 implies that $\xi_{1,t} = 0$ $(1 \leqq t \leqq p-1)$. If $m > 1$ we have, by $[14, 4.6]$, that

$$\pi_z(S_{p-2}^{2m}; p) \approx [2(m + t)(p-1) + 2m - 4, 2m - 1] + G(2(t + 1)(p-1) - 2(p - m))$$

$$(m + t \leqq p).$$

Since it follows from 0.3(a) that the first summand is trivial and from 3.3 that the second is trivial, we have proved 4.6. The formulae 3.9 now enable us to apply 0.2(a). We need merely observe that $\tau = E^{2mp-3}\alpha_t$ is of order p and that $E\tau$ generates $[2mp + 2t(p-1), 2mp + 1] \approx G(2t(p-1) - 1)$. Thus the conditions of 0.2(a) are satisfied with $w = s = 1$, $k = 0$ and it follows that the elements

$$\beta_{m,t} \in [2(m + t)(p - 1) + 2m - 1, 2m + 1]   (2 \leqq m + t \leqq p)$$

obtained by applying the Hopf construction to maps of type

$$(\iota_{2m}, [\iota_{2m}]^{p-1} \circ E^{2m(p-1)-4}\alpha_t)$$

are nonzero and of filtration $p - 1$.

Suppose that $1 \leqq t \leqq p - 1$. We proceed to examine the groups of the $(2t(p-1)-2)$-stem and prove

LEMMA 4.7.

$$[2t(p-1) + 2m - 1, 2m + 1] = \begin{cases} 0 & (m \geqq t), \\ Z_p & (m = 1, m = t-1). \end{cases}$$

The case $m \geqq t$ of 4.7 follows from 4.3 for here we are in the stable range. If $m = t - 1$ the remark following 4.1 implies that the kernel of $E^2 : [2t(p-1) + 2m - 1, 2m + 1] \to G(2t(p-1)-2) = 0$ is Z_p. If $m = 1$, 4.4 and 4.3 imply that $\pi_{2t(p-1)}(S^2_{p-1}; p) = Z_p$. Moreover, the image of a generator under the homomorphism

$$E_p : \pi_{2t(p-1)}(S^2_{p-1}; p) \to [2t(p-1) + 1, 3]$$

has to be $\beta_{1,t}$. The required result follows from the fact that E_p is an epimorphism which can be deduced from the vanishing of the appropriate group in the E-H-Q sequence.

Lemma 4.7 includes the cases $m = 1$, $m = t - 1$ of 0.3(b)(i). Let Γ be the product of the orders of the cokernels and K the product of the orders of the kernels of the homomorphisms

$$E^2 : [2t(p-1) + 2m - 1, 2m + 1] \to [2t(p-1) + 2m + 1, 2m + 3]$$

(4.8)
$$(1 \leqq m \leqq t - 2).$$

Since $\beta_{m,t-m}$ $(2 \leqq m \leqq t - 1)$ make contributions to the cokernels we have

$$\Gamma \geqq p^{t-2}.$$

The groups $G(w + 2)$ with $w = 2t(p-1)-2$, $t < p$ are trivial. Thus diagram 4.2 implies that the factors $i_{p-1}E$ are monomorphisms, and, since $G(2t(p-1)-1) = Z_p$, the orders of the kernels of the factors E_p are at most p. Hence we have

$$K \leqq p^{t-2}.$$

However 4.7 implies that $\Gamma = K$ and hence $\Gamma = K = p^{t-2}$. Thus the kernel and cokernel of each E^2 is Z_p. The remaining cases of 0.3(b)(i) follow immediately.

We now study the $(2p(p-1)-2)$-stem and obtain the following analogue of Lemma 4.7.

LEMMA 4.9.
$$[2p(p-1) + 2m - 1, 2m + 1] = \begin{cases} Z_p & (m \geqq p), \\ Z_{p^2} \text{ or } Z_p + Z_p & (m = p-1, m = 2). \end{cases}$$

If $m \geqq p$ we are in the stable range. The case $m = p - 1$ of 4.9 follows from the fact that the kernel of $E^2 : [2p^2 - 3, 2p - 1] \to [2p^2 - 1, 2p + 1]$ is Z_p and hence $[2p^2 - 3, 2p - 1]$ is an extension of Z_p by Z_p.

The case $m = 2$ we postpone and remark that the assertion 0.3(b)(iv) follows from 4.9 and an argument (which we leave to the reader) exactly similar to the one given above showing that the kernels and cokernels of the intermediate double suspensions are all Z_p.

To complete the proof of 4.9 we first prove

LEMMA 4.10. (a) $[2p(p-1)+1, 3] = Z_p = \{\beta_{1,p-1}\}$, (b) $\beta_{1,p-1} = \lambda \alpha_1 \circ E^{2p-3} \alpha_{p-1}$, where $\lambda \not\equiv 0 \pmod{p}$.

By 4.4 and 4.3, $\pi_{2p(p-1)}(S_{p-1}^2; p) \approx G(2(p-1)^2 - 1) = Z_p$, which is generated by the stable suspension of α_{p-1}. 4.10(b) now follows from the fact that the isomorphism 4.4 is induced by composition with the element α' of $\pi_{2p-1}(S_{p-1}^2)$ such that $E_p \alpha' = \alpha_1$. Moreover on examining the appropriate group in the E-H-Q sequence we find that E_p maps on to $[2p(p-1) + 1, 3]$ which proves 4.10(a) and 0.3(b)(iii).

We next observe from diagram 4.2 using our knowledge of the stable groups that

$$i_{p-1}E : [2p(p-1) + 1, 3] \to \pi_{2p(p-1)+2}(S_{p-1}^4; p) = G \text{ (say)}$$

is a monomorphism with cokernel no larger than Z_p. Since $\beta_{2,p-2}$ has filtration $p - 1$ it follows that the cokernel is precisely Z_p and hence that G is an extension of Z_p by Z_p. It only remains to prove that $E_p : G \to [2p(p-1) + 2, 5]$ is a monomorphism (it is certainly onto) or equivalently that the image of $Q_p : G(2(p-2)(p-1) - 1) = Z_p \to G$ is the zero of G. Now the iterated suspension of α_{p-2} generates $G(2(p-2)(p-1) - 1)$. Thus, by Lemma 3.3, it will be sufficient to prove:

LEMMA 4.11. $[\iota_4]^p \circ E^{4p-4}\alpha_{p-2} = 0 \in G$.

Suppose 4.11 is false. Then 4.10(b) implies that $E^2\alpha_1 \circ E^{2p-1}\alpha_{p-1} = 0$. Thus the toric element

$$\gamma = \{E^2\alpha_1, E^{2p-1}\alpha_{p-1}, p\iota_{k+3}\} \in \pi_{k+4}(S^5),$$

where $k = 2p(p-1)$, is defined. Moreover by [15, 4.17(ii)] its stable suspension is a generator of $G(k-1) = Z_{p^2}$, which implies that the order of γ is at least p^2. Now $[k + 2, 3]$ contains the nonzero element α_p and since $\pi_{k+1}(S_{p-1}^2; p) = 0$ and $[k + 2, 2p + 1] = Z_p$ it follows that $[k + 2, 3] = Z_p = \{\alpha_p\}$. On examining the relevant groups in 4.2 we find that $i_{p-1}E : [k+2, 3] \to \pi_{k+3}(S_{p-1}^4; p)$ is an isomorphism and hence that $\pi_{k+3}(S_{p-1}^4; p) = Z_p$. We can conclude that

$$H_p\gamma \neq 0 \in [k + 4, 4p + 1] = \{E^{4p-2}\alpha_{p-2}\}$$

and hence that $Q_p E^{4p-2}\alpha_{p-2} = [\iota_4]^p \circ E^{4p-4}\alpha_{p-2} = 0$, which contradicts the assumption. The proof of 4.9 and 0.3(b) is now complete.

It only remains to prove 0.3(c). We have already seen that $[2t(p-1) + 2, 3] = Z_p$ in the case $t = p$. The case $1 < t < p$ which is similar we leave to the reader and the result is well known if $t = 1$. 0.3(c)(i) and (ii) may be deduced from 4.3 and the lemma:

LEMMA 4.12.   If $t = p$ and $m \geq 3$ or if $t < p$ and $m \geq 2$ then $E^2 : [2t(p-1) + 2m - 2, 2m - 1] \to [2t(p-1) + 2m, 2m + 1]$ is an isomorphism.

If $m > t$, 4.12 follows from 4.1. If $m \leq t$ we may refer to diagram 4.2 from which we easily obtain that the factors $i_{p-1}E$ are isomorphisms and that the

factors E_p are monomorphisms. However in our study of the $(2t(p-1)-2)$-stems $(t \leqq p)$ we saw that the kernel of

$$E_p: \pi_{2t(p-1)+2m-2}(S_{p-1}^{2m}; p) \to [2t(p-1)+2m-1, 2m+1]$$

is Z_p if $t < p$ and $2 \leqq m \leqq t-1$ or if $t = p$ and $3 \leqq m \leqq p-1$. Since $G(2(t-m)(p-1)-1) = Z_p$, the exactness of the E-H-Q sequence implies that the corresponding homomorphisms Q_p are nontrivial and that the E_p factors in the $(2t(p-1)-1)$stems are epimorphisms and hence isomorphisms, which proves 4.12.

BIBLIOGRAPHY

1. A. L. Blakers and W. S. Massey, *Products in homotopy theory*, Ann. of Math. (2) **58** (1953), 295–324.

2. K. A. Hardie, *Note on the Hopf-James invariants*, Proc. Cambridge Philos. Soc. **57** (1961), 746–753.

3. ———, *A generalization of the Hopf construction*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 196–204.

4. ———, *Higher Whitehead products*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 241–249.

5. P. J. Hilton, *Suspension theorems and the generalized Hopf invariant*, Proc. London Math. Soc. (3) **1** (1951), 462–493.

6. I. M. James, *Reduced product spaces*, Ann. of Math. (2) **62** (1955), 170–197.

7. ———, *On the suspension triad*, Ann. of Math. (2) **63** (1956), 191–247.

8. ———, *Filtration of the homotopy groups of spheres*, Quart. J. Math. Oxford Ser. (2) **9** (1958), 301–309.

9. A. Liulevicius, *The factorization of cyclic reduced powers*, Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 978–981.

10. W. S. Massey, *Exact couples in algebraic topology. I, II*, Ann. of Math. (2) **56** (1952), 363–396.

11. J. C. Moore, *Some applications of homology theory to homotopy problems*, Ann. of Math. (2) **58** (1953), 325–350.

12. E. H. Spanier, *Duality and the suspension category*, pp. 259–272, Symposium Internacional de Topologia Algebraica, Unesco, 1958.

13. H. Toda, *Generalised Whitehead products and homotopy groups of spheres*, J. Inst. Polytech. Osaka City Univ. Ser. A **3** (1952), 43–82.

14. ———, *On the double suspension* E^2, J. Inst. Polytech. Osaka City Univ. Ser. A **7** (1956), 103–145.

15. ———, *p-Primary components of homotopy groups. IV*, Mem. Coll. Sci. Univ. Kyoto Ser. **32** (1959), 297–332.

16. ———, *On unstable homotopy of spheres and classical groups*, Proc. Nat. Acad. Sci. U.S.A. **46** (1960) 1102–1105.

17. G. W. Whitehead, *A generalization of the Hopf invariant*, Ann. of Math. (2) **51** (1950), 192–237.

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