ERRATA AND ADDENDA TO "A SUBGROUP THEOREM FOR FREE NILPOTENT GROUPS"

BY

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In Moran [1] we analyzed the structure of subgroups of free nilpotent groups. We gave necessary conditions for a group to be isomorphic to a subgroup of a free nilpotent group. Then we went on in Theorem 3.7 to show that under certain circumstances these conditions are also sufficient. However, while trying to prove this latter result, we fell into a number of errors which make the proof far from convincing. As it is not possible to rectify these errors in a concise manner, we give a new proof of a slightly modified theorem which we hope will stand up to the test of time.

Throughout we shall use the results and notation of our paper [1]. We introduce one new concept which will help to simplify the proof. Let B be a group generated by a set

$$B_1, B_2, \cdots, B_n$$

of its subgroups, where these subgroups satisfy the conditions

$$[B_i, B_j] \subseteq \{B_{i+j}, \cdots, B_n\} \text{ for } i+j \leqq n,$$

while $[B_i, B_j] = 1$, for $i + j > n$. Then we shall say that the above system of subgroups is an *nth basic system of subgroups*. If B is a subgroup of a free nth nilpotent group and the subgroups

$$B_1, B_2, \cdots, B_n$$

are constructed as in the proof of Theorem 3.1, then we shall say that

$$B_1, B_2, \cdots, B_n$$

is an *nth basic system of subgroups of a free nth nilpotent group*. Two nth basic systems of subgroups are said to be isomorphic if the following two conditions are satisfied. The group generated by one nth basic system of subgroups is isomorphic to the group generated by the other nth basic system of subgroups. Secondly this isomorphism can be obtained by isomorphisms of the corresponding subgroups which make up the two basic systems.

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We say that an abelian group is a "bounded by free" abelian group if it is an extension of a bounded abelian group by a free abelian group. It is a well-known result in the theory of abelian groups that only one such extension exists, namely, the direct product of these two groups.

We are now in a position to give the following converse to the partial Subgroup Theorem 3.4.

THEOREM 3.7([1]). *Let B be a group generated by a set of n subgroups $B_1, B_2, \cdots, B_n$ which satisfy the conditions of Theorem* 3.4. *Suppose further that*

$$B_l \cdot \{B_{l+1}, \cdots, B_n\}/(\{[B_i, B_j]; i + j = l\} \cdot \{B_{l+1}, \cdots, B_n\})$$

is a "bounded by free" abelian group for $l = 2, 3, \cdots, n$. Then B is isomorphic to a subgroup of a free nth nilpotent group.

Proof. This proceeds by induction on n. In fact, we shall prove more and our induction hypothesis will be as follows. If $\{B_1, B_2, \cdots, B_r\}$ is an rth basic system of subgroups satisfying all the conditions of Theorem 3.4 and $r < n$, then $\{B_1, B_2, \cdots, B_r\}$ is isomorphic to an rth basic system of subgroups of a free rth nilpotent group. The result is trivially true for $r = 1$.

We consider the given group

$$B = \{B_1, B_2, \cdots, B_n\}.$$

It can easily be verified that the group

$$B/B_n = \{B_1 \cdot B_n/B_n, \cdots, B_{n-1} \cdot B_n/B_n\}$$

satisfies the four conditions of Theorem 3.4. Here it is necessary to use the fact that the natural homomorphism

$$B_l B_n \rightarrow B_l \cdot B_n/B_n$$

extends uniquely to the homomorphism ([2])

$$M(B_l) \cdot M(B_n) \rightarrow M(B_l \cdot B_n/B_n)$$

with kernel $M(B_n)$ for all l. Hence, by the induction hypothesis, there exists an isomorphism ϕ that maps B/B_n onto an $(n-1)$th basic system of subgroups of $\bar{A} \cdot Z(A)/Z(A)$. It is convenient to take A to be a free nth nilpotent group freely generated by a_α, $\alpha \in M$, and the index set M is taken large enough to permit all necessary embeddings. The subgroup $\bar{A}$ of A is the free nth nilpotent subgroup freely generated by a_α^η, $\alpha \in M$, where the integer η will be specified at stage (c) of the proof.

We shall show how to extend ϕ to an isomorphism Φ of B into A.

We choose the free generators of the subgroup B_l in the way indicated by

([2]) Cf. initial part of proof of Lemma 3.5.

condition (iv) of Theorem 3.4 for $l = 1,2,\cdots,n-1$. Let $C(l)$ denote the free nilpotent subgroup of class $[n/l]$ freely generated by the original commutators of weight l, for $l = 1,2,\cdots,n-1$. Further let $K(l)$ denote the corresponding subgroup of $\{B_l,B_{l+1},\cdots,B_n\}$ generated by the nonoriginal commutators of weight l, for $l = 2,3,\cdots,n$.

A result which we shall constantly use is that

$$I(K(l)\cdot\{B_{l+1},\cdots,B_n\}) = I(\{[B_i,B_j];\ i+j=l\}\cdot\{B_{l+1},\cdots,B_n\}),$$

where I denotes the isolator of the enclosed subgroup in $\{B_l,B_{l+1},\cdots,B_n\}$. This follows from condition (iv) and the well-known fact that if H is a subgroup of B, then

$$M(H)\cap B = I(H) \text{ in } B.$$

It is also necessary to use Lemma 3.5.

From the assumptions of our theorem and Definition 3.2, we have that

$$\begin{aligned} B_l\cdot\{B_{l+1},\cdots,B_n\}/\{B_{l+1},\cdots,B_n\} &= I(K(l)\cdot\{B_{l+1},\cdots,B_n\})/\{B_{l+1},\cdots,B_n\} \\ &\quad\times C(l)\cdot\{B_{l+1},\cdots,B_n\}/\{B_{l+1},\cdots,B_n\}\end{aligned}$$

for $l = 2,3,\cdots,n$. We have not yet defined the subgroup $C(n)$ which is generated by a maximal set of original commutators of weight n. We can obviously fix these and a set of free generators for the free abelian group B_n so as to include them in the basis.

(a) We define the mapping Φ by first defining its effect on the original commutators of B. Let b_l be an original commutator of weight l. By the induction hypothesis, ϕ maps $b_l\cdot B_n$ onto an element $x_l\cdot Z(A)$, where x_l belongs to ${}^l\bar{A}$ but does not belong to ${}^{l+1}\bar{A}$. We take x_l to be the coset representative of $x_l\cdot Z(A)$ in A. We repeat this process for all original commutators of weight l and $l = 1,2,\cdots,n-1$. We define

$$\Phi(b_l) = x_l$$

for $l = 1,2,\cdots,n-1$. We define $\Phi(b_n)$ to be such that the resulting elements of $Z(\bar{A})$ are linearly independent and secondly they are power products of certain free generators a_α^n, all of which do not occur in the elements x_l ($l \leqq n-1$).

(b) By Lemma 3.5 and the Theorem of Širšov 2.3, the mapping which we have defined above gives rise to a Lie algebra isomorphism $\tilde{\Phi}$ of

$$M(B) = \{M(B_1),M(B_2),\cdots,M(B_n)\}$$

onto a subalgebra of $M(\bar{A}) = M(A)$. As the multiplication in $M(B)$ and $M(A)$ is given by the Campbell-Hausdorff formula, $\tilde{\Phi}$ is also a group isomorphism. We have to show that this isomorphism when restricted to B is a group isomorphism into A. We shall denote $\tilde{\Phi}$ restricted to B by Φ.

(c) We shall now fix the number η which occurs in the definition of the subgroup $\bar{A}$. Let b_l be a free generator of B_l which is not an original commutator. Then, by the initial part of our proof, there exists a least positive integer $\eta(l)$ such that

$$(*)\qquad b_l^{\eta(l)} \text{ belongs to } K(l)\cdot\{B_{l+1},\cdots,B_n\}.$$

Let η be the least common multiple of all the numbers $\eta(l)$ which arise as b_l varies over the free generators of B_l which are not original commutators and $l = 2,3,\cdots,n$. This number exists as

$$I(K(l)\cdot\{B_{l+1},\cdots,B_n\})/(K(l)\cdot\{B_{l+1},\cdots,B_n\})$$

is a bounded torsion abelian group, for $l = 2,3,\cdots,n$. In fact, one of the assumptions of our theorem is that

$$I(\{[B_i,B_j];\ i+j=l\}\cdot\{B_{l+1},\cdots,B_n\})/(\{[B_i,B_j];\ i+j=l\}\cdot\{B_{l+1},\cdots,B_n\})$$

is a bounded torsion abelian group for $l = 2,3,\cdots,n$. It remains to show that

$$\{[B_i,B_j];\ i+j=l\}\cdot\{B_{l+1},\cdots,B_n\}/(K(l)\cdot\{B_{l+1},\cdots,B_n\})$$

is a bounded torsion abelian group for $l = 2,3,\cdots,n$. We do this by induction on n. Thus we have to consider

$$\{[B_i,B_j];\ i+j=n\}/K(n)$$

and show that it is a bounded torsion abelian group. If b_i and b_j are free generators of B_i and B_j $(i,j < n)$ respectively, then, by the induction hypothesis, there exists a positive integer λ independent of i and j such that $[b_i,b_j]$ is the λth root of some element of $K(n)$. One has to use the direct decomposition given in the initial stage of our proof and also the fact that any commutator of the form $[u,v]$, where u and v are original or nonoriginal commutators of weight i and j respectively will be contained in $K(n)$. The latter result follows by an argument similar to that given in the first part of the proof of the main theorem of M. Hall [3].

(d) It remains to show that $\tilde{\Phi}(b_l)$ belongs to A, for all free generators of B_l which are not original commutators and $l = 2,3,\cdots,n$. Now

$$\Phi(b_n) = \tilde{\Phi}(b_n) = \tilde{\Phi}(b_n^{\eta(n)})^{1/\eta(n)},$$

$b_n^{\eta(n)}$ belongs to $K(n)$ and hence $\tilde{\Phi}(b_n^{\eta(n)})$ belongs to $Z(\bar{A})$. Thus $\Phi(b_n)$ belongs to $Z(A)$ for all free generators of the subgroup B_n. Suppose that

$$\Phi(b_k) = \tilde{\Phi}(b_k)$$

belongs to A for all $k > l$, so that $\tilde{\Phi}$ induces a group isomorphism of

$\{B_{l+1}, \cdots, B_n\}$ into ${}^{l+1}A$. Let b_l be a free generator of B_l which is not an original commutator. Then, by relation (*), it follows that

$$b_l^{\eta(l)} = u_l \cdot w_{l+1},$$

where u_l and w_{l+1} belong to $K(l)$ and $\{B_{l+1}, \cdots, B_n\}$ respectively. Further we have that

$$\tilde{\Phi}(b_l^{\eta(l)}) = \tilde{\Phi}(u_l) \cdot \tilde{\Phi}(w_{l+1})$$

which is an element of ${}^{l}A$. This gives that the value of $\Phi(b_l)$ is determined by

$$\Phi(b_l) = \tilde{\Phi}(b_l) = (\tilde{\Phi}(b_l^{\eta(l)}))^{1/\eta(l)}.$$

By the induction hypothesis,

$$(\phi(b_l^{\eta(l)} \cdot B_n))^{1/\eta(l)} \text{ belongs to } \bar{A} \cdot Z(A)/Z(A),$$

that is,

$$(\tilde{\Phi}(b_l^{\eta(l)} \cdot B_n))^{1/\eta(l)} \cdot Z(A) \text{ is contained in } \bar{A} \cdot Z(A)/Z(A).$$

Thus

$$(\tilde{\Phi}(b_l^{\eta(l)}))^{1/\eta(l)} \cdot (\tilde{\Phi}(B_n))^{1/\eta(l)} \text{ is contained in } \bar{A} \cdot Z(A)$$

giving that

$$(\tilde{\Phi}(b_l^{\eta(l)}))^{1/\eta(l)} \text{ belongs to } {}^{l}A.$$

We have to use the fact that

$$(\tilde{\Phi}(I(K(n))))^{1/((n-1)\cdot\eta)} \text{ is contained in } Z(A)$$

Hence $\Phi(b_l)$ belongs to ${}^{l}A$ for all free generators of the subgroup B_l and $l = 1, 2, \cdots, n$. Thus the fact that Φ maps B isomorphically into A is a direct consequence of the unique representation given by Lemma 3.3. In order to see that Φ maps $\{B_1, B_2, \cdots, B_n\}$ onto an nth basic system of subgroups of A, it is sufficient, by the induction hypothesis, to show that $\Phi(B_n)$ is contained in $Z(A)$. This we know to be true by construction and thus our proof by induction is completed.

Added in proof (February 1964). Alternative formulations of the subgroup theorem will be found in the forthcoming paper to be published in *Algebra*.

Reference

1. S. Moran, *A subgroup theorem for free nilpotent groups*, Trans. Amer. Math. Soc. **103** (1962), 495–515.

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