GROTHENDIECK GROUPS OF ORDERS IN SEMISIMPLE ALGEBRAS

BY

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Introduction. Let R be a noetherian domain with quotient field F, and let A be an R-algebra which is finitely generated and torsion free as R-module. Define the F-algebra A^* to be $F \otimes_R A$. We may form the Grothendieck groups $K^0(A)$, $K^0(A^*)$, $K_t^0(A)$, the last of which is obtained from the category of R-torsion A-modules (see §1 for the definitions of these groups).

On the other hand, we may define a Whitehead group $K^1(A^*)$. We shall set up a homomorphism $\Delta: K^1(A^*) \to K_t^0(A)$. If A^* is semisimple, we obtain an exact sequence

$$K^1(A^*) \xrightarrow{\Delta} K_t^0(A) \to K^0(A) \to K^0(A^*) \to 0.$$

This result is applied to the case where $A = RG$, the group ring of a finite group G over a Dedekind ring R of characteristic 0. If F is a splitting field for G, we are able to compute $K^0(A)$ explicitly in terms of the arithmetic of R and the decomposition matrices of G.

In a recent paper [5], Swan (using different methods) has independently obtained a number of striking results on the structure of $K^0(A)$.

Throughout this paper, all rings are left noetherian and have unity elements. All modules are left, finitely generated modules. The ring of rational integers is denoted by Z.

1. **Grothendieck groups.** 1. Let A be a ring, and let $\mathscr{A}$ be the free abelian group generated by the symbols (M), where M ranges over all A-modules. Define $\mathscr{A}_0$ as the subgroup of $\mathscr{A}$ generated by elements of the form

$$(M) - (M') - (M''),$$

where $0 \to M' \to M \to M'' \to 0$ ranges over all short exact sequences of A-modules. Then set $K^0(A) = \mathscr{A}/\mathscr{A}_0$, the *Grothendieck group* of A. We use $[M]$ to denote the image of M in $K^0(A)$.

2. If A is a ring with minimum condition, then the Jordan-Hölder theorem is valid for A-modules. Consequently, if $\{M_1, \cdots, M_n\}$ is a full set of irreducible A-modules, then $K^0(A)$ is the free Z-module with free Z-basis $[M_1], \cdots, [M_n]$.

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3. Returning to the general case, we wish to show that if M and N are A-modules, then $[M] = [N]$ in $K^0(A)$ if and only if M and N have the same composition factors, in some sense. More precisely, we prove

LEMMA 1. *Let M and N be A-modules. Then $[M] = [N]$ in $K^0(A)$ if and only if there exist two exact sequences*

$$(1) \qquad 0 \to U \to M \oplus W \to V \to 0, \quad 0 \to U \to N \oplus W \to V \to 0,$$

for some choice of A-modules U, V and W.

Proof. If there exist modules U, V, W for which the sequences in (1) are exact then clearly $[M] = [N]$ in $K^0(A)$.

Conversely, suppose that $[M] = [N]$ in $K^0(A)$, and write $K^0(A) = \mathscr{A}/\mathscr{A}_0$, using the notation of §1.1. Then

$$(M) - (N) = \sum_X \pm \{(X) - (X') - (X'')\},$$

where $0 \to X' \to X \to X'' \to 0$ is exact. Therefore

$$(2) \qquad (M) + \sum_i \{(X_i') + (X_i'')\} + \sum_j (Y_j) = (N) + \sum_i (X_i) + \sum_j \{(Y_j') + (Y_j'')\}$$

holds true in $\mathscr{A}$, with $0 \to X_i' \to X_i \to X_i'' \to 0$ exact for each i, and $0 \to Y_j' \to Y_j \to Y_j'' \to 0$ exact for each j. It follows from the definition of $\mathscr{A}$ that any term (T) which occurs on the left-hand side of equation (2) with some multiplicity t, say, must also occur on the right-hand side with multiplicity t. Set $X = \sum^{\oplus} X_i$, $X' = \sum^{\oplus} X_{i'}$, and so on. The preceding shows that

$$M \oplus X' \oplus X'' \oplus Y \cong N \oplus X \oplus Y' \oplus Y''.$$

Let W be a module isomorphic to both of the above.

Since $W \cong N \oplus X \oplus Y' \oplus Y''$, there is an embedding of $X' \oplus Y'$ in W with quotient module $N \oplus X'' \oplus Y''$. Thus there exists an exact sequence

$$0 \to X' \oplus Y' \to M \oplus W \to M \oplus N \oplus X'' \oplus Y'' \to 0.$$

Analogously, there exists another such exact sequence with M and N interchanged. This completes the proof of the lemma.

4. We next introduce Bass' version of the Whitehead group $K^1(A)$ (see [1]). Let A be a ring, and consider the category whose objects are pairs (M, μ) consisting of an A-module M and an automorphism μ of M. By a map $\phi: (M, \mu) \to (N, \nu)$ of one such object into another, we mean an element $\phi \in \mathrm{Hom}_A(M, N)$ such that $\phi\mu = \nu\phi$. Consider a sequence

$$(3) \qquad 0 \longrightarrow (L, \lambda) \xrightarrow{\phi} (M, \mu) \xrightarrow{\psi} (N, \nu) \longrightarrow 0$$

of objects and maps in this category. Then the sequence is exact in this category if and only if $0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$ is exact in the usual sense.

(For the orientation of the reader, we remark that if one regards ϕ as an embedding of L in M, and ψ as the canonical projection of M onto M/L, then the exactness of (3) simply means that μ is an automorphism of M which maps L onto itself, thereby inducing an automorphism λ of L and an automorphism ν of the factor module M/L.)

Let $\mathscr{B}$ be the free abelian group with generators (M,μ), where M ranges over all A-modules, and μ ranges over all automorphisms of M. Define $\mathscr{B}_0$ as the subgroup of $\mathscr{B}$ generated by the elements

$$(M,\mu)-(L,\lambda)-(N,\nu)$$

gotten from all exact sequences given by (3), together with all elements of the form

$$(M,\mu\mu')-(M,\mu)-(M,\mu').$$

Now let $K^1(A)=\mathscr{B}/\mathscr{B}_0$. We denote by $[M,\mu]$ the image of (M,μ) in $K^1(A)$.

If 1_M is the identity automorphism of M, then trivially

$$[M,1_M]=0,\quad [M,\mu^{-1}]=-[M,\mu].$$

Thus every element of $K^1(A)$ is of the form $[M,\mu]$ for some M and some automorphism μ thereof.

If A is a direct sum of the rings $A_1,\cdots,A_n$, then clearly

$$K^1(A)\cong K^1(A_1)\oplus\cdots\oplus K^1(A_n).$$

5. Let F be a field, and let $F^{\#}$ be the multiplicative group of nonzero elements of F. For an F-module V, an automorphism ϕ of V is just a nonsingular linear transformation on V. Let $\det\phi$ denote the determinant of this transformation. We have $K^1(F)\cong F^{\#}$, where $K^1(F)$ is written additively, $F^{\#}$ multiplicatively. The isomorphism is given by $[V,\phi]\to\det\phi$.

Now suppose that A is a full matrix algebra over F, and let X be a fixed irreducible A-module. Each A-module is isomorphic to $X^{(n)}$ for some n, where $X^{(n)}$ denotes the direct sum of n copies of X. Furthermore, $\mathrm{Hom}_A(X,X)\cong F$. Hence if $M=X^{(n)}$, and if μ is an automorphism of M, then μ may be represented by a nonsingular $n\times n$ matrix $T(\mu)$ with entries in F. The categories of A-modules and F-modules are isomorphic, and we have also

$$K^1(A)\cong F^{\#},$$

the isomorphism being given by $[M,\mu]\to\det T(\mu)$.

2. **Algebras over noetherian domains.** 1. Let R be a noetherian commutative integral domain, with quotient field F. If M is a torsion free R-module, we may form the F-module $F\otimes_R M$, denoted by FM for brevity. Let A be an R-algebra which is finitely generated and torsion free as R-module, and set $A^*=FA$, an F-algebra.

The additive groups $K^0(A)$, $K^0(A^*)$ and $K^1(A^*)$ have already been defined in §1. If $\{X_1^*, \cdots, X_n^*\}$ is a full set of irreducible A^*-modules, then $K^0(A^*)$ is just the free Z-module with Z-basis $[X_1^*], \cdots, [X_n^*]$.

2. Let C_f denote the category of R-torsion-free A-modules. If we restrict ourselves to this category, we obtain a Grothendieck group $K_f^0(A)$. To each $M \in C_f$ corresponds an element $[M]_f \in K_f^0(A)$. The proof of Lemma 1, §1.3, remains unchanged. Hence if $M, N \in C_f$, then $[M]_f = [N]_f$ in $K_f^0(A)$ if and only if there exist exact sequences (1) for some choice of U, V, $W \in C_f$.

Using a procedure due to Swan [4], we show at once that $K_f^0(A) \cong K^0(A)$. The desired isomorphism $K_f^0(A) \to K^0(A)$ is given by $[M]_f \to [M]$, $M \in C_f$, and the inverse map $\eta_0 : K^0(A) \to K_f^0(A)$ may be obtained as follows: Let M be any A-module, and choose an exact sequence

$$0 \to X \to Y \to M \to 0$$

with Y a projective A- module. Then X and Y are in C_f, and we define

$$\eta_0[M] = [Y]_f - [X]_f.$$

By Schanuel's lemma, the image $\eta_0[M]$ is independent of the choice of X and Y.

It is easily seen that if

$$0 \to U \to V \to M \to 0 \tag{4}$$

is exact, with U, $V \in C_f$, then also

$$\eta_0[M] = [V]_f - [U]_f.$$

3. To each $M \in C_f$ there corresponds an A^*-module FM. It is easily verified that the map $[M]_f \to [FM]$ gives a mapping θ of $K_f^0(A)$ onto $K^0(A^*)$.

4. Next, we introduce the category C_t of all R-torsion A-modules. If we restrict ourselves to this category, we obtain a Grothendieck group $K_t^0(A)$. To each $M \in C_t$ corresponds an element $[M]_t \in K_t^0(A)$. Since each short exact sequence from C_t is a short exact sequence of A-modules, the map $[M]_t \to [M]$ gives a mapping of $K_t^0(A)$ into $K^0(A)$. Composing this map with the map η_0 defined above, we obtain a mapping $\eta : K_t^0(A) \to K_f^0(A)$. Indeed, if $M \in C_t$, choose any exact sequence (4) with U, $V \in C_f$, and then

$$\eta([M]_t) = [V]_f - [U]_f.$$

5. Suppose hereafter that A^* is semisimple. Following Swan [4], we show the exactness of

$$K_t^0(A) \xrightarrow{\eta} K_f^0(A) \xrightarrow{\theta} K^0(A^*) \longrightarrow 0. \tag{5}$$

Indeed, it is trivial that $\theta\eta = 0$. On the other hand, let $x \in \ker \theta$, and write $x = [M]_f - [N]_f$ for some $M, N \in C_f$. From $\theta x = 0$ we obtain $[FM] = [FN]$ in $K^0(A^*)$. Since A^* is semisimple, this implies that $FM \cong FN$. Replacing N by a module isomorphic to it does not change $[N]_f$, so we may assume that $FM = FN$, and that $N \subset M$. But then M/N is an R-torsion module, and there is an exact sequence

$$0 \to N \to M \to M/N \to 0.$$

Therefore

$$x = [M]_f - [N]_f = \eta([M/N]_t) \in \text{image of } \eta.$$

This completes the proof of the exactness of (5).

6. Now let $M, N \in C_f$ be any modules for which $FM = FN$. Define

$$[M//N] = \left[\frac{M}{M \cap N}\right]_t - \left[\frac{N}{M \cap N}\right]_t \in K_t^0(A), \tag{6}$$

which is meaningful since $F(M \cap N) = FM = FN$. For any module $X \subset M \cap N$ such that $FX = F(M \cap N)$, we have

$$\left[\frac{M}{M \cap N}\right]_t = \left[\frac{M}{X}\right]_t - \left[\frac{M \cap N}{X}\right]_t,$$

which readily implies that

$$[M//N] = \left[\frac{M}{X}\right]_t - \left[\frac{N}{X}\right]_t. \tag{7}$$

LEMMA 2. *Let $L, M, N \in C_f$ be such that $FL = FM = FN$. Then $[L//M] + [M//N] = [L//N]$.*

Proof. Choose $X = L \cap M \cap N$. Then $[L//M] = [L/X]_t - [M/X]_t$, with analogous formulas for $[M//N]$ and $[L//N]$. The result now follows from formula (7).

LEMMA 3. *Let there be given modules L_i, M_i, $N_i \in C_f$ and exact sequences*

$$0 \longrightarrow L_i \xrightarrow{\phi_i} M_i \xrightarrow{\psi_i} N_i \longrightarrow 0, \qquad i = 1,2.$$

Let $L_i^* = FL_i$, and so on. Suppose there exist isomorphisms $\lambda : L_1^* \cong L_2^*$, $\mu : M_1^* \cong M_2^*$, $\nu : N_1^* \cong N_2^*$ for which the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1^* & \xrightarrow{\phi_1^*} & M_1^* & \xrightarrow{\psi_1^*} & N_1^* & \longrightarrow & 0 \\ & & \lambda\downarrow & & \mu\downarrow & & \nu\downarrow & & \\ 0 & \longrightarrow & L_2^* & \xrightarrow[\phi_2^*]{} & M_2^* & \xrightarrow[\psi_2^*]{} & N_2^* & \longrightarrow & 0. \end{array}$$

Then

$$[M_2//\mu M_1] = [L_2//\lambda L_1] + [N_2//\nu N_1].$$

Proof. The map ϕ_2^* induces a mapping $L_2 \to M_2/(M_2 \cap \mu M_1)$, and the kernel of this mapping is easily found to be $L_2 \cap \lambda L_1$. Thus, there is an isomorphism of

$L_2/(L_2 \cap \lambda L_1)$ into $M_2/(M_2 \cap \mu M_1)$. Analogously, there is a homomorphism of this latter module onto $N_2/(N_2 \cap \nu N_1)$. A routine computation then shows the exactness of

$$0 \to \frac{L_2}{L_2 \cap \lambda L_1} \to \frac{M_2}{M_2 \cap \mu M_1} \to \frac{N_2}{N_2 \cap \nu N_1} \to 0.$$

Consequently

$$\left[\frac{M_2}{M_2 \cap \mu M_1}\right]_t = \left[\frac{L_2}{L_2 \cap \lambda L_1}\right]_t + \left[\frac{N_2}{N_2 \cap \nu N_1}\right]_t.$$

An analogous formula holds with the numerators M_2, L_2, N_2 replaced by μM_1, λL_1, νN_1, respectively. This implies the desired result.

7. We shall proceed to construct a homomorphism $\Delta : K^1(A^*) \to K_t^0(A)$. Using the notation of §1.4, write $K^1(A^*) = \mathcal{B}/\mathcal{B}_0$, and define

$$\Delta(M^*, \mu^*) = [\mu^* M // M],$$

where $M \in C_f$ is chosen so that $FM = M^*$. Then Δ is well defined, since if also $FN = M^*$, $N \in C_f$, then

$$[\mu^* M//M] - [\mu^* N//N] = [\mu^* M//\mu^* N] - [M//N] = 0,$$

the latter equality true because μ^* is an automorphism of M^*.

We now prove that Δ annihilates $\mathcal{B}_0$, and hence induces a map of $K^1(A^*)$ into $K_t^0(A)$. Consider first a generator of $\mathcal{B}_0$ of the form

$$(M^*, \mu_1^* \mu_2^*) - (M^*, \mu_1^*) - (M^*, \mu_2^*).$$

Choose $M \in C_f$ such that $FM = M^*$. Then Δ maps the above generator onto

$$[\mu_1^* \mu_2^* M//M] - [\mu_1^* M//M] - [\mu_2^* M//M],$$

which is zero because $[\mu_1^* \mu_2^* M//\mu_1^* M] = [\mu_2^* M//M]$.

Second, consider a generator of $\mathcal{B}_0$ of the form

$$b_0 = (M^*, \mu^*) - (L^*, \lambda^*) - (N^*, \nu^*),$$

where

$$(8) \qquad 0 \longrightarrow (L^*, \lambda^*) \xrightarrow{\phi} (M^*, \mu^*) \xrightarrow{\psi} (N^*, \nu^*) \longrightarrow 0$$

is exact. Let $M \in C_f$ be such that $FM = M^*$, and set $L = \phi^{-1} M$, $N = \psi M$. Then we have the exact sequence

$$0 \longrightarrow L \xrightarrow{\phi} M \xrightarrow{\psi} N \longrightarrow 0,$$

which (when tensored with F) gives the exact sequence

$$0 \longrightarrow L^* \xrightarrow{\phi} M^* \xrightarrow{\psi} N^* \longrightarrow 0.$$

Since the sequence (8) is exact, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^* & \xrightarrow{\phi} & M^* & \xrightarrow{\psi} & N^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle \mu^*} & & \downarrow{\scriptstyle \nu^*} & & \\ 0 & \longrightarrow & L^* & \xrightarrow[\phi]{} & M^* & \xrightarrow[\psi]{} & N^* & \longrightarrow & 0. \end{array}$$

By Lemma 3, §2.7, we have

$$[\mu^*M//M] - [\lambda^*L//L] - [\nu^*N//N] = 0.$$

But the left-hand side of the above equation is precisely $\Delta(b_0)$, which completes the proof that $\Delta(\mathscr{B}_0) = 0$.

We shall use the same symbol Δ to denote the map of $K^1(A^*)$ into $K_t^0(A)$.

8. Let us now prove the exactness of

$$\text{(9)} \qquad K^1(A^*) \xrightarrow{\Delta} K_t^0(A) \xrightarrow{\eta} K_f^0(A) \xrightarrow{\theta} K^0(A^*) \longrightarrow 0.$$

To begin with, we verify that $\eta\Delta = 0$. For let $[M^*, \mu^*] \in K^1(A)$, and choose $M \in C_f$ such that $FM = M^*$. By definition,

$$\Delta[M^*, \mu^*] = [\mu^*M//M].$$

Choose $X = M \cap \mu^*M$, so that $FX = FM$. Then

$$\begin{aligned} \eta[\mu^*M//M] &= \eta\left\{\left[\frac{\mu^*M}{X}\right]_t - \left[\frac{M}{X}\right]_t\right\} \\ &= [\mu^*M]_f - [X]_f - [M]_f + [X]_f \in K_f^0(A) \\ &= 0, \end{aligned}$$

since $\mu^*M \cong M$.

On the other hand, let us show that $\ker\eta \subset$ image of Δ. For let $x \in \ker\eta$, and write $x = [M]_t - [N]_t$ for $M, N \in C_t$. Choose exact sequences

$$0 \to X' \to X \to M \to 0, \quad 0 \to Y' \to Y \to N \to 0,$$

with X, X', Y, $Y' \in C_f$. Then

$$0 = \eta x = [X]_f - [X']_f - [Y]_f + [Y']_f,$$

so $[X \oplus Y']_f = [X' \oplus Y]_f$ in $K_f^0(A)$. By §2.2, there exist modules U, V, $W \in C_f$ and exact sequences

$$0 \to U \to X \oplus Y' \oplus W \to V \to 0, \quad 0 \to U \to X' \oplus Y \oplus W \to V \to 0.$$

Tensoring with F, and setting $U^* = FU$, $V^* = FV$, we obtain the exact sequences

$$0 \to U^* \to F(X \oplus Y' \oplus W) \to V^* \to 0,$$

$$0 \to U^* \to F(X' \oplus Y \oplus W) \to V^* \to 0.$$

But all short exact sequences of A^*-modules must split, since A^* is assumed semisimple, and so there is an automorphism μ^* of $F(X \oplus Y' \oplus W)$ for which the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U^* & \longrightarrow & F(X \oplus Y' \oplus W) & \longrightarrow & V^* & \longrightarrow & 0 \\ & & \downarrow 1_{U^*} & & \downarrow \mu^* & & \downarrow 1_{V^*} & & \\ 0 & \longrightarrow & U^* & \longrightarrow & F(X' \oplus Y \oplus W) & \longrightarrow & V^* & \longrightarrow & 0, \end{array}$$

the 1's denoting identity maps. Since the restriction of an identity map is again an identity map, it now follows from Lemma 3, §2.6, that

$$[\mu^*(X \oplus Y' \oplus W)//(X' \oplus Y \oplus W)] = 0.$$

But then

$$[(X \oplus Y' \oplus W)//\mu^*(X \oplus Y' \oplus W)] = [(X \oplus Y' \oplus W)//(X' \oplus Y \oplus W)].$$

Now the left-hand expression lies in the image of Δ, while that on the right is equal to

$$\left[\frac{X \oplus Y' \oplus W}{X' \oplus Y' \oplus W}\right]_t - \left[\frac{X' \oplus Y \oplus W}{X' \oplus Y' \oplus W}\right]_t = \left[\frac{X}{X'}\right]_t - \left[\frac{Y}{Y'}\right]_t = [M]_t - [N]_t = x.$$

This completes the proof that the sequence (9) is exact.

3. **Group rings.** 1. In this section we choose R as a Dedekind domain of characteristic 0, with quotient field F. (For example, R might be the ring of all algebraic integers in an algebraic number field F.) Let G be a finite group, and set $A = RG$, its group ring. Assume throughout this section that F is a splitting field for G, so that $A^*(= FG)$ is a direct sum of full matrix algebras over F. We may choose A-modules $Z_1, \cdots, Z_n \in C_f$ (the category of R-torsion-free A-modules) such that if we set $Z_i^* = FZ_i$, then $\{Z_1^*, \cdots, Z_n^*\}$ is a full set of irreducible A^*-modules.

2. Let P be a (nonzero) prime ideal of R, and set $\bar{R} = R/P$, $\bar{A} = A/PA$. Then $\bar{A}$ is a $\bar{R}$-algebra, and $K^0(\bar{A})$ is a free Z-module with free Z-basis $[\bar{Y}_1], \cdots, [\bar{Y}_m]$, where $\bar{Y}_1, \cdots, \bar{Y}_m$ are a full set of irreducible $\bar{A}$-modules.

The *decomposition numbers* d_{ij}^P are non-negative integers such that $\bar{Y}_j$ occurs with multiplicity d_{ij}^P as composition factor of the $\bar{A}$-module Z_i/PZ_i. Therefore

$$\left[\frac{Z_i}{PZ_i}\right] = \sum_j d_{ij}^P [\bar{Y}_j] \text{ in } K^0(\bar{A}), \qquad 1 \leqq i \leqq n. \tag{10}$$

When P does not divide the order of G, the decomposition matrix (d_{ij}^P) is just the identity matrix.

For arbitrary P, Brauer [2; 3] has shown that $m \leqq n$, and that the G.C.D. of the $m \times m$ minors of the decomposition matrix (d_{ij}^P) is equal to 1. Therefore we may solve equations (10) for the $[\bar{Y}_j]$ in terms of the $[Z_i/PZ_i]$, and so there exist rational integers e_{ij}^P (not necessarily unique) such that

$$[\bar{Y}_j] = \sum_i e_{ij}^P \left[\frac{Z_i}{PZ_i}\right] \text{ in } K^0(\bar{A}), \quad 1 \leqq j \leqq m.$$

Furthermore, we have $[Z_i//P^k Z_i] = k\,[Z_i/PZ_i]$ in $K^0(\bar{A})$, for each rational integer k. Therefore every element of $K^0(\bar{A})$ is expressible as a sum

$$\sum_{i=1}^{n} [P^{k_i} Z_i//Z_i].$$

3. Now let P range over the prime ideals of R, and as in §2, let $K_t^0(A)$ be the Grothendieck group of the category of R-torsion A-modules. Since each such module is a direct sum of its P-primary components, we have

$$K_t^0(A) \cong \sum_P^{\oplus} K^0\left(\frac{A}{PA}\right).$$

Hence, using the results of the preceding paragraph, every element of $K_t^0(A)$ is expressible as a sum

$$\sum_{i=1}^{n} [J_i Z_i//Z_i], \qquad J_i = \text{fractional } R\text{-ideal in } K.$$

4. We set $\mathscr{J}$ = multiplicative group of fractional R-ideals in K, and let $\mathscr{J}^n = \mathscr{J} \times \cdots \times \mathscr{J}$ (n factors). Then there is a homomorphism $\tau : \mathscr{J}^n \to K_t^0(A)$ given by

$$\tau(J_1, \cdots, J_n) = [J_1 Z_1//Z_1] + \cdots + [J_n Z_n//Z_n],$$

and we have just shown that τ is a surjection.

Using the notation of the exact sequence (9), let us set $\sigma = \eta\tau$. Then

$$\sigma(J_1, \cdots, J_n) = \sum_{i=1}^{n} \{[J_i Z_i]_f - [Z_i]_f\},$$

and the kernel of θ equals the image of η, which in turn equals the image of σ. Now $K^0(A^*)$ is a free Z-module, so by the exactness of (9), we have

$$K_f^0(A) \cong K^0(A^*) \oplus \text{image of } \sigma, \tag{11}$$

the above being an isomorphism of additive groups. Furthermore,

$$\text{image of } \sigma \cong \mathscr{J}^n/\ker \sigma.$$

Thus, to compute the additive structure of $K_f^0(A)$, it suffices to determine $\ker\sigma$. We shall compute this kernel explicitly.

5. If R is a principal ideal ring, then each $J_i \in \mathscr{J}$ is of the form Ra_i for some $a_i \in F$, and thus

$$[J_iZ_i]_f = [a_iZ_i]_f = [Z_i]_f,$$

since $a_iZ_i \cong Z_i$. In this case we see that the image of σ is 0, and so $K_f^0(A) \cong K^0(A^*)$ as additive groups.

6. If R is not necessarily a principal ideal ring, the above argument still shows that the kernel of σ contains $\mathscr{J}_0^n$, defined as

$$\mathscr{J}_0^n = \{(J_1,\cdots,J_n) \in \mathscr{J}^n\colon \text{each } J_i \text{ is principal}\}.$$

We now make use of the decomposition matrices (d_{ij}^P) defined in §3.2. When P divides the order of G, the matrix (d_{ij}^P) is not a square matrix, and so there exist rational integers $q_1,\cdots,q_n$ (not all zero) such that $\sum_i q_i d_{ij}^P = 0$ for all j. But then

$$\sum_i \,[Z_i//P^{q_i}Z_i] = \sum_i q_i \left[\frac{Z_i}{PZ_i}\right] = \sum_{i,j} q_i d_{ij}^P[\bar{Y}_j] = 0 \text{ in } K_t^0(A).$$

Set

$$D_P = \left\{(P^{q_1},\cdots,P^{q_n}) \in \mathscr{J}^n : \sum_i q_i d_{ij}^P = 0 \text{ for all } j\right\}.$$

Then the preceding remarks imply that $\tau(D_P) = 0$ for each P. Indeed, since $K_t^0(A) \cong \sum^{\oplus} K^0(A/PA)$, we have shown that

$$\ker\tau = \prod_P D_P.$$

Note that $D_P = \{1\}$ whenever P does not divide the order of G.

5. Next, from the relation $\sigma = \eta\tau$ we conclude that $\ker\sigma \supset \ker\tau$. Combining this fact with the observation of §3.6, we have

$$\ker\sigma \supset \mathscr{J}_0^n \cdot \ker\tau.$$

We shall now prove that in fact

$$\ker\sigma = \mathscr{J}_0^n \cdot \ker\tau. \tag{12}$$

To begin with, since F is a splitting field for G, we may write $A^* = A_1^* \oplus \cdots \oplus A_n^*$ where each A_i^* is a full matrix algebra over F. For each i, the A^*-module Z_i^* is then an irreducible A_i^*-module. Let $F^{\#}$ be the multiplicative group of the field F. By the discussion of §1.5, we have

$$K^1(A^*) \cong \sum_{i=1}^{n}{}^{\otimes} K^1(A_i^*) \cong F^{\#} \times \cdots \times F^{\#} \ (n \text{ factors}).$$

We may thus define a map $\rho : K^1(A^*) \to \mathscr{J}^n$ by

$$\rho(a_1,\cdots,a_n) = (Ra_1,\cdots,Ra_n), \quad a_i \in F^{\#}.$$

Indeed, a_1 (as element of $K^1(A_1^*)$) represents the pair $[Z_1^*, a_1^*]$, where a_1^* is the automorphism $z \to a_1 z$, $z \in Z_1^*$. Then

$$\begin{aligned}\Delta[Z_1^*, a_1^*] &= [a_1 Z_1 // Z_1] = \tau(Ra_1, R, \cdots, R)\\ &= \tau\rho(a_1, 1, \cdots, 1).\end{aligned}$$

Corresponding results hold for $a_2, \cdots, a_n$, which shows that $\Delta = \tau\rho$. We therefore have a commutative diagram

$$\begin{array}{ccccccccc} & & \mathscr{J}^n & & & & & & \\ & \overset{\rho}{\nearrow} & \downarrow{\scriptstyle\tau} & \overset{\sigma}{\searrow} & & & & & \\ K^1(A^*) & \xrightarrow[\Delta]{} & K_t^0(A) & \xrightarrow[\eta]{} & K^0(A) & \xrightarrow[\theta]{} & K_f^0(A^*) & \longrightarrow & 0. \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

Using this diagram, a routine argument shows that $\ker\sigma = (\ker\tau)$ (image of ρ). However, the image of ρ is precisely the group $\mathscr{J}_0^n$ defined in §3.6. This completes the proof of formula (12), and so we have determined the structure of $K_f^0(A)$ (and thus of $K_0(A)$) as additive group.

6. Let us investigate briefly what happens in the nonsplitting field case. Let R_0 be the ring of all algebraic integers in an algebraic number field F_0, and set $A_0 = R_0 G$, $A_0^* = F_0 G$. The semisimple algebra A_0^* need not be a direct sum of full matrix algebras. Nevertheless, there is an exact sequence

$$K^1(A_0^*) \to K_t^0(A_0) \to K_f^0(A_0) \xrightarrow{\theta_0} K^0(A_0^*) \to 0,$$

so again

$$K_f^0(A_0) \cong K^0(A_0^*) \oplus \ker\theta_0$$

as additive groups. We shall show that $\ker\theta_0$ is a finite abelian group.

To begin with, we observe that $K_f^0(A_0)$ is finitely generated as Z-module. For let $V_1^*, \cdots, V_s^*$ be a full set of irreducible A_0^*-modules. For each i, consider the set of A_0-modules W which are R_0-torsion-free and satisfy $F_0 W = V_i^*$. By the Jordan-Zassenhaus theorem, there are only a finite number of nonisomorphic A_0-modules in this set, say $W_{i1}, \cdots, W_{it_i}$. But then it is easily seen that the elements

$$\{[W_{ij}]_f \in K_f^0(A_0) \colon 1 \leqq j \leqq t_i, \quad 1 \leqq i \leqq s\}$$

are a set of generators of the Z-module $K_f^0(A_0)$. (They are surely not a Z-basis, however.)

It follows then that $\ker\theta_0$ is also finitely generated as Z-module, so we need only show that $\ker\theta_0$ is a torsion module. We begin by choosing a finite extension F of F_0 which is a splitting field for G, say $(F:F_0) = k$. Let R be the integral closure of R_0 in F; then R is a Dedekind ring with quotient field F, and we have

$$R \cong R_0 \oplus \cdots \oplus R_0 \oplus J \qquad (k \text{ summands})$$

as R_0-modules, where J is some ideal in R_0.

For each R_0-torsion-free A_0-module M, define $\alpha[M] = [R \otimes_{R_0} M]$, thereby obtaining a map $\alpha: K_f^0(A_0) \to K_f^0(A)$. Analogously, there is a map $\alpha^*: K^0(A_0^*) \to K^0(A^*)$. On the other hand, every A-module can be viewed as an A_0-module, so there are maps $\beta: K_f^0(A) \to K_f^0(A_0)$, $\beta^*: K^0(A^*) \to K^0(A_0^*)$, and we have a commutative diagram

$$\begin{array}{ccc} K_f^0(A) & \xrightarrow{\theta} & K^0(A^*) \\ \alpha\uparrow\ \downarrow\beta & & \alpha^*\uparrow\ \downarrow\beta^* \\ K_f^0(A_0) & \xrightarrow[\theta_0]{} & K^0(A_0^*). \end{array}$$

Let $x \in \ker \theta_0$; then $\alpha x \in \ker \theta$, so there exists a positive integer q such that $q \cdot \alpha x = 0$, and therefore $q \cdot \beta\alpha x = 0$. However,

$$\beta\alpha[M] = \beta[R \otimes_{R_0} M] = (k-1)[M] + [JM] \text{ in } K_f^0(A_0).$$

Choose a positive integer h such that J^h is principal. Then the above implies that $h \cdot \beta\alpha[M] = hk[M]$, and thus

$$0 = h \cdot q \cdot \beta\alpha x = qhkx.$$

This completes the proof that $\ker \theta_0$ is a finite abelian group. We shall not attempt to obtain an explicit computation for this group.

REMARK. Since K^1 is functorial the sequence (4) extends to a sequence

$$K^1(A) \to K^1(A^*) \xrightarrow{\Delta} K_t^0(A) \to K^0(A) \to K^0(A^*) \to 0.$$

This extended sequence is *not* in general exact. Indeed if $A = Z[t]/(t^2 - 1)$, the group ring of a group of order 2, then $K^1(A^*) \cong Q^{\#} \times Q^{\#}$ and the kernel of Δ is $\{(\pm 2^k, \pm 2^{-k})\}$. But $K^1(A)$ is easily seen to be just the four-group.

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