ON A CLASS OF PARTITIONS WITH DISTINCT SUMMANDS

BY

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I. INTRODUCTION

1. Let p be an odd prime, and let $a = \{a_1, a_2, \cdots, a_r\}$, where $1 \leqq a_j \leqq (p-1)/2$, be a set of r distinct integers. In an earlier paper [1] the author has obtained a convergent series representation and asymptotic formulas for $p_a(n)$, the number of partitions of a positive integer n into summands congruent to $\pm a_j \pmod{p}$. The necessary transformation equations as well as estimates of the magnitude of certain exponential sums were obtained using the procedures of Lehner [2], while the circle dissection method of Rademacher [3] was employed for the integration. In the present paper we impose an additional restriction on the partitions, namely that the summands be distinct. That is, we wish to find a convergent series and asymptotic formulas for $q_a(n)$, the number of partitions of a positive integer n into distinct summands which are congruent modulo p to elements of a or their negatives. The methods employed are essentially the same as in [1] and free use will be made of the results obtained in the earlier paper whenever they are applicable.

2. In the sequel several generating functions, each convergent in the interior of the unit circle, will be needed. For convenience we list these now.

$$\begin{aligned} F_a(x) &= \prod_{j=1}^{r} \left(\prod_{m=0}^{\infty} (1 - x^{pm+a_j})^{-1} \prod_{m=1}^{\infty} (1 - x^{pm-a_j})^{-1} \right) \\ &= 1 + \sum_{n=1}^{\infty} p_a(n) x^n. \end{aligned} \tag{2.1}$$

$$\begin{aligned} G_a(x) &= \prod_{j=1}^{r} \left(\prod_{m=0}^{\infty} (1 + x^{pm+a_j}) \prod_{m=1}^{\infty} (1 + x^{pm-a_j}) \right) \\ &= F_a(x)/F_a(x^2) = 1 + \sum_{n=1}^{\infty} q_a(n) x^n. \end{aligned} \tag{2.2}$$

Let k be a positive integer and let h satisfy the conditions $(h,k) = 1, 0 \leqq h < k$. If $p \mid k$ we define $b_j \equiv \pm h a_j \pmod{p}$, whichever yields $1 \leqq b_j \leqq (p-1)/2$, and also define

$$b = \{b_1, b_2, \cdots, b_r\}. \tag{2.3}$$

Received by the editors April 22, 1963.

Our third function is

$$(2.4)\qquad \begin{aligned} I_b(x) &= \prod_{j=1}^{r}\left(\prod_{m=1}^{\infty}(1-x^{(2m-1)p+2b_j})(1-x^{(2m-1)p-2b_j})\right)\\ &= 1+\sum_{n=1}^{\infty} s_b(n)x^n. \end{aligned}$$

If $p \nmid k$ we define α_j by

$$(2.5)\qquad \alpha_j k \equiv a_j \pmod p,\ 0<\alpha_j<p,$$

and

$$(2.6)\qquad \rho_j = \exp(2\pi i\alpha_j/p),\quad \bar\rho_j = \exp(-2\pi i\alpha_j/p).$$

Then, we define

$$(2.7)\qquad \begin{aligned} H_{ak}(x) &= \prod_{j=1}^{r}\left(\prod_{m=1}^{\infty}(1-\rho_j x^m)^{-1}(1-\bar\rho_j x^m)^{-1}\right)\\ &= 1+\sum_{n=1}^{\infty} c_{ak}(n)x^n, \end{aligned}$$

$$(2.8)\qquad \begin{aligned} J_{ak}(x) &= \prod_{j=1}^{r}\left(\prod_{m=1}^{\infty}(1+\rho_j x^m)(1+\bar\rho_j x^m)\right)\\ &= 1+\sum_{n=1}^{\infty} d_{ak}(n)x^n, \end{aligned}$$

$$(2.9)\qquad \begin{aligned} P_{ak}(x) &= \prod_{j=1}^{r}\left(\prod_{m=1}^{\infty}(1-\rho_j x^{2m-1})(1-\bar\rho_j x^{2m-1})\right)\\ &= 1+\sum_{n=1}^{\infty} e_{ak}(n)x^n. \end{aligned}$$

II. The Transformation Equations

3. In what follows it will be necessary to determine the behavior of $G_a(x)$ in the neighborhood of a rational point on a circle concentric to the unit circle and of radius less than 1. Therefore, we take

$$(3.1)\qquad x = \exp\{2\pi ih/k - 2\pi z/k\}$$

where $\mathscr{R}(z)>0$, $(h,k)=1, 0 \leqq h < k$.

Theorem 1 of [1] states that

$$(3.2)\qquad \begin{aligned} F_a(\exp\{2\pi ih/k - 2\pi z/k\}) &= \omega_a(h,k)\exp\{\pi(B/z - Az)/6pk\}\\ &\quad \cdot F_b(\exp\{2\pi ih'/k - 2\pi/zk\}) \end{aligned}$$

if $p \mid k$, and

$$F_a(\exp\{2\pi ih/k - 2\pi z/k\}) = 2^{-r}\chi_a(h,k)\prod_{j=1}^{r}\csc\pi\alpha_j/p$$

$$(3.3) \qquad \cdot\exp\{\pi(r/z - Az)/6pk\}H_{ak}(\exp\{2\pi iH'/k - 2\pi/zK\})$$

if $p \nmid k$.

Here,

$$(3.4) \qquad A = \sum_{j=1}^{r}(p^2 - 6a_jp + 6a_j^2);$$

$$(3.5) \qquad B = \sum_{j=1}^{r}(p^2 - 6b_jp + 6b_j^2);$$

$$(3.6) \qquad \omega_a(h,k) = \exp\{\pi i\sigma_a(h,k)\},$$

$$(3.7) \qquad \sigma_a(h,k) = \sum_{j=1}^{r}\sum_{\mu_j}((\mu_j/k))((h\mu_j/k)), 0 < \mu_j < k, \mu_j \equiv \pm a_j \pmod p);$$

$$(3.8) \qquad \chi_a(h,k) = \exp\{\pi it_a(h,k)\},$$

$$(3.9) \qquad t_a(h,k) = \sum_{j=1}^{r}\sum_{\mu_j}((\mu_j/K))((h\mu_j/k)), 0 < \mu_j < K, \mu_j \equiv \pm a_j \pmod p).$$

$((y)) = y - [y] - \frac{1}{2} + \frac{1}{2}\delta(y)$ where $\delta(y) = 1$ if y is an integer and 0 otherwise. b and α_j are given by (2.3) and (2.5) respectively.

With the aid of this theorem we can now derive the necessary transformation equations for $G_a(x)$, the generating function of $q_a(n)$. Four cases must be considered.

If $2p \mid k$ and x is given by (3.1) then $x^2 = \exp\{2\pi ih/k^* - 2\pi z/k^*\}$ where $2k^* = k$. Since $G_a(x) = F_a(x)/F_a(x^2)$ we have, applying (3.2),

$$G_a(x) = w_a(h,k)\exp\{\pi(Az - B/z)/6pk\}G_b(x'),$$

where

$$(3.10) \qquad x' = \exp\{2\pi ih'/k - 2\pi/zk\}.$$

h' is a fixed solution of

$$(3.11) \qquad hh' \equiv -1 \pmod k,$$

and

$$(3.12) \qquad w_a(h,k) = \omega_a(h,k)/\omega_a(h,k^*).$$

If $p \mid k$ and k is odd then $x^2 = \exp\{4\pi ih/k - 4\pi z/k\}$. From (3.2) it follows that

$$G_a(x) = W_a(h,k)z\ \exp\{\pi(Az + B/z - D/2z)/6pk\}F_b(x')/F_d(y')$$

where

$$(3.13) \qquad y' = \exp\{2\pi ig'/k - \pi/zk\},$$

g' is a fixed solution of

(3.14) $$2hg' \equiv -1 \pmod{k},$$

and

(3.15) $$W_a(h,k) = \omega_a(h,k)/\omega_a(2h,k).$$

$d = \{d_1, d_2, \cdots, d_r\}$ where $d_j \equiv \pm 2ha_j \pmod{p}$, whichever yields $1 \leqq d_j \leqq (p-1)/2$, and $D = \sum_{j=1}^{r}(p^2 - 6d_jp + 6d_j^2)$. Since $b_j \equiv \pm ha_j \pmod{p}$ it follows that $d_j = 2b_j$ or $d_j = p - 2b_j$. Thus, $D = \sum_{j=1}^{r}(p^2 - 12b_jp + 24b_j^2)$. From (3.11) and (3.14) we have $h' \equiv 2g' \pmod{k}$ so that $x' = y'^2$. We now easily verify that $F_b(x')/F_d(y') = I_b(y')$ where $I_b(x)$ is given by (2.4). We conclude that

$$G_a(x) = W_a(h,k)\exp\{\pi(Az + B^*/z)/6pk\}I_b(y')$$

where

(3.16) $$B^* = \sum_{j=1}^{r} (p^2/2 - 6b_j^2).$$

If k is even and $p \nmid k$ then from (3.3)

$$G_a(x) = z_a(h,k)\prod_{j=1}^{r} \csc \pi\alpha_j/p \sin \pi\beta_j/p \exp\{\pi(Az - r/z)/6pk\}$$
$$\cdot H_{ak}(X)/H_{ak^*}(X^2)$$

where

(3.17) $$X = \exp\{2\pi iH'/k - 2\pi/zK\}.$$

Here,

(3.18) $$HH' \equiv -1 \pmod{k},\ H = ph,\ K = pk;$$

(3.19) $$z_a(h,k) = \chi_a(h,k)/\chi_a(h,k^*);$$

and $\beta_jk^* \equiv a_j \pmod{p}$, $0 < \beta_j < p$.

Since $\alpha_jk \equiv a_j \pmod{p}$ and $2k^* = k$ it follows that $2\alpha_j \equiv \beta_j \pmod{p}$. From (2.6) and (2.7) we now see that $H_{ak^*}(X^2) = \prod_{j=1}^{r}(\prod_{m=1}^{\infty}(1 - \rho_j^2X^{2m})^{-1}(1 - \bar{\rho}_j^2X^{2m})^{-1})$, from which we easily deduce that $H_{ak}(X)/H_{ak^*}(X^2) = J_{ak}(X)$ where $J_{ak}(x)$ is given by (2.8). We conclude that

$$G_a(x) = (-1)^{\mu}2^{r}z_a(h,k)\prod_{j=1}^{r} \cos \pi\alpha_j/p \exp\{\pi(Az - r/z)/6pk\}J_{ak}(X)$$

where μ is the number of α_j such that $\alpha_j > p/2$.

If k is odd and $p \nmid k$ then from (3.3)

$$G_a(x) = Z_a(h,k)\exp\{\pi(Az + r/2z)/6pk\}H_{ak}(X)/H_{ak}(Y)$$

where

(3.20) $$Y = \exp\{2\pi i G'/k - \pi/zK\}.$$

Here

(3.21) $$2HG' \equiv -1 \pmod{k},$$

and

(3.22) $$Z_a(h,k) = \chi_a(h,k)/\chi_a(2h,k).$$

From (3.18) and (3.21) we have $2G' \equiv H' \pmod{k}$ so that $X = Y^2$. We now easily verify that $H_{ak}(X)/H_{ak}(Y) = P_{ak}(Y)$ where $P_{ak}(x)$ is given by (2.9). We conclude that

$$G_a(x) = Z_a(h,k)\exp\{\pi(Az + r/2z)/6pk\}P_{ak}(Y).$$

With x, x', y', X, Y given by (3.1), (3.10), (3.13), (3.17), (3.20) respectively we combine these results to obtain

THEOREM 1. *$G_a(x)$ satisfies the transformation equations*

(3.23) $$G_a(x) = w_a(h,k)\exp\{\pi(Az - B/z)/6pk\}G_b(x')$$

if $p \mid k$ and $2 \mid k$,

(3.24) $$G_a(x) = W_a(h,k)\exp\{\pi(Az + B^*/z)/6pk\}I_b(y')$$

if $p \mid k$ and $2 \nmid k$,

(3.25) $$G_a(x) = (-1)^u 2^r z_a(h,k) \prod_{j=1}^{r} \cos \pi\alpha_j/p \exp\{\pi(Az - r/z)/6pk\}J_{ak}(X)$$

if $p \nmid k$ and $2 \mid k$, and

(3.26) $$G_a(x) = Z_a(h,k)\exp\{\pi(Az + r/2z)/6pk\}P_{ak}(Y)$$

if $p \nmid k$ and $2 \nmid k$.

III. ESTIMATES OF FOUR EXPONENTIAL SUMS

4. In what follows it will be necessary to have information concerning the magnitude of certain exponential sums involving $w_a(h,k)$, $W_a(h,k)$, $z_a(h,k)$ and $Z_a(h,k)$. Since the trivial estimate $O(k)$ will not suffice we now undertake an investigation of these sums. The method used is essentially that of Lehner, and for further details the interested reader is referred to [1] and [2]. We first consider the case $p \mid k$.

With $\sigma_q(h,k)$ given by (3.7) we have by (5.3), (5.4), (5.5) of [1]

$$6pk\sigma_a(h,k) \equiv 0 \pmod 3 \text{ if } 3 \nmid k, \tag{4.1}$$

$$6pk\sigma_a(h,k) \equiv \sum_{j=1}^{r} (2p + 2a_j + 2c_j - 3) \pmod 4 \text{ if } 2 \nmid k, \tag{4.2}$$

$$\begin{aligned} 6hkp\sigma_a(h,k) = h^2\left(A + \sum_{j=1}^{r} \{2k^2 + 3k(2a_j - p)\}\right) + (B - rk^2) \\ - 3hk \sum_{j=1}^{r} (2c_j - p) - 12kpI. \end{aligned} \tag{4.3}$$

Here I is an integer, and $c_j = b_j$ if $b_j \equiv ha_j \pmod p), c_j = p - b_j$ if $b_j \equiv -ha_j \pmod p$.

Now let $12p = fG$ where f is the greatest divisor of $12p$ prime to k. If k is even we have $f = 3$, $G = 4p$ if $3 \nmid k$, and $f = 1$, $G = 12p$ if $3 \mid k$. If k is odd we have $f = 12$, $G = p$ if $3 \nmid k$, and $f = 4$, $G = 3p$ if $3 \mid k$. Taking h' so that $hh' \equiv -1 \pmod{Gk}$ we have after multiplying (4.3) by $-h'$

$$6pk\sigma_a(h,k) \equiv hs - h't - 3k \sum_{j=1}^{r} (2c_j - p) \pmod{Gk} \tag{4.4}$$

where

$$\begin{aligned} s &= A + \sum_{j=1}^{r} \{2k^2 + 3k(2a_j - p)\}, \\ t &= B - rk^2. \end{aligned} \tag{4.5}$$

From (4.1) and (4.2) we have

$$6pk\sigma_a(h,k) \equiv \sum_{j=1}^{r} (6p + 6a_j + 6c_j - 3) \pmod f. \tag{4.6}$$

If k is even we have $hh' \equiv -1 \pmod{Gk^*}$, and since (4.1), (4.2), (4.3) hold with k replaced by k^* we have

$$6pk^*\sigma_a(h,k^*) \equiv hS - h'T - 3k^* \sum_{j=1}^{r} (2c_j - p) \pmod{Gk^*} \tag{4.7}$$

where

$$\begin{aligned} S &= A + \sum_{j=1}^{r} \{2k^{*2} + 3k^*(2a_j - p)\}, \\ T &= B - rk^{*2}. \end{aligned} \tag{4.8}$$

Since, when k is even, $f = 1$ or $f = 3$ we have from (4.6) and the corresponding statement for k^*

$$6pk\sigma_a(h,k) \equiv 6pk^*\sigma_a(h,k^*) \equiv 0 \pmod f. \tag{4.9}$$

If we define ϕ by $f\phi \equiv 1 \pmod{Gk}$ we have also $f\phi \equiv 1 \pmod{Gk^*}$. It follows then from (4.4), (4.7), and (4.9) that

$$6pk\sigma_a(h,k) \equiv f\phi(sh - th' - 6kC^* + 3rpk) \pmod{12pk}$$

and

$$6pk^*\sigma_a(h,k^*) \equiv f\phi(Sh - Th' - 6k^*C^* + 3rpk^*) \pmod{12pk^*}$$

where $C^* = \sum_{j=1}^{r} c_j$. From (3.6)

$$\begin{aligned}\omega_a(h,k) &= \exp\{2\pi i \cdot 6pk\sigma_a(h,k)/12pk\} \\ &= \exp\{2\pi i(3\phi(rp - 2C^*)/G + \phi(sh - th')/Gk)\}, \\ \omega_a(h,k^*) &= \exp\{2\pi i(3\phi(rp - 2C^*)/G + \phi(Sh - Th')/Gk^*)\}.\end{aligned}$$

From (3.12), (4.5), (4.8) it follows that

$$w_a(h,k) = \exp\{2\pi i(\phi(uh + vh')/Gk)\}, \tag{4.10}$$

where

$$\begin{aligned} u &= rk^2 - A, \\ v &= B + 2rk^{*2}. \end{aligned} \tag{4.11}$$

5. With h replaced by $2h$ we have from (4.1), (4.2), (4.3) if k is odd

$$6pk\sigma_a(2h,k) \equiv 0 \pmod 3 \text{ if } 3 \nmid k, \tag{5.1}$$

$$6pk\sigma_a(2h,k) \equiv \sum_{j=1}^{r} (2p + 2a_j + 2e_j - 3) \pmod 4, \tag{5.2}$$

$$\begin{aligned} 12hkp\sigma_a(2h,k) = 4h^2\left(A + \sum_{j=1}^{r} \{2k^2 + 3k(2a_j - p)\}\right) + (D - rk^2) \\ - 6hk \sum_{j=1}^{r} (2e_j - p) - 12kpI. \end{aligned} \tag{5.3}$$

Here $e_j = d_j$ if $d_j \equiv 2ha_j \pmod p$, $e_j = p - d_j$ if $d_j \equiv -2ha_j \pmod p$. d_j and D are defined in §3.

If we now take g' so that $2hg' \equiv -1 \pmod{Gk}$ we have after multiplying (5.3) by $-g'$

$$6pk\sigma_a(2h,k) \equiv 2hs - g'Q - 3k \sum_{j=1}^{r} (2e_j - p) \pmod{Gk}, \tag{5.4}$$

where s is given by (4.5) and

$$Q = D - rk^2. \tag{5.5}$$

From (5.1) and (5.2) we have

$$(5.6)\qquad 6pk\sigma_a(2h,k) \equiv \sum_{j=1}^{r} (6p + 6a_j + 6e_j - 3)(\bmod f).$$

Defining ϕ as before and Γ by $Gk\Gamma \equiv 1 \ (\bmod f)$ we have from (4.4), (4.6), (5.4), (5.6) if k is odd

$$6pk\sigma_a(h,k) \equiv f\phi(sh - th' - 6kC^* + 3rpk)$$
$$+ Gk\Gamma(6A^* + 6C^* + 6rp - 3r)(\bmod 12pk)$$

and

$$6pk\sigma_a(2h,k) \equiv f\phi(2sh - Qg' - 6kE^* + 3rpk)$$
$$+ Gk\Gamma(6A^* + 6E^* + 6rp - 3r)(\bmod 12pk)$$

where $A^* = \sum_{j=1}^{r} a_j$, $E^* = \sum_{j=1}^{r} e_j$. From (3.6)

$$\omega_a(h,k) = \exp\{2\pi i(\Gamma(6A^* + 6C^* + 6rp - 3r)/f$$
$$- 3\phi(2C^* - rp)/G + \phi(sh - th')/Gk)\},$$
$$\omega_a(2h,k) = \exp\{2\pi i(\Gamma(6A^* + 6E^* + 6rp - 3r)/f$$
$$- 3\phi(2E^* - rp)/G + \phi(2sh - Qg')/Gk)\}.$$

Since $hh' \equiv -1(\bmod Gk)$ and $2hg' \equiv -1(\bmod Gk)$ we have $h' \equiv 2g'(\bmod Gk)$. It follows from (3.15), (4.5), and (5.5) that

$$(5.7)\quad W_a(h,k) = \exp\{2\pi i(\Gamma(6C^* - 6E^*)/f - 3\phi(2C^* - 2E^*)/G + \phi(u'h + v'g')/Gk)\},$$

where

$$(5.8)\qquad \begin{aligned} u' &= -A - 2rk^2 - 6kA^* + 3rpk, \\ v' &= rk^2 - 2B^*, \end{aligned}$$

and B^* is given by (3.16).

6. Turning to the case $p \nmid k$ we have by (6.2), (6.3), (6.4), (6.5) of [1], with $t_a(h,k)$ given by (3.9)

$$(6.1)\qquad 6pkt_a(h,k) \equiv 0 \ (\bmod 3) \text{ if } 3 \nmid k,$$

$$(6.2)\qquad 6pkt_a(h,k) \equiv \sum_{j=1}^{r} (p - pk + 2a_j + 2\alpha_j)(\bmod 4) \text{ if } 2 \nmid k,$$

$$(6.3)\qquad 6pkt_a(h,k) \equiv 0 \ (\bmod p),$$

$$(6.4)\qquad \begin{aligned} 6hkpt_a(h,k) = {} & h^2\left(A + \sum_{j=1}^{r} \{2K^2 + 3K(2a_j - p)\}\right) \\ & + r(1 - k^2) + 6hk \sum_{j=1}^{r} \alpha_j - 12kI, \end{aligned}$$

where I is an integer. If $p=3$ then (mod p) in (6.3) may be replaced by (mod 9).

Now let $12p = Fg$ where F is the greatest divisor of $12p$ prime to k. If k is even we have $F = 3p$, $g = 4$ if $3 \nmid k$, and $F = p$, $g = 12$ if $3 \mid k$. If k is odd we have $F = 12p$, $g = 1$ if $3 \nmid k$, and $F = 4p$, $g = 3$ if $3 \mid k$. Taking h' so that $hh' \equiv -1$ (mod gk) we have after multiplying (6.4) by $-h'$

$$6pkt_a(h,k) \equiv hs - h't + 6k\alpha^* \pmod{gk} \tag{6.5}$$

where

$$\begin{aligned} s &= A + \sum_{j=1}^{r} \{2K^2 + 3K(2a_j - p)\}, \\ t &= r(1-k^2), \end{aligned} \tag{6.6}$$

and $\alpha^* = \sum_{j=1}^{r} \alpha_j$. From (6.1), (6.2), (6.3) we have

$$6pkt_a(h,k) \equiv 9p^2 \sum_{j=1}^{r} (2a_j + 2\alpha_j + p - pk) \pmod{F}. \tag{6.7}$$

If k is even then $hh' \equiv -1 \pmod{gk^*}$ and since (6.1), (6.2), (6.3), (6.4) hold with k replaced by k^* and α_j replaced by β_j, where β_j is defined in §3, we have

$$6pk^*t_a(h,k^*) \equiv hS - h'T + 6k^*\beta^* \pmod{gk^*}, \tag{6.8}$$

where

$$\begin{aligned} S &= A + \sum_{j=1}^{r} \{2K^{*2} + 3K^*(2a_j - p)\}, \\ T &= r(1-k^{*2}). \end{aligned} \tag{6.9}$$

Here $K^* = pk^*$ and $\beta^* = \sum_{j=1}^{r} \beta_j$.

Since, when k is even, $F = p$ or $F = 3p$ we have from (6.7) and the corresponding statement for k^*

$$6pkt_a(h,k) \equiv 6pk^*t_a(h,k^*) \equiv 0 \pmod{F}. \tag{6.10}$$

If we define Φ by $F\Phi \equiv 1 \pmod{gk}$ then also $F\Phi \equiv 1 \pmod{gk^*}$. It follows then from (6.5), (6.8), (6.10) that

$$6pkt_a(h,k) \equiv F\Phi(sh - th' + 6k\alpha^*) \pmod{12pk},$$

$$6pk^*t_a(h,k^*) \equiv F\Phi(Sh - Th' + 6k^*\beta^*) \pmod{12pk^*}.$$

Then, from (3.8)

$$\chi_a(h,k) = \exp\{2\pi i(6\Phi\alpha^*/g + \Phi(sh - th')/gk)\},$$

$$\chi_a(h,k^*) = \exp\{2\pi i(6\Phi\beta^*/g + \Phi(Sh - Th')/gk^*)\}.$$

From (3.19), (6.6), (6.9) it then follows that

$$z_a(h,k) = \exp\{2\pi i(6\Phi(\alpha^* - \beta^*)/g + \Phi(Uh + Vh')/gk)\}, \tag{6.11}$$

where

$$\begin{aligned} U &= rK^2 - A, \\ V &= r + 2rk^{*2}. \end{aligned} \tag{6.12}$$

7. With h replaced by $2h$ we have from (6.1), (6.2), (6.3), (6.4) if k is odd

$$6pkt_a(2h,k) \equiv 0 \pmod 3 \text{ if } 3\nmid k, \tag{7.1}$$

$$6pkt_a(2h,k) \equiv \sum_{j=1}^{r} (p - pk + 2a_j + 2\alpha_j) \pmod 4, \tag{7.2}$$

$$6pkt_a(2h,k) \equiv 0 \pmod p, \tag{7.3}$$

$$\begin{aligned} 12hkpt_a(2h,k) \equiv\ & 4h^2\left(A + \sum_{j=1}^{r} \{2K^2 + 3K(2a_j - p)\}\right) + r(1 - k^2) \\ & + 12hk\alpha^* - 12kI. \end{aligned} \tag{7.4}$$

If $p = 3$ then $(\mathrm{mod}\, p)$ in (7.3) is replaced by (mod 9).

If we now take g' so that $2hg' \equiv -1 \pmod{gk}$ we have after multiplying (7.4) by $-g'$

$$6pkt_a(2h,k) \equiv 2hs - g't + 6k\alpha^* \pmod{gk}, \tag{7.5}$$

where s and t are given by (6.6). From (7.1), (7.2), (7.3)

$$6pkt_a(2h,k) \equiv 9p^2 \sum_{j=1}^{r} (2a_j + 2\alpha_j + p - pk) \pmod F. \tag{7.6}$$

Defining Φ as before and γ by $gk\gamma \equiv 1 \pmod F$ we have from (6.5), (6.7), (7.5), (7.6) if k is odd

$$6pkt_a(h,k) \equiv F\Phi(sh - th' + 6k\alpha^*) + 9gk\gamma p^2(2A^* + 2\alpha^* + rp - rpk) \pmod{12pk},$$

$$6pkt_a(2h,k) \equiv F\Phi(2sh - tg' + 6k\alpha^*) + 9gk\gamma p^2(2A^* + 2\alpha^* + rp - rpk) \pmod{12pk}.$$

From (3.8)

$$\chi_a(h,k) = \exp\{2\pi i(9p^2\gamma(2A^* + 2\alpha^* + rp - rpk)/F + 6\Phi\alpha^*/g + \Phi(sh - th')/gk)\},$$

$$\chi_a(2h,k) = \exp\{2\pi i(9p^2\gamma(2A^* + 2\alpha^* + rp - rpk)/F + 6\Phi\alpha^*/g + \Phi(2sh - tg')/gk)\}.$$

Since $h' \equiv 2g' \pmod{gk}$ we have from (3.22), (6.6) and the fact that $gk \mid 2K^2$,

$$Z_a(h,k) = \exp\{2\pi i\Phi(U'h + V'g')/gk\}, \tag{7.7}$$

where

$$(7.8)\qquad \begin{aligned} U' &= -A - 6KA^* + 3rpK, \\ V' &= rk^2 - r. \end{aligned}$$

8. THEOREM 2. *The sum*

$$S_1 = \sum_{h \bmod k}{}' w_a(h,k)\exp\{-2\pi i(hn - h'v)/k\},$$

where $h \equiv \zeta \pmod p$, $p \nmid \zeta$; $\sigma_1 \leqq h' < \sigma_2 \pmod k$, $0 \leqq \sigma_1 < \sigma_2 \leqq k$; $2p \mid k$, *is subject to the estimate* $O(n^{1/3}k^{2/3+\varepsilon})$ *uniformly in* $v, \zeta, \sigma_1, \sigma_2, a$.

THEOREM 3. *The sum*

$$S_2 = \sum_{h \bmod k}{}' W_a(h,k)\exp\{-2\pi i(hn - g'v)/k\},$$

where $h \equiv \zeta \pmod p$, $p \nmid \zeta$; $\sigma_1 \leqq h' < \sigma_2 \pmod k$, $0 \leqq \sigma_1 < \sigma_2 \leqq k$; $p \mid k$, $2 \nmid k$, *is subject to the estimate* $O(n^{1/3}k^{2/3+\varepsilon})$ *uniformly in* $v, \zeta, \sigma_1, \sigma_2, a$.

THEOREM 4. *The sum*

$$S_3 = \sum_{h \bmod k}{}' z_a(h,k)\exp\{-2\pi i(hn - H'v)/k\},$$

where $\sigma_1 \leqq h' < \sigma_2 \pmod k$, $0 \leqq \sigma_1 < \sigma_2 \leqq k$; $p \nmid k$, $2 \mid k$, *is subject to the estimate* $O(n^{1/3}k^{2/3+\varepsilon})$ *uniformly in* v, σ_1, σ_2, a.

THEOREM 5. *The sum*

$$S_4 = \sum_{h \bmod k}{}' Z_a(h,k)\exp\{-2\pi i(hn - G'v)/k\},$$

where $\sigma_1 \leqq h' < \sigma_2 \pmod k$, $0 \leqq \sigma_1 < \sigma_2 \leqq k$; $p \nmid k$, $2 \nmid k$, *is subject to the estimate* $O(n^{1/3}k^{2/3+\varepsilon})$ *uniformly in* v, σ_1, σ_2, a.

Here σ_1, σ_2 are integers and $\sum'$ indicates that h runs over integers prime to the modulus of the sum. h', g', H', G' are given by (3.11), (3.14), (3.18), (3.21) respectively.

Proof. $w_a(h,k)$, $W_a(h,k)$, $z_a(h,k)$, $Z_a(h,k)$, when viewed as functions of h, all have period k. Therefore, if in S_1 and S_2 we select h' so that $hh' \equiv -1 \pmod{Gk}$ and make use of (4.10) and (5.7) we can write

$$S_1 = G^{-1} \sum_{h \bmod Gk}{}' \exp\{2\pi i f(h)/Gk\},$$

$$S_2 = G^{-1} c_2(a,k,\zeta) \sum_{h \bmod Gk} \exp\{2\pi i g(h)/Gk\},$$

where $|c_2| = 1$, $f(h) = (\phi u - Gn)h + (\phi v + Gv)h'$, and $g(h) = (\phi u' - Gn)h + (\phi v' + Gv)g'$. Since $h' \equiv 2g' \pmod{Gk}$ and since an integer λ exists such that $2\lambda \equiv 1 \pmod{Gk}$ we see that $g(h) \equiv (\phi u' - Gn)h + \lambda(\phi v' + Gv)h' \pmod{Gk}$.

If in S_3 and S_4 we take h' so that $hh' \equiv -1 \pmod{gk}$ and make use of (6.11), (7.7) we can write

$$S_3 = g^{-1}c_3(a,k) \sum_{h \bmod gk}' \exp\{2\pi i F(h)/gk\},$$

$$S_4 = g^{-1} \sum_{h \bmod gk}' \exp\{2\pi i G(h)/gk\},$$

where $|c_3| = 1$, $F(h) = (\Phi U - gn)h + \Phi V h' + gvH'$, and $G(h) = (\Phi U' - gn)h + \Phi V g' + gvG'$. Since $h' \equiv pH' (\bmod gk)$ if $HH' \equiv -1 (\bmod gk)$ and since an integer δ exists such that $p\delta \equiv 1 (\bmod gk)$ we see that $F(h) \equiv (\Phi U - gn)h + (\Phi V + \delta g v)h' (\bmod gk)$. Also, since $h' \equiv 2g' (\bmod gk)$, and $h' \equiv 2pG' (\bmod gk)$ if $2HG' \equiv -1 (\bmod gk)$ and since integers σ and ω exist such that $2\sigma \equiv 1 (\bmod gk)$ and $2p\omega \equiv 1 (\bmod gk)$ we have $G(h) \equiv (\Phi U' - gn)h + (\sigma \Phi V' + \omega g v)h' (\bmod gk)$.

Following Lehner's procedure [2], these sums may now be written in the form $S_i = O(K_i \log k)$ where K_i is a complete Kloosterman sum. The desired estimates are now obtained as in the proof of Theorem 2 in [1].

IV. A Convergent Series for $q_a(n)$

9. By Cauchy's integral formula and (2.2) we have

$$q_a(n) = \frac{1}{2\pi i} \int_C x^{-n-1} G_a(x) dx = \sum_{h,k}' \frac{1}{2\pi i} \int_{\xi_{hk}} x^{-n-1} G_a(x) dx$$

where $0 \leqq h < k \leqq N$ and ξ_{hk} are the Farey arcs of order N of C, the circle $|x| = \exp\{-2\pi N^{-2}\}$. If, on ξ_{hk}, we introduce the variable ϕ by setting $x = \exp\{-2\pi N^{-2} + 2\pi i h/k + 2\pi i \phi\}$ and then write $w = N^{-2} - i\phi$, $z = wk$, we have

$$q_a(n) = \sum_{h\ k}' \exp\{-2\pi i n h/k\} \int_{-\theta'}^{\theta''} G_a(\exp\{2\pi i h/k - 2\pi z/k\}) \exp\{2\pi n w\}\, d\phi.$$

Here $\theta' = 1/k(k+k_1)$ and $\theta'' = 1/k(k+k_2)$ where $h_1/k_1 < h/k < h_2/k_2$ are consecutive terms in the Farey series of order N.

We now split the sum over k into four parts $q_a(n,1)$, $q_a(n,2)$, $q_a(n,3)$, $q_a(n,4)$ according as k satisfies the requirements given in (3.23), (3.24), (3.25), (3.26) respectively. Selecting β so that $a_1\beta \equiv 1 \ (\bmod p)$ we have by (3.23)

$$q_a(n,1) = \sum_{b_1=1}^{(p-1)/2} \sum_{h,k}' w_a(h,k) \exp\{-2\pi i n h/k\} \int_{-\theta'}^{\theta''} \sum_{v=0}^{\infty} q_b(v) \exp\{2\pi i h' v/k\}$$
$$\cdot \exp\{-(\pi/k^2 w)(2v + B/6p) + \pi w(2n + A/6p)\} d\phi,$$

where $h \equiv \pm b_1\beta (\bmod p)$.

If we split the sum over v into two parts according as $2v + B/6p$ is negative or nonnegative we find, using Rademacher's argument [3] and employing Theorem 2, that

$$\begin{aligned} q_a(n,1) &= 2\pi \sum_{k=1}^{N} \sum_{b_1=1}^{(p-1)/2} \sum_{v<-B/12p} q_b(v) A_{kb}(n,v) L_{kb}(n,v) \\ &\quad + O(n^{1/3} N^{-1/3+\varepsilon} \exp\{2\pi n N^{-2}\}) \end{aligned} \tag{9.1}$$

where $2p \mid k$. Here,

$$A_{kb}(n,v) = \sum_{h \bmod k}{}' w_a(h,k)\exp\{-2\pi i(nh - vh')/k\} \tag{9.2}$$

where $h \equiv \pm b_1\beta \pmod p$; and

$$L_{kb}(n,v) = \begin{cases} k^{-1}\{(-B-12vp)/(12np+A)\}^{1/2} \\ \quad \cdot I_1\{\pi(12np+A)^{1/2}(-B-12vp)^{1/2}/3pk\} \text{ if } n > -A/12p, \\ -(B+12vp)\pi/6pk^2 \text{ if } n = -A/12p, \end{cases} \tag{9.3}$$

where $I_1(z)$ is the Bessel function.

As an example of a case for which $n = -A/12p$, if we take $p = 71$, $a = \{22,23,25,26,27,28,29,30,31,32,33,34\}$, we find that $-A/12p = 30$. The case $n < -A/12p$ is of no interest since the final formula for $q_a(n)$ requires the calculation of $q_a(v)$ for $v < -A/12p$.

From (3.24) we have

$$q_a(n,2) = \sum_{b_1=1}^{(p-1)/2} \sum_{h,k}{}' W_a(h,k)\exp\{-2\pi inh/k\}\int_{-\theta'}^{\theta''} \sum_{v=0}^{\infty} s_b(v)\exp\{2\pi ig'v/k\}$$
$$\cdot\exp\{-(\pi/k^2w)(v - B^*/6p) + \pi w(2n + A/6p)\}d\phi,$$

where $h \equiv \pm b_1\beta \pmod p$.

Splitting the sum over v into two parts according as $v - B^*/6p$ is negative or nonnegative, utilizing Rademacher's method [3] and making use of Theorem 3 we have

$$\begin{aligned} q_a(n,2) &= 2\pi \sum_{k=1}^{N} \sum_{b_1=1}^{(p-1)/2} \sum_{v<B^*/6p} s_b(v)B_{kb}(n,v)M_{kb}(n,v) \\ &\quad + O(n^{1/3}N^{-1/3+\varepsilon}\exp\{2\pi nN^{-2}\}) \end{aligned} \tag{9.4}$$

where $p \mid k$, $2 \nmid k$.

$$B_{kb}(n,v) = \sum_{h \bmod k}{}' W_a(h,k)\exp\{-2\pi i(nh - vg')/k\} \tag{9.5}$$

where $h \equiv \pm b_1\beta \pmod p$, and

$$M_{kb}(n,v) = \begin{cases} k^{-1}\{(B^* - 6vp)/(12np+A)\}^{1/2} \\ \quad \cdot I_1\{\pi(12np+A)^{1/2}(B^*-6vp)^{1/2}/3pk\} \text{ if } n > -A/12p, \\ (B^*-6vp)\pi/6pk^2 \text{ if } n = -A/12p. \end{cases} \tag{9.6}$$

From (3.25) we have

$$q_a(n,3) = 2^r \sum_{h,k}{}' (-1)^\mu\left(\prod_{j=1}^{r}\cos\pi\alpha_j/p\right)z_a(h,k)\exp\{-2\pi inh/k\}$$
$$\cdot\int_{-\theta'}^{\theta''}\sum_{v=0}^{\infty} d_{ak}(v)\exp\{2\pi iH'v/k\}\exp\{-(\pi/Kkw)(2v + r/6) + \pi w(2n + A/6p)\}d\phi.$$

Since $2v + r/6 > 0$ we have by the argument detailed in [3], and employing Theorem 4

$$q_a(n,3) = O(n^{1/3} N^{-1/3+\varepsilon} \exp\{2\pi n N^{-2}\}). \tag{9.7}$$

From (3.26) we have

$$q_a(n,4) = \sum_{h,k}{}' Z_a(h,k)\exp\{-2\pi i n h/k\}\int_{-\theta'}^{\theta''} \sum_{v=0}^{\infty} e_{ak}(v)\exp\{2\pi i G' v/k\} \cdot \exp\{-(\pi/Kkw)(v - r/12) + \pi w(2n + A/6p)\}d\phi.$$

If we split the sum over v into two parts according as $v - r/12$ is negative or nonnegative we have, using Theorem 5,

$$q_a(n,4) = 2\pi \sum_{k=1}^{N}{}' \sum_{v<r/12} e_{ak}(v) C_k(n,v) N_k(n,v) + O(n^{1/3} N^{-1/3+\varepsilon}\exp\{2\pi n N^{-2}\}). \tag{9.8}$$

$$C_k(n,v) = \sum_{h \bmod k} Z_a(h,k)\exp\{-2\pi i(nh - vG')/k\}. \tag{9.9}$$

$$N_k(n,v) = \begin{cases} k^{-1}\{(r/2 - 6v)/(12np + A)\}^{1/2} \\ \quad \cdot I_1\{\pi(12np + A)^{1/2}(r/2 - 6v)^{1/2}/3pk\} \text{ if } n > -A/12p, \\ (r/2 - 6v)\pi/6pk^2 \text{ if } n = A/12p. \end{cases} \tag{9.10}$$

Letting $N \to \infty$ in (9.1), (9.4), (9.7), (9.8) we have

THEOREM 6. *The number, $q_a(n)$, of partitions of a positive integer n, $n \geqq -A/12p$, into distinct positive summands of the form $pm \pm a_j$, $a_j \in a$, is given by the convergent series*

$$q_a(n) = 2\pi \sum_{k=1}^{\infty} \sum_{b_1=1}^{(p-1)/2} \sum_{v<-B/12p} q_b(v) A_{kb}(n,v) L_{kb}(n,v) + 2\pi \sum_{k=1}^{\infty} \sum_{b_1=1}^{(p-1)/2} \sum_{v<B^*/6p} s_b(v) B_{kb}(n,v) M_{kb}(n,v) + 2\pi \sum_{k=1}^{\infty} \sum_{v<r/12} e_{ak}(v) C_k(n,v) N_k(n,v)$$

where $A_{kb}, L_{kb}, B_{kb}, M_{kb}, C_k, N_k$ are given by (9.2), (9.3), (9.5), (9.6), (9.9), (9.10) *respectively. In the first sum $2p \mid k$; in the second $p \mid k, 2 \nmid k$; in the third $p \nmid k$, $2 \nmid k$.*

V. ASYMPTOTIC FORMULAS

10. As in the case of $p_a(n)$ the dominant term in the series representing $q_a(n)$ is that for which $k = 1$. Letting $G(v) = (r/2 - 6v)^{1/2}$, $T = \pi(12np + A)^{1/2}/3p$, $W = \sum_{v<r/12} e_{a1}(v) G(v) I_1\{TG(v)\}$ we have

THEOREM 7. *As* $n \to \infty$

$$q_a(n) = 2\pi W(12np + A)^{-1/2}(1 + O(\exp\{-cn^{1/2}\}))$$

where $c > 0$.

The proof is essentially the same as that of Theorem 7 in [1] and is therefore omitted. We remark only that one must establish that if $-B > 0 (B^* > 0)$ and M is the maximum possible value of $-B\ (B^*)$ then $k^{-1}M^{1/2} - (r/2)^{1/2} < 0$ if $k \geqq p$. This is easily done as follows.

Since B is a decreasing function of each b_j the maximal value of $-B$ is $M = -rp^2 + 6p\sum j - 6\sum j^2$ where $(p+1)/2 - r \leqq j \leqq (p-1)/2$. An easy calculation yields $M = rp^2/2 + r(1-4r^2)/2 < rp^2/2$. (The maximal value of B^* is $M = rp^2/2 - 6\sum_{j=1}^{r} j^2 < rp^2/2$.) Since $k \geqq p$ we have $k^{-2}M \leqq p^{-2}M < r/2$.

Since $W \sim (r/2)^{1/2} I_1\{T(r/2)^{1/2}\}$ (see (11.13) in [1]) we have from Theorem 7

COROLLARY 7.1. *As* $n \to \infty$

$$q_a(n) = \pi(2r/(12np + A))^{1/2} I_1\{T(r/2)^{1/2}\}(1 + O(\exp\{-cn^{1/2}\})).$$

Finally, since $I_1(z) = e^z(2\pi z)^{-1/2}(1 + O(z^{-1}))$ as $z \to \infty$, we have

COROLLARY 7.2. *As* $n \to \infty$

$$q_a(n) = (6p)^{1/2}(r/2)^{1/4}(12np + A)^{-3/4}\exp\{T(r/2)^{1/2}\}(1 + O(n^{-1/2})).$$

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