THE EXISTENCE OF TOPOLOGIES ON FIELD EXTENSIONS

BY

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This work stems from the study of a special aspect of the problem of determining all topologies on the algebra $C(X)$, of all real-valued continuous functions on a topological space X. Specifically, we are interested in the following question, posed by M. Henriksen [4].

(I) Can $C(X)$ be topologized as an algebra in such a manner that every maximal ideal is closed?

We are able to settle this question completely only in certain special cases (see §1). However, we do show that the existence of such topologies is equivalent to the existence of (Hausdorff) algebra topologies on each of the residue class fields $C(X)/M$. Thus we are led to the general question, which occupies the main portion of this paper, namely:

(II) Let $(k,\mathscr{T})$ be a topological field and let K be an extension field of k. Does there exist a ring topology $\mathscr{T}_K$ for K such that $\mathscr{T}_K \mid k \subseteq \mathscr{T}$?

The attack on this question splits naturally into two cases: (a) K a purely transcendental extension of k, and (b) K an algebraic extension of k. In neither case are we able to offer a complete solution. In §2, however, using an extension of a technique employed by Williamson [7], we are able to provide an affirmative answer to (a) in the case where k is locally compact. An interesting by-product of this construction is an example of an additively generated field which is not a subfield of the quaternion field. In §3 we provide an affirmative answer to (b), under the restrictions that the degree of extension be at most countable and that the underlying field k be locally bounded.

1. **The role of the residue class fields.** We are able to simplify the problem of seeking algebra topologies on $C(X)$ by reducing it to that of finding algebra topologies on extension fields of the real field R. Explicitly, we have:

1.1. PROPOSITION. *Let X be an arbitrary topological space. The algebra $C(X)$ admits an algebra topology having all maximal ideals closed if and only if,*

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for each maximal ideal M, the residue class field $C(X)/M$ admits a (Hausdorff) algebra topology.

Proof. For each maximal ideal M, let π_M be the natural homomorphism of $C(X)$ onto $C(X)/M$. If each residue class field $C(X)/M$ admits a (Hausdorff) algebra topology $\mathcal{T}_M$, then the weak topology on $C(X)$ determined by the family $\{\pi_M\}$ is an algebra topology. Since $\mathcal{T}_M$ is Hausdorff, M is closed in the weak topology. Moreover, since $C(X)$ is semi-simple, $\{0\}$ is the intersection of all maximal ideals and hence the weak topology is Hausdorff.

Conversely, if $C(X)$ admits an algebra topology having all maximal ideals closed, then the quotient topology on each residue class field is an algebra topology. Since each maximal ideal M is closed, the quotient topology on $C(X)/M$ is Hausdorff.

Although the above result was stated for algebra topologies it is clear that "order-convex algebra," "locally-convex algebra," or "algebra with continuous inversion" can be substituted for "algebra".

The algebra $C(X)$ is said to be *pseudo-compact* in case every $f \in C(X)$ is bounded. It is well known that when X is pseudo-compact, the norm $\| \ \|$ defined by $\|f\| = \sup_{x \in X} |f(x)|$ defines an algebra topology having most of the properties that one might desire. The following theorem shows that if X is not pseudo-compact many of these properties are unattainable for an algebra topology on $C(X)$.

1.2. THEOREM. *For a topological space X the following statements are equivalent.*

(i) *X is pseudo-compact.*

(ii) *$C(X)$ admits an order-convex algebra topology having the property that every maximal ideal is closed.*

(iii) *$C(X)$ admits an algebra topology in which the units form an open subset.*

(iv) *$C(X)$ admits a locally-convex algebra topology having the property that every maximal ideal is closed and inversion is continuous where defined.*

Proof. If X is pseudo-compact, then the usual norm $\| \ \|$ defines an algebra topology satisfying (ii), (iii) and (iv). To see the reverse implications suppose that X is not pseudo-compact. Then there exists a hyper-real residue class field, say $F^M = C(X)/M$, cf. [3, p. 71].

To see that (ii) implies (i), suppose that $C(X)$ admits an order-convex algebra topology having all maximal ideals closed. Then the quotient topology on F^M is an order-convex Hausdorff algebra topology. This is impossible since in the presence of infinitely small elements, an order-convex algebra topology cannot be Hausdorff.

To see that (iii) implies (i), suppose that $\mathcal{T}$ is a topology for $C(X)$ satisfying (iii), F^M is a hyper-real residue class field and $\mathcal{T}_M$ is the quotient topology. In view of Proposition 1.1, $\mathcal{T}_M$ is a locally-convex field topology for F^M.

By a well-known theorem of Arens [1], F^M must be the real or complex field. Since F^M is a totally ordered field properly containing the real field, it is not either of these.

To complete the proof we suppose that $\mathscr{T}$ is an algebra topology for $C(X)$ in which the set of units is open. Let $\mathscr{V}$ be a basis for the neighborhood system of zero and let α be a positive real number. We claim that there exists $V \in \mathscr{V}$ such that $\|f\| < \alpha$ for every $f \in V$. Suppose false, then for each integer n and each $U \in \mathscr{V}$ there exists $f \in U$ and $x \in X$ such that $|f(x)| > n\alpha$. For ε a positive real number, set $V_\varepsilon = \{a \in R;\ |a| \leqq \varepsilon\}$. Hence, for each $\varepsilon > 0$, and $U \in \mathscr{V}$ there exists $f \in V_\varepsilon \cdot U$ and $x \in X$ such that $f(x) = -\alpha$. Let T be arbitrary in $\mathscr{V}$. Since $\mathscr{T}$ is an algebra topology there exists $\varepsilon > 0$, $U \in \mathscr{V}$ such that $V_\varepsilon \cdot U \subseteq T$. Thus $-\alpha + T$ contains a nonunit, contrary to the hypothesis that the set of units is open. Thus $\mathscr{T}$ is smaller than the norm topology. It is clear that no such topology can be an algebra topology if X is not pseudo-compact.

2. **Purely transcendental extensions.** The literature dealing with question (II) is quite meager. Irving Kaplansky in [5] poses the question of whether or not any proper extension field of the complex field admits a complex algebra topology. This question is answered in the affirmative by Williamson in [7]. He demonstrates the existence of at least two distinct complex algebra topologies on a simple transcendental extension of the complex field. The method used here is a generalization of the procedure used by Williamson in [7].

Let K be a purely transcendental extension of a locally-compact field k. Let A be a compact neighborhood of zero and μ_k a Haar measure for A such that $\mu_k(A) = 1$. Finally, let T be a transcendence base for K over k and set $X = A^T$. Then the polynomial ring $k[T]$ has a natural representation as the ring $k[X]$ of polynomial functions mapping $X = A^T$ into k. This in turn has a natural extension to an isomorphism mapping the field K of quotients of $k[T]$ onto the field of rational functions mapping X into k. We will topologise K by defining a topology on the field $k(X)$ of rational functions mapping X into k.

Let μ be the product μ_k-measure on X and let $\mathscr{V}$ be a basis for the neighborhood system of zero in k, consisting of symmetric, compact, closed sets. For $V \in \mathscr{V}$ and $h \in k(X)$, define $\rho(h, V)$ by

$$\rho(h, V) = \mu\{x \in X;\ \ h(x) \notin V\}.$$

Let $\mathscr{N}$ be the collection of all sets of the form

$$N(v, \varepsilon) = \{h \in k(X);\ \ \rho(h, V) < \varepsilon\},$$

where $V \in \mathscr{V}$ and ε is a positive real number. We will show that $\mathscr{N}$ defines a field topology for $k(X)$, which extends the topology on k.

To see that $\mathscr{N}$ defines a group topology for $k(X)$, let $N(V, \varepsilon) \in \mathscr{N}$ be given. Then there exists $W \in \mathscr{V}$ such that $W - W \subseteq V$. Suppose $f, g \in N(W, \varepsilon/2)$.

Then $\{x\in X;\ (f-g)(x)\notin V\}\subseteq\{x\in X;\ f(x)\notin W\}\cup\{x\in X;\ g(x)\notin W\}$. Thus $\rho(f-g,V)\leqq\rho(f,W)+\rho(g,W)<\varepsilon/2+\varepsilon/2=\varepsilon$. Consequently

$$N(W,\varepsilon/2)-N(W,\varepsilon/2)\subseteq N(V,\varepsilon)$$

and $\mathscr{N}$ defines a group topology for $k(X)$.

By a similar argument, it can be seen that if $W,\ V\in\mathscr{V}$ with $W\cdot W\subseteq V$, then $N(W,\varepsilon/2)\cdot N(W,\varepsilon/2)\subseteq N(V,\varepsilon)$. Since the constant functions $v\in V$ are in $N(V,\varepsilon)$, $\mathscr{N}$ defines a k-vector space topology in which multiplication is continuous at zero.

To see that multiplication is continuous everywhere, let $f\in k(X)$ and $N(V,\varepsilon)\in\mathscr{N}$ be given, Let $D_f=\{x\in X;\ f(x)\notin k\}$. It is easily seen that there exists a subset U of X, open in the product topology, such that $D_f\subseteq U$ and $\mu(U)<\varepsilon/2$. Since f is a continuous k-valued function on the compact set $X-U$, $f[X-U]$ is a compact subset of k. Hence, there exists $W\in\mathscr{V}$ such that $W\cdot f[X-U]\subseteq V$. If $g\in N(W,\varepsilon/2)$, then $\{x\in X;\ (fg)(x)\notin V\}\subseteq\{x\in X;\ f(x)\notin f[X-V]\}\cup\{x\in X;\ g(x)\notin W\}$. Therefore $\rho(fg,\cdot V)\leqq\mu(U)+\rho(g,W)<\varepsilon$. Thus $f\cdot N(W,\varepsilon/2)\subseteq N(V,\varepsilon)$ and consequently multiplication is continuous everywhere.

To show that inversion is continuous it is sufficient to show that if $0\neq f\in k(X)$ and $N(V,\varepsilon)\in\mathscr{N}$ are given, there exists $N(W,\varepsilon')\in\mathscr{N}$ such that $[f+N(W,\varepsilon')]^{-1}\subseteq f^{-1}+N(V,\varepsilon)$. Let $O_f=\{x\in X;\ f(x)=0\ \text{or}\ f(x)\notin k\}$. As above, there exists an open subset U of X such that $O_f\subseteq U$ and $\mu(U)<\varepsilon/2$. Since f is a continuous k-valued function on the compact set $X-U$, $L=f[X-U]$ is a compact subset of k disjoint from zero. Consequently there exists a finite open cover, say $V_1,\cdots,V_n$, of L such that each V_i has compact closure disjoint from zero. Set $L'=\bigcup_{i=1}^{n}\bar{V}_i$, so that L' is a compact neighborhood of L disjoint from zero. Choose $W_1\in\mathscr{V}$ such that $x+W_1\subseteq L'$, for every $x\in L$. Since the mapping $a\rightarrow a^{-1}$ is uniformly continuous on L', there exists $W_2\in\mathscr{V}$ such that $a,\ b\in L'$ and $a-b\in W_2$ implies $a^{-1}-b^{-1}\in V$. Set $W=W_1\cap W_2$. If $g\in k(X)$ and $x\in X$ such that $g(x)\in W=W_1\cap W_2$ and $f(x)\in L$, then $(f+g)(x)$, $f(x)\in L$ and $(f+g)(x)\in W_2$ so that $[(f+g)(x)]^{-1}-[f(x)]^{-1}=[(f+g)^{-1}-f^{-1}](x)\in V$. Thus, for $g\in k(X)$, $\{x\in X;\ [(f+g)^{-1}-f^{-1}](x)\notin V\}\subseteq\{x\in X; f(x)\notin W\}\cup\{x\in X; f(x)\notin L\}\subseteq U\cup\{x\in X;\ g(x)\notin W\}$. Consequently $[f+N(W,\varepsilon/2)]^{-1}\subseteq f^{-1}+N(V,\varepsilon)$. For if $g\in N(W,\varepsilon/2)$, then $\rho([(f+g)^{-1}-f^{-1}],V)\leqq\mu(U)+\mu\{x\in X;\ g(x)\notin W\}<\varepsilon/2+\varepsilon/2=\varepsilon$. Thus inversion is continuous.

Finally we note that $\mathscr{N}$ defines a Hausdorff topology. For if $f\in\bigcap\mathscr{N}$, then f vanishes on a dense subset of X and hence $f=0$.

The following theorem is a direct consequence of the foregoing construction.

2.1. Theorem. *Let $(k,\mathscr{T})$ be a subtopological field of a locally compact field and let K be a purely transcendental extension field of k. Then there exists a field topology $\mathscr{T}_K$ for K such that $\mathscr{T}_K\,|\,k=\mathscr{T}$.*

The topology defined above can be used to provide an example of an additively generated field which is not (algebraically) a subfield of the quaternions(2).

A topological field k is said to be *additively generated* in case there are no proper open subgroups; i.e., in case any nonvoid open subset additively generates the field.

Let T be a set of distinct indeterminants such that the cardinality of T is greater than the cardinality of the quaternion field. Let R be the real field, let $A = [-1,1]$, and let μ_R be Lebesgue measure. Let $R(X)$ be the topological field as constructed above. A basis for the neighborhood system of zero in $R(X)$ consists of all sets of the form

$$N(k,\varepsilon) = \{f \in R(X); \mu\{x \in X; |f(x)| \geqq k\} < \varepsilon\},$$

where k and ε are positive real numbers.

It is clear that $R(X)$ is not isomorphic to a subfield of the quaternions since the cardinality of the former is greater than that of the latter.

To see that $R(X)$ is additively generated, suppose that G is an open subgroup of $R(X)$. Then there exist k, $\varepsilon > 0$ such that $N(k,\varepsilon) \subseteq G$. Let f be an arbitrary element of $R(X)$. Since f is almost everywhere finite, there exists a positive real number m such that $\mu\{x \in X; |f(x)| \geqq m\} < \varepsilon$ If n is a positive integer such that $n > mk^{-1}$, then $(n^{-1})f \in N(k,\varepsilon)$. For if $|(n^{-1})f(x)| \geqq k$, then

$$|f(x)| \geqq nk \geqq (mk^{-1})k = m.$$

Thus f is in the subgroup generated by $N(k,\varepsilon)$. Since f was arbitrary, $G = R(X)$ and $R(X)$ is additively generated.

3. **Algebraic extensions.** In a commutative topological ring a subset A is said to be *bounded* in case, for each neighborhood U of zero, there exists a neighborhood V of zero such that $V \cdot A \subseteq U$. A topological ring is said to be *locally bounded* in case there exists a basis for the neighborhood system of zero, consisting of bounded sets.

3.1. LEMMA. *Let K be a simple algebraic extension of k. If k is a topological field, then the Cartesian topology on K, as a finite-dimensional vector space over k, is a field topology. Moreover, if k is locally bounded, then the Cartesian topology on K is locally bounded.*

Proof. It is well known that as a vector space over k, K is, with the Cartesian topology, a topological vector space. Moreover, we can define multiplication in K in terms of the basis $1 = X^0, X^1, \cdots, X^n$. Explicitly, there exist constants m_{ijk}, $0 \leqq i,j,k \leqq n$, such that $X^iX^j = \sum_k m_{ijk}X^k$ and hence

$$(*) \qquad (\textstyle\sum a_iX^i)(\sum b_jX^j) = \sum_k \left(\sum_{i,j} a_ib_jm_{ijk}\right)X^k.$$

(2) The question of the existence of such fields has been posed by M. Shanks in correspondence.

Since k is a topological field and K is a topological vector space over k, the mapping

$$\left(\sum_i a_iX^i, \sum_j b_jX^j\right) \longrightarrow \sum_k \left(\sum_{i,j} a_ib_jm_{ijk}\right)X^k$$

is continuous from $K \times K \to K$. Moreover, inversion is continuous since if $0 \neq a = \sum_i a_iX^i \in K$, $a^{-1} = [p(a_0,\cdots,a_n)] \cdot [q(a_0,\cdots,a_n)]^{-1}$ where p and q are polynomials in $(n+1)$-variables and q does not vanish except for $a_i \equiv 0$.

Finally, to show that the Cartesian topology on K is locally bounded, we will take as a basis for the neighborhood system of zero in K all sets of the form

$$U_K = \left\{\sum_i a_iX^i \in K;\ a_i \in U \text{ for } 0 \leqq i \leqq n\right\},$$

where U is a bounded neighborhood of zero in k. Given two such neighborhoods U_K and V_K we choose neighborhoods V_1 and V_2, of zero in k, such that $V_2 \cdot \{m_{ijk};\ 0 \leqq i,j,k \leqq n\} \subseteq V_1$ and $n \cdot V_1 \subseteq V$. It follows from (*) that $(V_1)_K \cdot U_K \subseteq V_K$.

In order to prove the main result of this section it is necessary to establish two technical lemmas. The first is a bound on the coefficients of products in K.

To this end, let $K = k(\alpha)$ be a simple algebraic extension of k and let $f = \sum_i f_iX^i$ be the minimal polynomial of α, of degree $n+1$. As in the proof of the last lemma we will write every element a of K in the form $\sum_{i=0}^n a_iX^i$. For such an element $a \in K$, we set $C_a = \{\pm a_i;\ i = 0,\cdots,n\} \cup \{0\}$ and $C'_a = \{a_1; i = 1,\cdots,n\} \cup \{0\}$. Finally, we will denote multiplication in $k[X]$ by $a * b$ and multiplication in K by $a \cdot b$. Then for $a, b \in K$, the product $a \cdot b$ is defined by the equation

$$a * b = f * g + a \cdot b, \quad \text{where degree } \ a \cdot b \leqq n.$$

Since, for $n < i \leqq 2n$, $(f * g)_i = (a * b)_i$, the coefficients of g must satisfy the n linear equations

$$\begin{aligned} y_0f_{n+1} + y_1f_n + \cdots + y_{n-1}f_2 &= (a*b)_{n+1} \\ y_1f_{n+1} + \cdots + y_{n-1}f_3 &= (a*b)_{n+2} \\ &\vdots \\ y_{n-1}f_{n+1} &= (a*b)_{2n} \end{aligned}$$

in the n unknowns $y_0,\cdots,y_{n-1}$. The value of the coefficient determinant is $f_{n+1}^n = 1$ so $g_k = \| G(k)_{ij} \|$, $0 \leqq k \leqq n-1$, where $G(k)_{ij}$ is the determinant obtained by replacing the kth column of the coefficient determinant by the column

$$\begin{bmatrix} (a*b)_{n+1} \\ \vdots \\ (a*b)_{2n} \end{bmatrix}.$$

Thus, each g_k is a sum of $n!$ terms; each term of which is a product of $(n-1)$ factors from C_f and one factor from C'_{a*b}. It follows that $C_g \subseteq (n!)C_f^{n-1} \cdot C'_{a*b}$. Since $C'_{a*b} \subseteq (n+1)[C'_a C_b \cup C_a C'_b]$, we have $C_g \subseteq (n+1)!\,C_f^{n-1}[C'_a C_b \cup C_a C'_b]$. Finally, since $(ab)_d = (a*b)_d - (f*g)_d$ we have

$$(a \cdot b)_0 \in \{a_0 \cdot b_0\} + (n+2)C_f C_g \subseteq \{a_0 b_0\} + (n+2)!\,C_f^n[C'_a C_b \cup C_a C'_b],$$

and for $i = 1, \cdots, n$

$$(a \cdot b)_i \in (n+1)[C'_a C_b \cup C_a C'_b] + (n+2)!\,C_f^n[C'_a C_b \cup C_a C'_b]$$
$$\subseteq [(n+2)!\,C_f^n + n + 1]\ [C'_a C_b \cup C_a C'_b].$$

We have established the following lemma.

3.2. Lemma. *Let $a, b \in K$. Then, in the above notation, we have the following bounds for the coefficients of the product*

$$a \cdot b\colon (a \cdot b)_0 \in \{a_0 \cdot b_0\} + (n+2)!C_f^n[C'_a C_b \cup C_a C'_b]$$

and, for $i = 1, \cdots, n$,

$$(a \cdot b)_i \in [(n+2)!\ C_f^n + n + 1]\ [C'_a C_b \cup C_a C'_b].$$

If A and B are subsets of K we define $\sigma(A,B)$ by

$$\sigma(A,B) = \left\{ g = \sum^n g_i X^i ; g_0 \in A \text{ and } g_i \in B \text{ for } 0 < i \leqq n \right\}.$$

3.3. Lemma. *Let K be a simple algebraic extension of k. Let k be a locally bounded field and $\mathscr{U}$ a basis, for the neighborhood system of zero in k, consisting of bounded sets. Suppose U, V, M, and W are in $\mathscr{U}$ with $V, V+V+M \subseteq U$. Then there exists $B \in \mathscr{U}$ such that $\sigma(V,B)$ and $\sigma(V,B) \cdot \sigma(V,B) + \sigma(B,B)$ are contained in $\sigma(U,W)$.*

Proof. We will use the notation of the previous lemma. Clearly, the sets $(n+2)!\,C_f^n \cdot V$ and $[(n+2)!C_f^n + n + 1]V$ are bounded. Let us choose B and B' in $\mathscr{U}$ such that $B, B'+B' \subseteq M \cap W$ and B, $[(n+2)!\,C_f^n] \cup [((n+2)!C_f^n + n + 1)]V \cdot B \subseteq B'$. Then for $a, b \in \sigma(V,B)$ and $c \in \sigma(B,B)$ we have $C_a, C_b \subseteq V$ and $C_c, C'_a, C'_b \in B$. From Lemma 3.2 it follows that

$$(a \cdot b + c)_0 = (a \cdot b)_0 + c_0 \in \{a_0 b_0\} + (n+2)!\,C_f^n[C'_a C_b \cup C_a C'_b] + C_c$$
$$\subseteq V \cdot V + (n+2)!\,C_f^n V \cdot B + B \subseteq V \cdot V + B + B \subseteq V \cdot V + M \subseteq U,$$

and for $1 \leqq i \leqq n$,

$$(a \cdot b + c)_i = (a \cdot b)_i + c_i \in [(n+2)!C_f^n + n + 1][C'_a C_b \cup C_a C'_b] + C_c$$
$$\subseteq [(n+2)!C_f^n + n + 1]\,V \cdot B + B \subseteq B + B \subseteq W.$$

Thus we have $\sigma(v,B)$, $\sigma(v,B) \cdot \sigma(v,B) + \sigma(B,B) \subseteq \sigma(V,W)$ as desired.

3.4. THEOREM. *Let K be an algebraic extension of k of countable degree. If k admits a locally bounded field topology $\mathcal{T}$, then K admits a field topology $\mathcal{T}_K$ such that $\mathcal{T}_K | k \subseteq \mathcal{T}$.*

Proof. In view of Theorem 3.1, it is sufficient to assume that the degree of K over k is countably infinite. Let $\{k_n, n \in N\}$ be a sequence of intermediate fields such that $k_0 = k$, k_{n+1} is a proper simple algebraic extension of k_n, and $K = \bigcup k_n$. Let $\mathcal{T}_0 = \mathcal{T}$ and inductively assume that for each $n \in N$, $\mathcal{T}_{n+1}$ is the uniform topology on k_{n+1} inherited from $(k_n, \mathcal{T}_n)$. In view of Theorem 3.1 and Lemma 3.2, each $\mathcal{T}_n$ is a locally bounded field topology for k_n. For each $n \in N$ let $\mathcal{U}_n$ be the family of all $\mathcal{T}_n$-open subsets of k_n containing zero. For subsets A, B of k_n let $\sigma_n(A,B)$ have the obvious interpretation.

We define a family $\mathcal{U}$ of subsets of K by $U \in \mathcal{U}$ if and only if

(i) $U \cap k_n \in \mathcal{U}_n$ for every $n \in N$.

(ii) For each $n \in N$, there exists $W_n \in \mathcal{U}_n$ such that

$$\sigma_n(U \cap k_n, W_n) \subseteq U \cap k_{n+1}.$$

It is easily seen that $\mathcal{U}$ is nonvoid and closed under finite intersection. Thus to see that $\mathcal{U}$ defines a group topology for K it is sufficient to see that if $U \in \mathcal{U}$ is given then there exists $V \in \mathcal{U}$ such that $V - V \subseteq U$. Let $U \in \mathcal{U}$ be given. By hypothesis, for each $n \in N$, there exists $W_n \in \mathcal{U}_n$ such that

$$\sigma_n(U \cap k_n, W_n) \subseteq U \cap k_{n+1}.$$

Using the axiom of choice pick such a W_n for each $n \in N$. Applying the axiom of choice again, for each $n \in N$ pick W_n' in $\mathcal{U}_n$ such that $W_n' + W_n' \subseteq W_n$. Inductively, we define a sequence $\{V_n\}_{n \in N}$ satisfying (i) and (ii) and such that $V_n - V_n \subseteq U \cap k_n$, for each $n \in N$. To begin, choose $V_0 \in \mathcal{U}_0$ such that $V_0 - V_0 \subseteq U \cap k_0$. For each $n \in N$, set $V_{n+1} = \sigma_n(V_n, W_n')$. Clearly $V = \bigcup_{n \in N} V_n$ satisfies (i) and (ii). The obvious inductive argument shows that, for each $n \in N$, $V_n - V_n \subseteq U \cap k_n$. It follows that $V \in \mathcal{U}$ and $V - V \subseteq U$.

To see that multiplication is continuous at zero, let $U \in \mathcal{U}$ be given. We will construct a $V \in \mathcal{U}$ such that $V \cdot V \subseteq \mathcal{U}$.

Let $\mathcal{F}$ be the family of all finite sequences $\{V_n; n \leqq p, n \in N\}$, $p \in N$, satisfying the following three conditions:

(a) $V_n \in \mathcal{U}_n$ for each $n \in N$.

(b) For each $n \in N$, there exists $W_n \in \mathcal{U}_n$ such that

$$V_n, V_n \cdot V_n + W_n \subseteq U \cap k_n.$$

(c) If $0 < n \leqq p$, there exists $W_{n-1} \in \mathcal{U}_{n-1}$ such that

$$\sigma_{n-1}(V_{n-1}, W_{n-1}') = V_n.$$

Note that $\mathcal{F}$ is not void since there exists $V_0 = \mathcal{U}_0$ satisfying (a) and (b).

We define a partial order on $\mathcal{F}$ by $\{A_n\}_{n \leqq p} \leqq \{B_n\}_{n \leqq q}$ in case $p \leqq q$ and $A_n = B_n$ for $n \leqq p$. By the maximum principle there exists a maximal chain in $\mathcal{F}$, say $\mathcal{M}$. The maximal chain $\mathcal{M}$ must be infinite. For if not, $\mathcal{M}$ has a largest element, say $\{V_n\}_{n \leqq p}$. By Lemma 3.3, this sequence can be extended to a sequence $\{V_n\}_{n \leqq p+1}$. Adjoining this sequence to $\mathcal{M}$ yields a chain properly containing $\mathcal{M}$, contrary to the maximality of $\mathcal{M}$. Let $\{V_n\}_{n \in N} = \{V_n; V_n \in M$ for some $M \in \mathcal{M}\}$. Clearly $\{V_n\}$ satisfies (i), (ii), (a), (b), and (c). Consequently $V = \bigcup_{n \in N} V_n \in \mathcal{U}$ and $V \cdot V \subseteq U$.

To complete the proof that multiplication is continuous, it is sufficient to see that if $U \in \mathcal{U}$ and $h \in K$ are given, there exists $V \in \mathcal{U}$ such that $hV \subseteq U$. Thus suppose $U \in \mathcal{U}$ and $h \in K$ are given. Let p be the smallest integer such that $h \in k_n$. There exists $V_p \in \mathcal{U}_p$ such that $hV_p \subseteq U \cap k_p$. As above, for each $n \in N$, $n \geqq p$, use the axiom of choice to pick W_n and $W_n' \in \mathcal{U}_n$ such that $\sigma_n(U \cap k_n, W_n) \subseteq U \cap k_{n+1}$ and $hW_n' \subseteq W_n$. For $n < p$, set $V_n = V_p \cap k_n$ and inductively for $n > p$, set $V_n = \sigma_{n-1}(V_{n-1}, W_{n-1}')$. It is easily verified that $V = \bigcup_{n \in N} V_n \in \mathcal{U}$ and $hV \subseteq U$.

The proof that U defines a Hausdorff topology is similar. It is sufficient to see that if $0 \neq g \in K$, then there exists a $V \in \mathcal{U}$ such that $g \notin V$. Let p be the smallest integer n such that $g \in k_n$. Since $\mathcal{T}_p$ is Hausdorff, there exists $V_p \in \mathcal{U}_p$ such that $g \notin V_p$. For $n < p$, set $V_n = V_p \cap k_n$ and for $n > p$, set $V_n = \sigma_{n-1}(V_{n-1}, V_{n-1})$. It is clear that $V = \bigcup_{n \in N} V_n \in U$ and $g \notin V$.

In [2] it is shown that any ring topology on a field can be weakened to a field topology. Thus the theorem follows.

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