GENERATORS IN $l_1$ (I)

BY

D. J. NEWMAN

In an earlier paper (see [4]) the authors investigated the following problem: Let $f(z) = \sum C_n z^n$ where $\sum |C_n| < \infty$ (i.e. $f \in l_1$) and where $f(z)$ is schlicht in $|z| \leq 1$, does it follow that $f(z)$ generates $l_1$? The reader is referred to [4] for a complete explanation of this problem and all the terms involved. For our purposes it suffices to state the problem as follows: Given $\epsilon > 0$, and $f$ satisfying the above hypotheses, is it true that

(1) There exists a polynomial $P$ such that, writing $P(f(z)) = \sum a_n z^n$, we have

$$|a_1 - 1| + \sum_{n=2}^{\infty} |a_n| < \epsilon.$$ 

The previously cited paper treats in particular those $f(z)$ which map $|z| \leq 1$ onto a Jordan domain with rectifiable boundary. Thus $f(z)$ is of bounded variation in the sense of Hardy, namely $f'(z) \in H^1$, and such $f$ are automatically $l_1$ by Hardy's inequality (see [6]). Even in this case, [4] gives a rather incomplete answer when $f'(z)$ has a nontrivial inner factor in the sense of Beurling (see [1]). It is our purpose to supply the affirmative answer in all cases of an $f(z)$ of bounded variation. Thus our main result is as follows:

**Theorem.** If $f(z)$ is schlicht in $|z| \leq 1$ and of bounded variation, then $f$ generates $l_1$ (i.e. (1) holds).

In what follows we let $I(z)$ denote the inner factor of $f'(z)$ (we allow the case where $I(z)$ is trivial, i.e. a constant). We also denote $\| \sum b_n z^n \| = \sum |a_n|/(n + 1)$ and we write $A$ to represent a positive constant, not always the same. ($A$ may depend on $f(z)$ but is otherwise absolute.)

**Lemma 1.** $|I(z)| > A(1 - |z|)^2$ for $|z| < 1$.

**Proof.** This is proved in [4] but we repeat the proof for the sake of completeness. $f(z)$ is schlicht, and so, by the distortion theorem $|f'(z)| > A(1 - |z|)$, also

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(1) This paper is an addendum to a previous paper published in the Trans. Amer. Math. Soc. 107 (1964), 466–484 by D. Newman, J. T. Schwartz and H. S. Shapiro entitled On generators of the Banach algebras $l_1$ and $L_1(0, \infty)$.

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\[
f'(z) \in H^1
\]

and so

\[
\left| \frac{f'(z)}{I(z)} \right| < \frac{A}{1 - |z|}.
\]

The result now follows by division of these two inequalities.

**Lemma 2.**

\[
\|F(z)\| \leq A \left( \max_{|z| \leq r} |F(z)| + \sqrt{(1 - r) \sup_{|z| < 1} |F(z)|} \right)
\]

for any \( r, 0 < r < 1 \).

**Proof.** The estimate

\[
(2) \quad \|F(z)\| \leq A \sup_{|z| < 1} |F(z)|
\]

is an obvious one, resulting e.g. from Schwarz' inequality. Now, writing

\[
F(z) = \sum C_n z^n,
\]

we have

\[
\sum_{n \leq 1/(1-r)} \frac{|C_n|}{n + 1} \leq A \sum_{n \leq 1/(1-r)} \frac{|C_n|}{n + 1} r^n \leq A \|F(rz)\|
\]

so that, applying (2), we obtain

\[
(3) \quad \sum_{n \leq 1/(1-r)} \frac{|C_n|}{n + 1} \leq A \max_{|z| \leq r} |F(z)|.
\]

Next the Schwarz inequality gives

\[
\sum_{n > 1/(1-r)} \frac{|C_n|}{n + 1} \leq \sqrt{\left( \sum_{n > 1/(1-r)} |C_n|^2 \cdot \sum_{n > 1/(1-r)} \frac{1}{(n + 1)^2} \right)}
\]

\[
\leq A \sqrt{(1 - r) \sqrt{(\sum |C_n|^2)}}
\]

and since \( \sqrt{(\sum |C_n|^2)} \leq \sup_{|z| < 1} |F(z)| \) we obtain

\[
(4) \quad \sum_{n > 1/(1-r)} \frac{|C_n|}{n + 1} \leq A \sqrt{(1 - r) \sup_{|z| < 1} |F(z)|}
\]

adding (2) and (4) now yields the result.

**Corollary.** If \( (1 - |z|) |F(z)| \leq \delta \) and \( |F(z)| \leq M \) in \( |z| < 1 \) then

\[
\|F(z)\| \leq A M^{2/3} \delta^{1/3}.
\]

**Proof.** Choose \( r = 1 - (\delta/M)^{2/3} \) in Lemma 2.

**Lemma 3.** Let \( 0 < \delta < 1, \ \rho_i = 1 - \delta^{4i}, \ for \ i = 1, 2, \ldots, 8, \)

\[
g(z) = [I(\rho_1 z) I(\rho_2 z) \cdots I(\rho_8 z)]^{-1/8},
\]

then \( \|g(z) I(z) - 1\| < 4^{5/\delta} \).
Proof. For convenience we write \( U(z) = I^{1/8}(z) \). Then \( g(z)I(z) - 1 = \sum_{i=1}^{8} F_i(z) \), where

\[
F_i(z) = [U(\rho_1 z) \cdots U(\rho_i z)]^{-1} U^i(z) - [U(\rho_1 z) \cdots U(\rho_{i-1} z)]^{-1} U^{i-1}(z).
\]

We will now estimate each of the \( \|F_i(z)\| \) by our corollary to Lemma 2. To do so notice that

\[
|F_i(z)| \leq |U(\rho_1 z) \cdots U(\rho_i z)|^{-1} |U(z) - U(\rho_i z)|
\]

and since, by Lemma 1, \( |U(\rho_i z)| \geq A(1 - \rho_i)^{1/4} \), we obtain

\[
|F_i(z)| \leq A \left[ \sum_{(1 - \rho_1) \cdots (1 - \rho_i)} \right]^{-1/4} |U(z) - U(\rho_i z)|.
\]

In particular, then, since \( |U(z)| \leq 1 \), we have

\[
|F_i(z)| \leq A \left[ \sum_{(1 - \rho_1) \cdots (1 - \rho_i)} \right]^{-1/4}.
\]

Also, since \( |U(z)| \leq 1 \), it follows that \( |U'(z)| \leq 1/(1 - |z|) \) (see [2]), and so

\[
|(z) - U(\rho_i z)| = \left| z \int_{\rho_i z}^{z} U'(z) d\zeta \right| \leq \frac{1 - \rho_i}{1 - |z|}.
\]

Combining (6) and (8) gives

\[
(1 - |z|)|F_i(z)| \leq A \left[ \sum_{(1 - \rho_1) \cdots (1 - \rho_i)} \right]^{-1/4}(1 - \rho_i)^{3/4}.
\]

Estimates (7) and (9) now allow us to apply the corollary to Lemma 2. The conclusion is

\[
\|F_i(z)\| \leq A \left[ \sum_{(1 - \rho_1) \cdots (1 - \rho_{i-1})} \right]^{-1/4}(1 - \rho_i)^{1/2} = A^{3/4}.
\]

Since (10) holds for every \( i = 1, 2, \cdots, 8 \), our lemma follows.

Proof of Theorem. The remainder of the proof now follows the lines set down in [4], but again we will present it here for the sake of completeness.

Let \( g(z) \) be the function given by Lemma 4 (clearly \( g \in H^1 \)). Since \( f'(z)/I(z) \) is outer and \( H^1 \) there exists a polynomial \( R(z) \) such that \( R(f'/I) - g \) is small in the \( H^1 \) norm (see [3]). Thus

\[
\int \left| \frac{Rf'}{I-g} \right| < \delta
\]

or, equivalently,

\[
\int \left| Rf' - gI \right| < \delta.
\]

By the theorem of Carathéodory-Walsh (see [5]), however, there is a polynomial in \( f \), which we can denote as \( P'(f) \), such that, uniformly,

\[
\left| P'(f) - R \right| < \delta.
\]
Thus, since $f' \in H^1$,

$$
(13) \quad \int |P'(f)f' - Rf'| \, < A\delta.
$$

Combining (11) and (13) gives

$$
(14) \quad \int |P'(f)f' - gI| \, < A\delta.
$$

Hardy's inequality (see [6]), however, states that $\|h\| \leq A \int |h|$ and so we may conclude from (14) that

$$
(15) \quad \|P'(f)f' - gI\| < A\delta.
$$

From (15) and Lemma 3 it follows that

$$
(16) \quad \|P'(f)f' - 1\| < A^3/\delta,
$$
or,

$$
(17) \quad \left\| \frac{d}{dz}(P(f(z)) - z) \right\| < A^{3/\delta} < \varepsilon
$$

by choosing $\delta$ small enough. The statement (17), however, is exactly the statement (1) and the proof is complete.

REFERENCES


Yeshiva University,
New York, New York