

GENERATORS IN l_1 (1)

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In an earlier paper (see [4]) the authors investigated the following problem: Let $f(z) = \sum C_n z^n$ where $\sum |C_n| < \infty$ (i.e. $f \in l_1$) and where $f(z)$ is schlicht in $|z| \leq 1$, does it follow that $f(z)$ generates l_1 ? The reader is referred to [4] for a complete explanation of this problem and all the terms involved. For our purposes it suffices to state the problem as follows: Given $\varepsilon > 0$, and f satisfying the above hypotheses, is it true that

(1) There exists a polynomial P such that, writing $P(f(z)) = \sum a_n z^n$, we have

$$|a_1 - 1| + \sum_{n=2}^{\infty} |a_n| < \varepsilon.$$

The previously cited paper treats in particular those $f(z)$ which map $|z| \leq 1$ onto a Jordan domain with rectifiable boundary. Thus $f(z)$ is of bounded variation in the sense of Hardy, namely $f'(z) \in H^1$, and such f are automatically l_1 by Hardy's inequality (see [6]). Even in this case, [4] gives a rather incomplete answer when $f'(z)$ has a nontrivial inner factor in the sense of Beurling (see [1]). It is our purpose to supply the affirmative answer in *all* cases of an $f(z)$ of bounded variation. Thus our main result is as follows:

THEOREM. *If $f(z)$ is schlicht in $|z| \leq 1$ and of bounded variation, then f generates l_1 (i.e. (1) holds).*

In what follows we let $I(z)$ denote the inner factor of $f'(z)$ (we allow the case where $I(z)$ is trivial, i.e. a constant). We also denote $\| \sum b_n z^n \| = \sum |b_n| / (n+1)$ and we write A to represent a positive constant, not always the same. (A may depend on $f(z)$ but is otherwise absolute.)

LEMMA 1. $|I(z)| > A(1 - |z|)^2$ for $|z| < 1$.

Proof. This is proved in [4] but we repeat the proof for the sake of completeness. $f(z)$ is schlicht, and so, by the distortion theorem $|f'(z)| > A(1 - |z|)$, also

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$$\frac{f'(z)}{I(z)} \in H^1$$

and so

$$\left| \frac{f'(z)}{I(z)} \right| < \frac{A}{1 - |z|}.$$

The result now follows by division of these two inequalities.

LEMMA 2.

$$\|F(z)\| \leq A \left(\text{Max}_{|z| \leq r} |F(z)| + \sqrt{(1-r)} \text{Sup}_{|z| < 1} |F(z)| \right)$$

for any $r, 0 < r < 1$.

Proof. The estimate

$$(2) \quad \|F(z)\| \leq A \text{Sup}_{|z| < 1} |F(z)|$$

is an obvious one, resulting e.g. from Schwarz' inequality. Now, writing $F(z) = \sum C_n z^n$, we have

$$\sum_{n \leq 1/(1-r)} \frac{|C_n|}{n+1} \leq A \sum_{n \leq 1/(1-r)} \frac{|C_n|}{n+1} r^n \leq A \|F(rz)\|$$

so that, applying (2), we obtain

$$(3) \quad \sum_{n \leq 1/(1-r)} \frac{|C_n|}{n+1} \leq A \text{Max}_{|z| \leq r} |F(z)|.$$

Next the Schwarz inequality gives

$$\begin{aligned} \sum_{n > 1/(1-r)} \frac{|C_n|}{n+1} &\leq \sqrt{\left(\sum_{n > 1/(1-r)} |C_n|^2 \cdot \sum_{n > 1/(1-r)} \frac{1}{(n+1)^2} \right)} \\ &\leq A \sqrt{(1-r)} \sqrt{(\sum |C_n|^2)} \end{aligned}$$

and since $\sqrt{(\sum |C_n|^2)} \leq \text{Sup}_{|z| < 1} |F(z)|$ we obtain

$$(4) \quad \sum_{n > 1/(1-r)} \frac{|C_n|}{n+1} \leq A \sqrt{(1-r)} \text{Sup}_{|z| < 1} |F(z)|$$

adding (2) and (4) now yields the result.

COROLLARY. If $(1 - |z|) |F(z)| \leq \delta$ and $|F(z)| \leq M$ in $|z| < 1$ then $\|F(z)\| \leq AM^{2/3} \delta^{1/3}$.

Proof. Choose $r = 1 - (\delta/M)^{2/3}$ in Lemma 2.

LEMMA 3. Let $0 < \delta < 1$, $\rho_i = 1 - \delta^{4^i}$, for $i = 1, 2, \dots, 8$, $g(z) = [I(\rho_1 z) I(\rho_2 z) \dots I(\rho_8 z)]^{-1/8}$, then $\|g(z)I(z) - 1\| < A\sqrt[3]{\delta}$.

Proof. For convenience we write $U(z) = I^{1/8}(z)$. Then $g(z)I(z) - 1 = \sum_{i=1}^8 F_i(z)$, where

$$F_i(z) = [U(\rho_1 z) \cdots U(\rho_i z)]^{-1} U^i(z) - [U(\rho_1 z) \cdots U(\rho_{i-1} z)]^{-1} U^{i-1}(z).$$

We will now estimate each of the $\|F_i(z)\|$ by our corollary to Lemma 2. To do so notice that

$$(5) \quad |F_i(z)| \leq |U(\rho_1 z) \cdots U(\rho_i z)|^{-1} |U(z) - U(\rho_i z)|$$

and since, by Lemma 1, $|U(\rho_i z)| \geq A(1 - \rho_i)^{1/4}$, we obtain

$$(6) \quad |F_i(z)| \leq A[(1 - \rho_1) \cdots (1 - \rho_i)]^{-1/4} |U(z) - U(\rho_i z)|.$$

In particular, then, since $|U(z)| \leq 1$, we have

$$(7) \quad |F_i(z)| \leq A[(1 - \rho_1) \cdots (1 - \rho_i)]^{-1/4}.$$

Also, since $|U(z)| \leq 1$, it follows that $|U'(z)| \leq 1/(1 - |z|)$ (see [2]), and so

$$(8) \quad |U(z) - U(\rho_i z)| = \left| \int_{\rho_i z}^z U'(\zeta) d\zeta \right| \leq \frac{1 - \rho_i}{1 - |z|}.$$

Combining (6) and (8) gives

$$(9) \quad (1 - |z|) |F_i(z)| \leq A[(1 - \rho_1) \cdots (1 - \rho_{i-1})]^{-1/4} (1 - \rho_i)^{3/4}.$$

Estimates (7) and (9) now allow us to apply the corollary to Lemma 2. The conclusion is

$$(10) \quad \|F_i(z)\| \leq A[(1 - \rho_1) \cdots (1 - \rho_{i-1})]^{-1/4} (1 - \rho_i)^{1/12} = A \sqrt[3]{\delta}.$$

Since (10) holds for every $i = 1, 2, \dots, 8$, our lemma follows.

Proof of Theorem. The remainder of the proof now follows the lines set down in [4], but again we will present it here for the sake of completeness.

Let $g(z)$ be the function given by Lemma 4 (clearly $g \in H^1$). Since $f'(z)/I(z)$ is outer and H^1 there exists a polynomial $R(z)$ such that $R(f'/I) - g$ is small in the H^1 norm (see [3]). Thus

$$\int \left| \frac{Rf'}{I-g} \right| < \delta$$

or, equivalently,

$$(11) \quad \int |Rf' - gI| < \delta.$$

By the theorem of Carathéodory-Walsh (see [5]), however, there is a polynomial in f , which we can denote as $P'(f)$, such that, uniformly,

$$(12) \quad |P'(f) - R| < \delta.$$

Thus, since $f' \in H^1$,

$$(13) \quad \int |P'(f)f' - Rf'| < A\delta.$$

Combining (11) and (13) gives

$$(14) \quad \int |P'(f)f' - gI| < A\delta.$$

Hardy's inequality (see [6]), however, states that $\|h\| \leq A \int |h|$ and so we may conclude from (14) that

$$(15) \quad \|P'(f)f' - gI\| < A\delta.$$

From (15) and Lemma 3 it follows that

$$(16) \quad \|P'(f)f' - 1\| < A^3\sqrt{\delta},$$

or,

$$(17) \quad \left\| \frac{d}{dz}(P(f(z)) - z) \right\| < A^3\sqrt{\delta} < \varepsilon$$

by choosing δ small enough. The statement (17), however, is exactly the statement (1) and the proof is complete.

REFERENCES

1. A. Beurling, *Linear transformations in Hilbert space*, Acta Math. **81** (1945), 239–255.
2. C. Carathéodory, *Funktionentheorie*, Band II, Verlag Birkhäuser, Basel, 1950, p. 17.
3. K. de Leeuw and W. Rudin, *Extreme points and extremum problems in H_1* , Pacific J. Math. **8** (1958), 467–485.
4. D. J. Newman, J. T. Schwartz and H. S. Shapiro, *Generators of the Banach algebras I_1 and $L_1(0, \infty)$* , Trans. Amer. Math. Soc. **107** (1964), 466–484.
5. J. L. Walsh, *Interpolation and approximation in the complex domain*, Amer. Math. Soc. Colloq. Publ. Vol. 20, Amer. Math. Soc., Providence, R. I., 1956.
6. A. Zygmund, *Trigonometrical series*, 2nd ed., Vol. I, Cambridge Univ. Press, Cambridge, 1959, p. 286.

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