A DIFFERENCE PROPERTY FOR POLYNOMIALS AND EXPONENTIAL POLYNOMIALS ON ABELIAN LOCALLY COMPACT GROUPS (1)

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1. Introduction. Let \( f \) be a complex-valued function on \( E^m \), Euclidean \( m \)-space, which has the property that for each \( h \in E^m \), the function \( \Delta_h f: \Delta_h f(x) = f(x + h) - f(x) \) is continuous (or is a polynomial, or an exponential polynomial). Then \( f \) itself need not be continuous (or a polynomial, or an exponential polynomial), for there exist nonmeasurable additive functions on \( E^m \), that is, nonmeasurable solutions \( \Gamma \) of the functional equation \( \Gamma(x + y) = \Gamma(x) + \Gamma(y) \). However, de Bruijn [1], [2] (for \( E^1 \)) and Kemperman [7], [8] (for \( E^m \)) showed that, among many others, the classes of continuous functions, polynomials, and certain classes of trigonometric and exponential polynomials have the property that if \( \Delta_h f \) is in the class for each \( h \in E^m \), then there exists an additive function \( \Gamma \) on \( E^m \) such that \( f - \Gamma \) is in the class.

Let \( G \) be an abelian topological group, and let \( \Omega \) be a class of complex-valued functions on \( G \) which contains the constant functions, and such that \( f, g \in \Omega \) implies \( f - g \in \Omega \) and \( f_h \in \Omega \) for each \( h \in G \), where \( f_h(x) = f(x + h) \). The class \( \Omega \) is said to have the difference property if the following implication holds: let \( f \) be a complex-valued function on \( G \) such that \( \Delta_h f \in \Omega \) for each \( h \in G \). Then there is an additive function \( \Gamma \) on \( G \) such that \( f - \Gamma \in \Omega \).

Except where the contrary is explicitly stated, \( G \) will denote an abelian locally compact group. The product of \( m \) copies of the reals will be denoted by \( E^m \), and the group of integers by \( C \). All functions considered are complex-valued. It is known [3] that the class of continuous functions on \( G \) has the difference property. The principal results of this paper are Theorems 1 and 2, which give necessary and sufficient conditions on \( G \) in order that the class of polynomials on \( G \) (as defined, for instance, in [6]) and the class of exponential polynomials on \( G \) have the difference property.

2. The difference property for polynomials. A function \( P \) on \( G \) is a polynomial of degree \( N \) \( (N < \infty) \), provided

\[ (P_1) \text{ for each } (a, b) \in G \times G, \text{ the function } \]

\[ P_{ab}: P_{ab}(\lambda) = P(a + \lambda b) \quad (\lambda \in C) \]

is a polynomial on \( C \).

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(P2) if \( N_{a,b} \) denotes the degree of \( P_{a,b} \), then \( N = \max \{ N_{a,b}; (a,b) \in G \times G \} \), and

(P3) \( P \) is continuous on \( G \).

An element \( b \in G \) is said to be compact if the closure of the subgroup generated by \( b \) is compact. The set \( B \) of all compact elements is a closed subgroup \([11]\). Since \( G = E^m + G' \), where \( G' \) contains a compact open subgroup \([12]\), it follows that \( G/B = E^m + G_1 \), where \( G_1 \) is discrete. If a function \( P \) has properties (P1) and (P3) on \( G \), if \( a \in G \) and \( b \in B \), then the set of polynomial values \( P_{a,b}(\lambda) : \lambda \in C \) is bounded, since \( \{ a + \lambda b : \lambda \in C \} \) has compact closure. Thus \( P_{a,b}(\lambda) = P(a) \), so that \( P \) is constant on cosets of \( B \), and is essentially a function on \( G/B \).

A set \( H \subset G \) is a Hamel basis for \( G \) provided that for no finite subset \( \{ b_0, b_1, \ldots, b_k \} \) of \( H \) do there exist integers \( N \neq 0, n_1, \ldots, n_k \) such that \( N b_0 = n_1 b_1 + \cdots + n_k b_k \), and provided that \( H \) is maximal with respect to this property. It follows from the Hausdorff maximal principle that every abelian group has a (possibly empty) Hamel basis. If \( x \in G \), then there exist uniquely determined elements \( b_1, \ldots, b_k \) in \( H \) and integers \( N \neq 0, n_1, \ldots, n_k \) such that

\[
(2.1) \quad Nx = n_1 b_1 + \cdots + n_k b_k.
\]

If (2.1) holds, we shall write

\[
x \sim (n_1/N)b_1 + \cdots + (n_k/N)b_k = r_1(x)b_1 + \cdots + r_k(x)b_k.
\]

In this way there is associated with each \( b \in H \) a rational-valued function \( r_\alpha \) on \( G \), and \( r_\alpha \) is easily seen to be additive.

**Theorem 1.** Let \( G \) be an abelian locally compact group, and let \( B \) be the group of compact elements of \( G \). A necessary and sufficient condition in order that the class of polynomials on \( G \) have the difference property is that \( G/B = E^m + G_1 \), where \( G_1 \) has a finite Hamel basis.

The following example shows the necessity of the condition.

**Example 1.** Let \( G/B = G_1 + E^m \) where \( G_1 \) is discrete and has an infinite Hamel basis. Then there exists a continuous function on \( G \) which is not a polynomial, but each of whose differences is a polynomial. It suffices to show that such a function \( f \) exists on \( G_1 \), for then the function

\[
f_1: f_1(x) = f(P_2P_1x),
\]

where \( P_1 \) and \( P_2 \) are the natural mappings from \( G \) to \( G/B \) and from \( G/B \) to \( G_1 \) respectively, is an extension of \( f \) which has the required properties on \( G \). Let \( b_1, b_2, \ldots \) be a countably infinite subset of the Hamel basis of \( G_1 \), let \( r_1, r_2, \ldots \) be the corresponding rational-valued functions, and let

\[
f(x) = \sum_{i=1}^{\infty} (r_\alpha(x))' \quad (x \in G_1).
\]
If $h$ is an arbitrary fixed element of $G_1$, there is an integer $N = N(h)$ such that $\nu > N$ implies $r_\nu(h) = 0$. Thus, from the additivity of the $r_\nu$,
\[
f(x + h) - f(x) = \sum_{\nu=1}^{N} \left[ (r_\nu(x) + r_\nu(h)) - (r_\nu(x)) \right],
\]
a polynomial.

It may be noticed that the function $f$ in Example 1 satisfies (P1). In fact, the following is true: if $G$ is an arbitrary abelian group, and if $P$ is a function on $G$ such that, for each $(a, b) \in G \times G$, the function given by $\Delta \lambda P(a + \lambda b)$ ($\lambda \in C$) is a polynomial on $C$, then $P$ has property (P1). For, given $(a, b)$, it is easy to construct a polynomial $Q_{a, b}$ on $C$ such that
\[
Q_{a, b}(\lambda + 1) - Q_{a, b}(\lambda) = \Delta \lambda P(a + \lambda b) \quad (\lambda \in C),
\]
\[
Q_{a, b}(0) = P(a).
\]
Then $P(a + \lambda b) = Q_{a, b}(\lambda)$.

Example 1 shows that, in general, the degree $N_{a, b}$ of $P_{a, b}$ is not a bounded function on $G \times G$. Even on $C + C$, there exists a function satisfying (P1) with $N_{a, b}$ unbounded [4].

**Proof of Theorem 1. Sufficiency.** Let $f$ be a function on $G$ such that $\Delta \alpha$ is a polynomial for each $h \in G$. Then $f$ may be taken to be a continuous function, since otherwise there is an additive function $\Gamma_1$ on $G$ such that $f - \Gamma_1$ is continuous, and the differences of $f - \Gamma_1$ will be polynomials. Also, $f$ has property (P1) and therefore can (and will) be considered simply as a function on $G/B$.

Clearly, $G/B$ contains a dense subgroup $G'$ which has a finite Hamel basis, viz., $G' = R^n + G_1$, where $R$ denotes the subgroup of $E$ consisting of the rational numbers. If $H = \{h_1, \ldots, h_p\}$ is a Hamel basis for $G'$, and if $G''$ is the group generated by $H$, then $G''$ is isomorphic to $C^p$. The isomorphism $\phi$ can be chosen so that $\phi h_i = \epsilon_i$ has $\delta_{ij}$ as its $j$th coordinate. Let polynomials $f_i$ ($i = 1, \ldots, p$) be defined on $C^p$ by
\[
f_i(\phi(x)) = \Delta_{h_i}f(x) \quad (x \in G'').
\]
Then, clearly,
\[
(2.2) \quad \Delta_{h_i}f_j = \Delta_{h_i}f_i \quad (i, j = 1, \ldots, p).
\]
But (2.2) implies that there exists a polynomial $g^*$ on $C^p$ such that $g^*(x + \epsilon_i) - g^*(x) = f_i(x)$ ($i = 1, \ldots, p$). Explicitly, $g^* = g_{p, k}$, where $g_k$ is given on $C^k$ ($k = 1, 2, \ldots$) by
\[
g_0 \equiv 0,
\]
\[
g_k(n_1, \ldots, n_{k-1}, n_k) = g_{k-1}(n_1, \ldots, n_{k-1})
\]
\[
+ \sum_{m=0}^{M} c_{m,k}(n_1, \ldots, n_{k-1}) (B_{m+1}(n_k) - B_{m+1}) / (m+1).
\]
Here, the \( c_{m, k} \) are the polynomials obtained from

\[
f_k(n_1, \ldots, n_{k-1}, n_k, 0, \ldots, 0) = \sum_{m=0}^{M} c_{m,k}(n_1, \ldots, n_{k-1})(n_k)^m,
\]

while \( B_m(x) \) and \( B_n \) are the \( m \)th Bernoulli polynomial and number respectively. This assertion can be proved by induction on \( p \). (Note that the well-known identity

\[
(B_{m+1}(x + 1) - B_{m+1}(x))/(m + 1) = x^m
\]

[10] implies that the sum (2.3) is

\[
(2.4) \quad \left( -\sum_{j=n_k}^{n_k-1} + \sum_{j=0}^{n_k-1} \right) f_k(n_1, \ldots, n_{k-1}, j),
\]

the first (respectively, second) sum in (2.4) being empty if \( n_k \geq 0 \) (resp., if \( n_k \leq 0 \)).

Also, \( g^* \) has a unique extension as a polynomial \( g^{**} \) on all of \( \mathbb{R}^p \). Let \( g \) denote the function on \( G' \) given as follows: if

\[
x \sim \sum_{i=1}^{p} r_i h_i, \quad \text{then } g(x) = g^{**}(r_1, \ldots, r_p).
\]

Clearly, \( g \) satisfies (P1) and is of bounded degree. Since \( \Delta_h(f - g)(x) = 0 \) for each \( x \in G'' \) (i = 1, \ldots, p), it follows that \( f - g \) is constant on \( G'' \). If \( x \) is any point in \( G' \), then there exists a positive integer \( K \) such that \( \mu K x \in G'' \) for all \( \mu \in C \). Then the polynomial in \( \lambda \) given by \( f(\lambda x) - g(\lambda x) \), being constant for \( \lambda = \mu K \) (\( \mu \in C \)), is a constant on \( C \), so that \( f \) is a polynomial on \( G' \). If \( N - 1 \) is its degree, then \( \Delta^N f(a) \) vanishes for all \( (a, b) \in G' \times G' \), and therefore, by continuity, for all \( (a, b) \in G \times G \). But this implies that \( f \) is a polynomial of degree \( N - 1 \) on \( G \):

\[
f(a + \lambda b) = \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{\mu} (-1)^{\nu} \binom{\mu}{\nu} f(a + \nu b) \quad (\lambda, \mu \in C; a, b \in G).
\]

3. The difference property for exponential polynomials. A function \( z \) on \( G \) is a generalized character if \( z \) is a continuous homomorphism from \( G \) to the multiplicative group of nonzero complex numbers. A function \( e \) on \( G \) is an exponential polynomial if

\[
e = \sum_{i=1}^{n} P_i z_i,
\]

where each \( P_i \) is a polynomial and each \( z_i \) is a generalized character. If each \( P_i \) is a constant, and each \( z_i \) is an ordinary character, then \( e \) is said to be a trigonometric polynomial.
Theorem 2. Let $G$ be an abelian locally compact group. A necessary and sufficient condition in order that the class of exponential polynomials on $G$ have the difference property is that $G$ be compactly generated.

Let $f$ be a function on $G$ such that
\begin{equation}
\Delta_h f = \sum_{\alpha \in A} P^\alpha_h z_\alpha \quad (h \in G)
\end{equation}
where $\{z_\alpha : \alpha \in A\}$ is the set of all generalized characters on $G$, each $P^\alpha_h$ is a polynomial on $G$, and, for each fixed $h \in G$,
\begin{equation}
P^\alpha_h = 0 \quad \text{for } \alpha \neq \alpha_1(h), \ldots, \alpha_k(h).
\end{equation}
Since the class of continuous functions on $G$ has the difference property, $f$ may be taken to be continuous.

Distinct generalized characters are linearly independent over the ring of polynomials on $G$ (Lemma 3.1, below). Hence, $f$ will be an exponential polynomial if and only if there exist polynomials $\{Q^\alpha : \alpha \in A\}$ such that
\begin{equation}
\Delta_h (Q^\alpha z_\alpha) = P^\alpha_h z_\alpha \quad (h \in G, \alpha \in A),
\end{equation}
and
\begin{equation}
Q^\alpha = 0 \quad \text{for } \alpha \neq \alpha_1, \ldots, \alpha_k,
\end{equation}
for then it is clear that
\[ f = \sum_{\alpha \in A} Q^\alpha z_\alpha. \]

The proof of the sufficiency portion of Theorem 3 consists in constructing polynomials $Q^\alpha$ satisfying (3.3), and showing that (3.4) also holds. If $G$ is not compactly generated, however, then it is not true that (3.2) and (3.3) imply (3.4), even when $|\Delta_h f : h \in G|$ are all trigonometric polynomials. This is shown in Theorem 3, from which the necessity of the condition in Theorem 2 will follow.

Lemma 3.1. Let $G$ be an arbitrary group, let $z_1, \ldots, z_n$ be distinct homomorphisms of $G$ into the multiplicative group of nonzero complex numbers, and let $P_1, \ldots, P_n$ be complex functions satisfying (P1). If
\begin{equation}
P_1 z_1 + \cdots + P_n z_n = 0,
\end{equation}
then $P_1 = \cdots = P_n = 0$.

Proof. First, consider the special case $G = C$. Since $z_1, \ldots, z_n$ are distinct, the complex numbers $z_1(1), \ldots, z_n(1)$ are necessarily distinct. If not all $P_j$ are identically zero, then, reordering if necessary, it may be assumed that for some integer $p$, $1 \leq p \leq n$, ...
$z_j(1)/z_1(1) = e^{i\theta} \neq 1 \quad (\beta_j \text{ real, } 2 \leq j \leq p),$

$|z_1(1)| > |z_j(1)| \quad (p + 1 \leq j \leq n),$

$P_j(\lambda) = c_j \lambda^m + O(\lambda^{m-1}) \quad \text{as } \lambda \to \infty \quad (1 \leq j \leq p; c_1 \neq 0).$

It follows upon division of the terms of (3.5) by $\lambda^n z_1(\lambda)$ that

\begin{equation}
(3.6) \quad c_1 = -\sum_{j=2}^{p} c_j e^{i\theta_j} + O(\lambda^{-1}) \quad \text{as } \lambda \to \infty.
\end{equation}

Thus, taking $\lambda = 1, 2, \ldots, N$ in (3.6), adding, and dividing by $N$, it is seen that

\begin{equation}
(3.7) \quad c_1 = -\left(1/N\right) \sum_{j=2}^{p} c_j e^{i\theta_j} \frac{(e^{i\theta_j N} - 1)}{e^{i\theta_j} - 1} + O(N^{-1} \log N).
\end{equation}

Letting $N$ approach infinity in (3.7), it follows that $c_1 = 0$, a contradiction.

In the general case, we prove by induction on $n$ that $P_1(x) = 0$. This is obvious for $n = 1$, since $z_1$ is never zero. Now let $n > 1$, and suppose the result holds for all $p < n$. Choose $x_0 \in G$ such that $z_1(x_0) \neq z_1(x_0)$; this is possible since the $z$'s are distinct. Suppose that $z_1(x_0) = z_2(x_0) = \ldots = z_p(x_0)$ for some integer $p$, $1 \leq p < n$, while $z_j(x_0) \neq z_1(x_0)$ for $p < j \leq n$. Then

\begin{equation*}
\sum_{j=1}^{n} P_j(x + \lambda x_0)z_j(x + \lambda x_0) = \left[ \sum_{j=1}^{p} P_j(x + \lambda x_0)z_j(x) \right] z_1(\lambda x_0)
\end{equation*}

\begin{equation*}
+ \sum_{j=p+1}^{n} P_j(x + \lambda x_0)z_j(x + \lambda x_0).
\end{equation*}

For each fixed $x$, this expression can be considered as an exponential polynomial on $C$, and $z_1(\lambda) = z_1(\lambda x_0)$ is distinct from the other generalized characters. Its coefficient is therefore zero for each $\lambda \in C$. Hence, for each $x \in G$

\begin{equation*}
\sum_{j=1}^{p} P_j(x)z_j(x) = 0,
\end{equation*}

so that $P_1(x) \equiv 0$, by the induction assumption.

**Lemma 3.2.** Let $G$ be an abelian topological group, let $z \neq 1$ be a generalized character on $G$, and let $\{P_h : h \in G\}$ be a collection of polynomials on $G$ such that

\begin{equation}
P_h(x + h')z(h') - P_h(x) = P_h(x + h)z(h) - P_h(x),
\end{equation}

for all $h, h', x \in G$. Then there exists a polynomial $Q$ on $G$ such that

\begin{equation}
\Delta_h(Qz) = P_hz \quad (h \in G).
\end{equation}

**Proof.** Let $h \in G$ be such that $|z(h)| < 1$, or, if there is no $h$ with this property, i.e., if $|z| \equiv 1$, let $z(h) \neq 1$. Consider the expression
\[ \lim_{r \to 1^-} \left\{ -\sum_{n=0}^{\infty} (z(h)r)^n P_n(x + nh) \right\} \quad (x \in G). \]

If \(|z(h)| < 1\), then the \(r\) in the summation may be replaced by 1, and the \(\lim\) omitted. Since \(P_n\) is a polynomial,

\[ P_n(x + nh) = \sum_{k=0}^{N} c_k(x)n^k \quad (x \in G), \]

where each \(c_k(x)\) is a polynomial on \(G\), obtainable explicitly from setting \(n = 0, 1, \ldots, N\) in (3.11) and solving by Cramer's rule. The sum in (3.10) is given for \(0 < r < 1\) by

\[ -\sum_{k=0}^{N} c_k(x) \sum_{n=0}^{\infty} n^k(z(h)r)^n = -\sum_{k=0}^{N} c_k(x) \left\{ \frac{dy}{d(y)} \right\}^k \left( \frac{1}{1-y} \right) \bigg|_{y = r(z(h))}. \]

Hence the limit in (3.10) exists and yields a polynomial on \(G\); let this polynomial be denoted by \(Q\). Let \(h'\) and \(x\) be arbitrary elements of \(G\). Then

\[ Q(x + h')z(h') - Q(x) \]

(3.12)

\[ = \lim_{r \to 1^-} \left\{ -\sum_{n=0}^{\infty} (z(h)r)^n [P_n(x + h' + nh)z(h') - P_n(x + nh)] \right\}. \]

From (3.8), the right-hand side of (3.12) is

\[ \lim_{r \to 1^-} \left\{ -\sum_{n=0}^{\infty} (z(h)r)^n [P_n(x + (n+1)h)z(h) - P_n(x + nh)] \right\} \]

(3.13)

\[ = P_n(x) - \lim_{r \to 1^-} (1 - r) \sum_{n=1}^{\infty} P_n(x + nh)z(h)^n r^{n-1} = P_n(x), \]

so that (3.9) follows from (3.12) and (3.13).

**Proof of Theorem 2.** Sufficiency. Let \(f\) be a function (which may and will be assumed to be continuous) such that (3.1) and (3.2) hold. From (3.1), Lemma 3.1, and the identity

\[ \Delta_h \Delta_z f = \Delta_z \Delta_h f, \]

it follows that (3.8) holds for each generalized character \(z\). For each \(z\) except \(z_0 = 1\), it follows from Lemma 3.2 that there exists a polynomial \(Q^z\) satisfying (3.9); in particular, if \(P_n^z = 0\) for all \(h\), then \(Q^z = 0\). But only finitely many \(Q^z\) can be nonzero. For suppose that \(Q^z \neq 0\) \((i = 1, 2, \ldots)\), with \(P_n^z\) and \(z_i\) the corresponding polynomials and generalized characters. For each \(h \in G\), (3.2) shows that there exists an integer \(i(h)\) such that \(P_n^z = 0\) for \(i \geq i(h)\). Hence, from (3.3),

\[ Q^z(x + h)z_i(h) - Q^z(x) = 0 \quad (i \geq i(h), x \in G), \]
whence it follows that \( z_i(h) = 1 \) for \( i \geq i(h) \). Let
\[
H_j = \{ h : h \in G, z_i(h) = 1 \text{ for all } i \geq j \}.
\]

\( H_j \) is a closed subgroup of \( G \), \( H_{j+1} \subset H_j \) and \( G = \bigcup H_n \), so that at least one \( H_j \) is of positive Haar measure, and thus open \([5]\). Therefore \( H_k \) is open for all \( k \geq j \). Let \( A \) be a compact neighborhood of 0 which generates \( G \). Then \( \bigcup \{ H_k : k \geq j \} \) covers \( G \), so that \( A \) is covered by some \( H_N \), whence \( H_N = G \). Therefore \( z_i = 1 \) for all \( i \geq N \), contradicting the distinctness of the \( z_i \). Since \( Q^a = 0 \) for all but finitely many \( a \), the function given by \( g = \sum Q^a z_a \) (the summation taken for all \( a \) such that \( z_a \neq 1 \)) is an exponential polynomial, and, for each \( h, \Delta_h(f - g) \) is clearly a polynomial on \( G \). But the function \( f - g \) is continuous, and the class of polynomials on \( G \) has the difference property from Theorem 1, since \( G \) is compactly generated only if \( G_i \) is finitely generated. Hence, \( f - g \) is a polynomial on \( G \).

In the proof just given, use was made of the fact that a compactly generated group \( G \) is not the countable union of a strictly increasing sequence of closed subgroups. Conversely,

**Lemma 3.3.** If the locally compact abelian group \( G \) is not compactly generated, there is a sequence \( \{ H_j \} \) of closed subgroups of \( G \), such that \( H_j \subset H_{j+1} \) (strictly) and \( \bigcup H_j = G \).

**Proof.** There is a compact subgroup \( G' \) of \( G \) such that \( G/G' = E^p + G_2 \), with \( G_2 \) discrete. Since \( G \) is not compactly generated, it follows that \( G_2 \) is not finitely generated. It is known \([9]\) that
\[
G_2 = \bigcup_{n=1}^\infty S_n,
\]
where each \( S_n \) is a direct sum of cyclic groups, and \( S_n \subset S_{n+1} \). If the inclusion is proper for infinitely many \( n \), the choice of the \( H_j \) is clear, and the lemma follows. Otherwise, \( G_2 \) is itself a direct sum of infinitely many cyclic groups:
\[
G_2 = \sum_a A_a.
\]

Let \( \{ A_{a1}, A_{a2}, \ldots \} \) be a countably infinite subset of \( \{ A_a \} \), and let
\[
H_j = E^p + \sum_a A_a : \alpha \neq \alpha_{j+1}, \alpha_{j+2}, \ldots \}.
\]

Then \( H_j \subset H_{j+1} \) properly, and their union is \( G \).

**Theorem 3.** Let \( G \) be an abelian locally compact group. The class of trigonometric polynomials on \( G \) has the difference property if and only if \( G \) is compactly generated.

**Proof.** The sufficiency is clearly a corollary of the sufficiency proof of Theorem 2. If \( G \) is not compactly generated, let \( \{ H_j \} \) be the sequence given
by Lemma 3.3, and for each $j$ let $z_j$ be a character identically 1 on $H_j$ but
not identically 1 on $G$; such characters exist [11]. Let $\sum a_j$ be a convergent
infinite series of positive numbers, and let $f$ be defined by

\begin{equation}
(3.14)\quad f = \sum_{j=1}^{n} a_j z_j.
\end{equation}

If $h \in G$ is given, there exists an integer $k = k(h)$ such that $h \in H_{k+1}$. Then

$$f(x + h) - f(x) = \sum_{j=1}^{k} a_j (z_j(h) - 1) z_j(x),$$

a trigonometric polynomial.

Suppose now that $f$, given by (3.14) is also given by

\begin{equation}
(3.15)\quad f = \sum_{i=1}^{n} P_i z_{a_i} + \Gamma,
\end{equation}

with polynomials $P_i$, generalized characters $z_{a_i}$, and an additive function
$\Gamma$. Let $z_j$ be a character appearing in (3.14) but not in (3.15), and let $h \in G$
be chosen such that $z_j(h) \neq 1$. Then $z_j$ appears in the expression for the
exponential polynomial $\Delta f$ obtained from (3.14) but not in that obtained
from (3.15). This contradicts Lemma 3.1. Thus the necessity portions of
both Theorem 2 and of Theorem 3 are established.

References

1. N. G. de Bruijn, Functions whose differences belong to a given class, Nieuw Arch. Wisk. 23 (1961), 194-218.
2. ______, A difference property for Riemann integrable functions and for some similar classes
3. F. W. Carroll, Difference properties for continuity and Riemann integrability on a locally
28-56.
8. ______, On exponential polynomials, unpublished manuscript.
12. A. Weil, L'integration dans les groupes topologiques et ses applications, Hermann, Paris,
1953, p. 110.

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