PERTURBATIONS OF THE SHIFT OPERATOR(1)

BY
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Introduction. The lattice of invariant subspaces of the shift operator

$$S:(x_0,x_1,x_2,\cdots)\rightarrow(0,x_0,x_1,\cdots)$$

on $l^2(0,\infty)$ has been characterized by Beurling [1]. In [3] Duren shows that certain "tridiagonal" operators on $l^2(0,\infty)$ have lattices of invariant subspaces isomorphic to that of S. In this paper we show that a large class of lower-triangular matrices represent operators on $l^p(0,\infty)$ $(1\leqq p\leqq\infty)$ which are actually similar to S. Beurling's function algebra representation of the shift operator will not be used.

Our methods are motivated by the following considerations. Let P be a bounded operator on $l^p(0,\infty)$. We ask for conditions on P which insure the similarity of $S+P$ and S, i.e., the existence of a nonsingular operator X such that $S=X(S+P)X^{-1}$. Defining $\Delta X=SX-XS$, this equation becomes

$$\Delta X = XP. \tag{1}$$

It is easily seem that Δ is a derivation;

$$\Delta(XY)=(\Delta X)Y+X\Delta Y,$$

and hence that two nonsingular solutions X and Y of (1) must differ by a multiplicative nonsingular "constant", i.e., $Y=CX$ with $\Delta C=0$.

Thus, up to this point, (1) is completely analogous to the differential equation

$$\frac{d}{dt}X(t)=X(t)P(t) \tag{2}$$

in a Banach algebra. (2) is solved by considering the corresponding integral equation

$$X(t)=I+\int_0^t X(\tau)P(\tau)\,d\tau.$$

Under certain integrability conditions on P, this equation has a unique solution which is expressible either as a Peano series

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$$X(t) = I + \int_0^t P(\tau)d\tau + \int_0^t \int_0^{\tau_2} P(\tau_1)P(\tau_2)d\tau_1 d\tau_2 + \cdots$$

or as the product integral (see [7])

$$X(t) = \hat{\int_0^t} [I + P(\tau)d\tau].$$

It is not unnatural then to attempt to solve the "differential equation" (1) by first defining an "indefinite integral" $\Gamma(P)$ of "integrable" operators P;

$$\Delta\Gamma(P) = P,$$

and then considering the "integral equation"

$$X = I + \Gamma(XP). \tag{3}$$

This we will do. It turns out that there is a rather striking formal parallel of our results with the classical theory of the differential equation (2). The solution of (3) will be given by the Peano series

$$X = I + \Gamma(P) + \Gamma(\Gamma(P)P) + \cdots$$

and also be expressible as a "product integral,"

$$X = \hat{\Gamma}(I + P).$$

1. **The matrix algebra $\mathscr{A}$ and its ideal $\mathscr{L}$.** Let $A = (a_{nm})$ be a doubly infinite matrix of complex numbers. Then

$$A : (x_0, x_1, x_2, \cdots) \to (y_0, y_1, y_2, \cdots),$$

$$y_n = \sum_{m=0}^{\infty} a_{nm} x_m, \tag{4}$$

defines a bounded operator on $l^1(0, \infty)$ if and only if

$$\sup_m \sum_{n=0}^{\infty} |a_{nm}| = M < \infty,$$

and (4) defines a bounded operator on $l^\infty(0, \infty)$ if and only if

$$\sup_n \sum_{m=0}^{\infty} |a_{nm}| = N < \infty.$$

The operator norms are respectively $||A||_1 = M$ and $||A||_\infty = N$.

THEOREM 1.1 (M. RIESZ). *If $||A||_1 < \infty$ and $||A||_\infty < \infty$, then* (4) *defines a bounded operator on $l^p(0, \infty)$ $(1 \leqq p \leqq \infty)$. Moreover*

$$||A||_p \leqq \max(||A||_1, ||A||_\infty).$$

We denote by $\mathscr{A}$ the class of matrices A with

$$||A|| = \max(||A||_1, ||A||_\infty) < \infty.$$

It is easily seen that $||AB|| \leqq ||A||\ ||B||$ when $A, B \in \mathscr{A}$ and hence that $\mathscr{A}$ is a Banach algebra (with unit) under $||\cdot||$. $\mathscr{L}$ will be the class of matrices P such that

$$|P| = \sum_{n,m=0}^{\infty} |p_{nm}| < \infty.$$

Then $\mathscr{L}$ is a Banach space and, as a linear space, $\mathscr{L} \subset \mathscr{A}$.

LEMMA 1.2. *$\mathscr{L}$ is an idela in $\mathscr{A}$, i.e., if $A, B \in \mathscr{A}$ and $P \in \mathscr{L}$, then $APB \in \mathscr{L}$ and $|APB| \leqq ||A||\ |P|\ ||B||$. The matrices $P \in \mathscr{L}$ represent completely continuous operators on all l^p-spaces.*

Proof. Since $AP = [\sum_{k=0}^{\infty} a_{nk}p_{km}]$ we have

$$|AP| \leqq \sum_{n,m=0}^{\infty} \sum_{k=0}^{\infty} |a_{nk}p_{kn}| = \sum_{m,k=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nk}| \right) |p_{km}| \leqq ||A||_1 |P|.$$

Similarly, $|PB| \leqq |P|\ ||B||_\infty$. The first part of the lemma follows. To see that the matrices P represent completely continuous operators via (4) we let P_n be the matrix obtained from P by replacing all but the first n columns of P by zeros. The corresponding operator is of finite rank and

$$||P - P_n||_p \leqq ||P - P_n|| \leqq |P - P_n| \to 0$$

as $n \to \infty$.

The matrices representing shift right,

$$S\colon\ (x_0, x_1, x_2, \cdots) \to (0, x_0, x_1, \cdots),$$

and shift left,

$$S^*\colon\ (x_0, x_1, x_2, \cdots) \to (x_1, x_2, \cdots)$$

are

$$S = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot \\ 1 & 0 & 0 & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \\ \cdot & & & \cdot & \cdot \end{bmatrix} \quad \text{and} \quad S^* = \begin{bmatrix} 0 & 1 & 0 & 0 & & & \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & \cdot & & \\ \cdot & \cdot & & & & \cdot & \\ \cdot & \cdot & & & & & \cdot \end{bmatrix}.$$

The operation $A \to AS$ shifts a matrix left one column and $A \to S^*A$ shifts up one row. Hence $A \to S^{*k}AS^k$ shifts a matrix k units diagonally upwards.

LEMMA 1.3. *If $P \in \mathscr{L}$, then the series $\sum_{k=0}^{\infty} S^{*k}PS^k$ converges in $\mathscr{A}$; its sum is*

$$\gamma(P) = [\,\mathrm{tr}(S^{*n}PS^m)\,]$$

(*where* $\mathrm{tr}(P) = \sum_i p_{ii}$) *and* $\|\gamma(P)\| \leqq |P|$.

Proof. We first show that $\gamma(P)$ above belongs to $\mathscr{A}$. We have

$$\|\gamma(P)\|_1 = \sup_m \sum_{n=0}^{\infty} |\,\mathrm{tr}(S^{*n}PS^m)\,|$$

$$\leqq \sup_m \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |p_{n+k,m+k}| \leqq |P|$$

and similarly $\|\gamma(P)\|_\infty \leqq |P|$. Thus $\|\gamma(P)\| \leqq |P|$. From the remarks above, the series has partial sums

$$\gamma_N(P) = \sum_{k=0}^{N} S^{*k}PS^k = \left[\,\sum_{k=0}^{N} p_{n+k,m+k}\right]$$

and hence $\gamma(P) - \gamma_{N-1}(P) = [\,\mathrm{tr}(S^{*N+n}PS^{N+m}] = \gamma(S^{*N}PS^N)$. Thus, replacing P by $S^{*N}PS^N$ in the above inequality, we have

$$\|\gamma(P) - \gamma_{N-1}(P)\| = \|\gamma(S^{*N}PS^N)\| \leqq |S^{*N}PS^N|$$

and the latter tends to 0 as $N \to \infty$.

Applying now Lemma 1.2 and Theorem 1.1 yields

$$|\gamma(Q)P| \leqq \|\gamma(Q)\|\ |P| \leqq |Q|\ |P|$$

for $P,\ Q \in \mathscr{L}$. Thus the operator

$$\gamma_P : Q \to \gamma(Q)P$$

maps $\mathscr{L}$ boundedly into itself and for its iterates we have $|\gamma_P^n(Q)| \leqq |P|^n|Q|$. The following lemma shows that this estimate can be considerably improved if we assume that P is *strictly lower-triangular*, i.e., $p_{nm} = 0$ unless $n > m$. The subspace of $\mathscr{L}$ consisting of such P will be denoted by $\mathscr{L}_0$.

LEMMA 1.4. *If* $P,\ Q \in \mathscr{L}_0$, *then* $\gamma_P^n(Q) \in \mathscr{L}_0$ *and*

$$|\gamma_P^n(Q)| \leqq \frac{|P|^n}{n!}|Q|.$$

Proof. That γ_P maps $\mathscr{L}_0$ into itself follows from 1.3. A straightforward inductive argument using Lemma 1.3 shows that the (n,m)-entry $(n > m)$ of $\gamma_P^N(Q)$ is

$$\sum_{m \leqq k_1 \leqq \cdots \leqq k_N < n}\ \sum_{0 \leqq i_1 \leqq \cdots \leqq i_N < \infty} q_{n+i_N,k_N+i_N}\, p_{k_N+i_{N-1},k_{N-1}+i_{N-1}} \cdots p_{k_2+i_1,k_1+i_1}\, p_{k_1,m}.$$

Hence

$$
\begin{aligned}
|\gamma_P^N(Q)| & \\
\leqq & \sum_{0\leqq m\leqq k_1\leqq\cdots\leqq k_N\leqq n<\infty}\ \sum_{0\leqq i_1\leqq\cdots\leqq i_N<\infty} |q_{n+i_N,k_N+i_N}p_{k_N+i_{N-1},k_{N-1}} \\
& \qquad\qquad\qquad\qquad \cdots p_{k_2+i_1,k_1+i_1}\,p_{k_1,m}| \\
= & \sum_{0\leqq m\leqq k_1\leqq\cdots\leqq k_N<\infty}\ \sum_{0\leqq i_1\leqq\cdots\leqq i_{N-1}<\infty} \\
& \left[\sum_{n=k_N}^{\infty}\ \sum_{i_N=i_{N-1}}^{\infty} |q_{n+i_N,k_N+i_N}|\right] |p_{k_N+i_{N-1},k_{N-1}+i_{N-1}}\cdots p_{k_2+i_1,k_1+i_1}p_{k_1,m}| \\
\leqq |Q| & \sum_{0\leqq m\leqq k_1\leqq\cdots\leqq k_N<\infty}\ \sum_{0\leqq i_1\leqq\cdots\leqq i_{N-1}<\infty} |p_{k_N+i_{N-1},k_{N-1}+i_{N-1}}\cdots p_{k_2+i_1,k_1+i_1}p_{k_1,m}|.
\end{aligned}
$$

Since P is *strictly* lower-triangular, the nonzero terms of the above sum are products of the form

$$(*)\qquad |p_{r_N s_N}\cdots p_{r_2 s_2}p_{r_1 s_1}|$$

with $r_N > s_N \geqq \cdots \geqq r_2 > s_2 \geqq r_1 > s_1$. Such a product contains any given entry p_{ij} of P at most once. Moreover, each product (*) occurs at most once in the above sum, since the indices $m, k_1, \cdots, k_N$ and $i_1, \cdots, i_{N-1}$ are determined recursively by $r_1, \cdots, r_N$ and $s_1, \cdots, s_N$; $m = s_1$, $k_1 = r_1$, $k_1 + i_1 = s_2$, $k_2 + i_1 = r_2$, etc. But each (*) occurs exactly $N!$ times when the product $|P|^N = (\sum_{n,m}|p_{nm}|)^N$ is expanded. Lemma 1.4 now follows.

Given matrices $A_0, A_1, \cdots, A_N$, we write(2)

$$\prod_{k=0}^{N} A_k = A_N \cdots A_1 A_0.$$

LEMMA 1.5. *If $Q \in \mathscr{S}$ and Q is lower-triangular (i.e., $Q \in S^*\mathscr{S}_0$), then for all N*

$$\left|\left|\prod_{k=0}^{N}(I + S^{*k}QS^k) - I\right|\right| \leqq \exp|Q| - 1.$$

Proof. Multiplying out the product gives

$$
\begin{aligned}
\prod_{k=0}^{N}(I + S^{*k}QS^k) - I = \sum_{k=0}^{N} S^{*k}QS^k + \sum_{0\leqq k_1<k_2\leqq N} (S^{*k_2}QS^{k_2})(S^{*k_1}QS^{k_1}) \\
+ \cdots + (S^{*N}QS^N)\cdots(S^*QS)Q.
\end{aligned}
$$

Very much as in Lemma 1.4 it can be shown that any absolute column [row] sum of the nth matrix sum above ($1 \leqq n \leqq N$) is dominated by a sum of products of the form

(2) All products, finite or infinite, are to be expanded toward the left.

$$|q_{r_n s_n} \cdots q_{r_2, s_2} q_{r_1 s_1}|,$$

where $r_n \geqq s_n > \cdots > r_2 \geqq s_2 > r_1 \geqq s_1$, each such occurring at most once. Thus (exactly as in Lemma 1.4) we have

$$\left\| \prod_{k=0}^{N} (I + S^{*k} Q S^k) - I \right\| \leqq |Q| + \frac{|Q|^2}{2} + \cdots + \frac{|Q|^N}{N!} \leqq \exp|Q| - 1.$$

2. **The "indefinite integral"** $\Gamma(P)$. We denote by Δ the derivation $\Delta X = SX - XS$ on $\mathscr{A}$ and for $P \in \mathscr{L}$ we define

$$\Gamma(P) = \sum_{k=0}^{\infty} S^{*k+1} P S^k,$$

so that $\Gamma(P) = S^* \gamma(P) \in \mathscr{A}$ (see Lemma 1.3).

PROPOSITION 2.1 *$\Gamma(P)$ satisfies the equation $\Delta\Gamma(P) = P$ if and only if $\operatorname{tr}(PS^n) = 0$ for $n = 0, 1, 2, \cdots$. In particular, this condition is fulfilled if $P \in \mathscr{L}_0$.*

Proof. Since $S^*S = I$ and $SS^* = E$ where $E = \operatorname{diag}(0, 1, 1, \cdots)$ we have

$$\begin{aligned} S\Gamma(P) - \Gamma(P)S &= E\gamma(P) - S^*\gamma(P)S = \gamma(P) - S^*\gamma(P)S - (I - E)\gamma(P) \\ &= P - (I - E)\gamma(P). \end{aligned}$$

But by Lemma 1.2 we have

$$(I - E)\gamma(P) = \begin{bmatrix} \operatorname{tr}(P) & \operatorname{tr}(PS) & \operatorname{tr}(PS^2) & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{bmatrix}$$

from which the condition follows. If P is strictly lower-triangular, then $\operatorname{tr}(PS^n)$, being the sum along the nth super diagonal, is trivially zero.

3. **The "integral equation"** $X = I + \Gamma(XP)$. We want now to restrict attention to the subalgebra $\mathscr{A}_0$ of the Banach algebra $\mathscr{A}$ consisting of those matrices $A \in \mathscr{A}$ which are (not necessarily strictly) lower-triangular. Recall that $\mathscr{L}_0$ is the class of *strictly* lower-triangular $P \in \mathscr{L}$. The following theorem is essentially a summary of some of the results in the earlier sections.

THEOREM 3.1. (a) *$\mathscr{L}_0$ is an ideal in $\mathscr{A}_0$, i.e., for $A, B \in \mathscr{A}_0$ and $P \in \mathscr{L}_0$ we have $APB \in \mathscr{L}_0$ and*

$$|APB| \leqq ||A|| \; |P| \; ||B||.$$

(b) Γ *maps* $\mathscr{L}_0 \to \mathscr{A}_0$ *and for* $P \in \mathscr{L}_0$,

$$||\Gamma(P)|| \leqq |P|,$$

$$\Delta\Gamma(P) = P.$$

THEOREM 3.2. *If* $P \in \mathscr{L}_0$, *then the equation*

$$X = I + \Gamma(XP)$$

is uniquely solvable for $X \in \mathscr{A}_0$. *The solution is given by the Peano series*

$$X = I + \Gamma(P) + \Gamma(\Gamma(P)P) + \Gamma(\Gamma(\Gamma(P)P)P) + \cdots$$

which converges absolutely in $\mathscr{A}_0$. *Moreover*

$$\Delta X = XP.$$

Proof. Assuming the existence of a solution $X \in \mathscr{A}_0$ we have by successive substitutions

$$X = I + \Gamma(P) + \Gamma(\Gamma(P)P) + \cdots + \Gamma(\Gamma_P^n(P)) + \Gamma(\Gamma_P^{n+1}(XP)),$$

where Γ_P: $\mathscr{L}_0 \to \mathscr{L}_0$ is defined by $\Gamma_P(Q) = \Gamma(Q)P$. Since $\Gamma(Q) = S^*\gamma(Q) = \gamma(S^*Q)$, it follows that $\Gamma_P^n(Q) = S^{*n}\gamma_P^n(Q)$. Hence by Lemma 1.3 and Theorem 3.1 we have

$$||\Gamma(\Gamma_P^n(Q))|| \leqq |\Gamma_P^n(Q)| \leqq \frac{|P|^n}{n!} |Q|.$$

Taking $Q = XP$ above gives the uniqueness of the solution. The absolute convergence of the Peano series follows taking $Q = P$ in the same inequality. That the series satisfies the "integral equation" is clear. That it then must satisfy the "differential equation" $\Delta X = XP$ follows (from Theorem 3.1) by applying Δ to both sides.

4. **The "product integral"** $\hat{\Gamma}(I + P)$. Crucial to this section is the notion of infinite product in a Banach algebra (with unit). Given a sequence $A_0, A_1, \cdots \in \mathscr{A}$, the product $\prod_{k=0}^{\infty} A_k$ will be said to be convergent provided that there exist an N_0 such that A_k is nonsingular ($A_k^{-1} \in \mathscr{A}$) for $k \geqq N_0$ and $\prod_{k=N_0}^{N} A_k$ converges to a nonsingular element of $\mathscr{A}$ as $N \to \infty$. If this is the case we write $\prod_{k=0}^{\infty} A_k = \lim_{N\to\infty} \prod_{k=0}^{N} A_k$. It is easy to show that if $\prod_{k=0}^{\infty} A_k$ is convergent, then so are all of the tail products $\prod_{k=N}^{\infty} A_k$ and $\prod_{k=0}^{\infty} A_k$ can be factored like a finite product;

$$\prod_{k=0}^{\infty} A_k = \left(\prod_{k=N}^{\infty} A_k\right) \left(\prod_{k=0}^{N-1} A_k\right).$$

Using this we show that

$$\lim_{N\to\infty} \prod_{k=N}^{\infty} A_k = I.$$

We can assume that all A_k are nonsingular. Then $X = \prod_{k=0}^{\infty} A_k$ and $X_N = \prod_{k=0}^{N-1} A_k$ are nonsingular and $X_N \to X$. By a standard theorem of Banach algebras, it follows that $X_N^{-1} \to X^{-1}$ and hence

$$I = \lim_{N\to\infty} XX_N^{-1} = \lim_{N\to\infty} \prod_{k=N}^{\infty} A_k.$$

We need finally the

Convergence condition. The product $\prod_{k=0}^{\infty} A_k$ converges provided that

$$(*) \qquad \lim_{M\to 0} \left\| \prod_{k=M}^{M+N} A_k - I \right\| = 0 \text{ uniformly in } N.$$

Proof. We choose N_0 so large that $\|\prod_{k=N_0}^{N} A_k - I\| \leqq 1/2$ for all $N \geqq N_0$. It is then clear that A_k^{-1} exists for $k \geqq N_0$ and $\lim_{N\to\infty} \prod_{k=N_0}^{N} A_k$ is invertible if it exists. We show that it does (and we are done).

$$\prod_{k=N_0}^{M+N} A_k - \prod_{k=N_0}^{M} A_k = \left[\prod_{k=M+1}^{M+N} A_k - I \right] \prod_{k=N_0}^{M} A_k$$

so that (*) and the boundedness of $\prod_{k=N_0}^{M} A_k$ imply that $\lim_{N\to\infty} \prod_{k=N_0}^{N} A_k$ exists.

PROPOSITION 4.1. *If $P \in \mathscr{L}_0$, then $\prod_{k=0}^{\infty}(I + S^{*k+1}PS^k)$ converges in $\mathscr{A}_0$.*

Proof. Since $Q = S^*P$ and $S^{*M}QS^M$ are lower-triangular, the estimate of Lemma 1.5 applies. Thus

$$\left\| \prod_{k=M}^{M+N} (I + S^{*k}QS^k) - I \right\| \leqq \exp |S^{*M}QS^M| - 1.$$

Since $|S^{*M}QS^M| \to 0$, the convergence condition is fulfilled.

Before passing to the proof of the main result of this section we show that the operation $'$: $\mathscr{A}_0 \to \mathscr{A}_0$ defined by $A' = S^*AS$ is multiplicative on $\mathscr{A}_0$.

LEMMA 4.2. *$(AB)' = A'B'$ for $A, B \in \mathscr{A}_0$.*

Proof. First observe that if C is *strictly* lower-triangular then $EC = C$ where $E = SS^*$, and that if B is lower-triangular then BS is strictly lower-triangular. Hence

$$A'B' = S^*ASS^*BS = S^*AE(BS) = S^*ABS = (AB)'.$$

THEOREM 4.3. *If* $P \in \mathscr{L}_0$, *then* $X = I + \Gamma(XP)$ *is solved by*

$$\hat{\Gamma}(I+P) = \prod_{k=0}^{\infty} (I + S^{*k+1}PS^k)$$

Proof. Let $\Pi_n = \prod_{k=n}^{\infty} (I + S^{*k+1}PS^k)$. Then $\Pi_n - \Pi_{n+1} = \Pi_{n+1} S^{*n+1}PS^n$. Adding these equations over $0 \leqq n \leqq N$, the left-hand side telescopes yielding

$$\Pi_0 - \Pi_{N+1} = \sum_{n=0}^{N} \Pi_{n+1} S^{*n+1}PS^n.$$

But by Lemma 4.2 we have $\Pi_n = S^{*n}\Pi_0 S^n$ and hence (again by Lemma 4.2)

$$\Pi_{n+1}S^{*n+1}PS^n = S^{*n}[S^*\Pi_0 SS^*P]S^n = S^{*n+1}(\Pi_0 P)S^n.$$

Thus $\Pi_0 - \Pi_{N+1} = \sum_{n=0}^{N} S^{*n+1}(\Pi_0 P)S^n$ from which, letting $N \to \infty$, we have

$$\Pi_0 - I = \sum_{n=0}^{\infty} S^{*n+1}(\Pi_0 P)S^n = \Gamma(\Pi_0 P).$$

This is the assertion of the theorem.

5. **The similarity of** $S + P$ **and** S. We now have at our disposal the tools needed for proving the

MAIN THEOREM. *If* $P = (p_{nm})$ *is a strictly lower-triangular matrix with* $|P| = \sum_{n,m=0}^{\infty} |p_{nm}| < \infty$ *and* $p_{n+1,n} \neq -1$, *then* $S + P$ *and* S *represent similar operators on* $l^p(0, \infty)$ $(1 \leqq p \leqq \infty)$. *Moreover, the "product integral"* $\hat{\Gamma}(I+P)$ *represents an operator which implements the similarity* $S + P \sim S$.

Proof. By the remarks of the introduction, $S + P \sim S$ is equivalent to the solvability of $\Delta X = XP$ by a nonsingular operator X. By hypothesis $P \in \mathscr{L}_0$, so that the matrix equation $\Delta X = XP$ is solved by $X = \hat{\Gamma}(I+P) \in \mathscr{A}_0$. Thus it remains only to show that $\hat{\Gamma}(I+P)$ represents a nonsingular operator on $l^p(0, \infty)$. By §4,

$$\hat{\Gamma}(I+P) = X_{N_0} \prod_{k=0}^{N_0} (I + S^{*k+1}PS^k)$$

where $X_{N_0}^{-1} \in \mathscr{A}_0$. By the Riesz Theorem 1.1 these matrices all represent bounded operators on $l^p(0, \infty)$. Thus it remains only to show that the finite product $\prod_{k=0}^{N_0} (I + S^{*k+1}PS^k)$ represents a nonsingular operator on $l^p(0, \infty)$. This is where the assumption $p_{n+1,n} \neq -1$ enters. Then the matrices $(I + S^{*k+1}PS^k)$ are lower-triangular with nonvanishing diagonal entries. It follows that -1 is not an eigenvalue of $S^{*k+1}PS^k$. Since $S^{*k+1}PS^k \in \mathscr{L}$, they represent completely continuous operators (see Lemma 1.2). Thus by the Fredholm alternative -1 is a regular point of $S^{*k+1}PS^k$. Thus the finite product above is nonsingular.

Conclusion. Our starting point was the formal analogy between $\Delta X = XP$ and the classical differential equation (2). That this analogy proved here to be fruitful in obtaining similarity results is no accident depending on very special properties of the shift operator. This author has used it also to obtain sharp results for the Volterra operator $Jf = \int_0^x f(y)dy$ (see [5]). Earlier work of Friedrichs and Schwartz on perturbations of self-adjoint operators follows similar lines.

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