ON NOETHERIAN PRIME RINGS(1)

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A ring S with identity element is a *classical right quotient ring of a ring* R in case: (i) $S \supseteq R$; (ii) R contains nondivisors of zero, called regular elements, and each regular $d \in R$ has a two-sided inverse $d^{-1} \in S$; and (iii)

$$S = \{ab^{-1} | a, \text{regular } b \in R\}.$$

Classical left quotient rings are defined symmetrically. R is *right* (resp. *left*) *quotient-simple* in case R has a classical right (resp. left) quotient ring S which is isomorphic to a complete ring D_n of $n \times n$ matrices over a (not necessarily commutative) field D. R is *quotient-simple* if R is both left and right quotient-simple.

Goldie [2] has determined that a ring R is right quotient-simple if and only if R is a prime ring satisfying the maximum conditions on complement and annihilator right ideals. In particular, any right noetherian prime ring is right quotient-simple. (See also Lesieur-Croisot [1].)

A (not necessarily commutative) integral domain K is a *right Ore domain* in case K possesses a classical right quotient field $\hat{K}$. Observe that if K is a right Ore domain, then, for each natural number n, the ring K_n of all $n \times n$ matrices over K is right quotient-simple, and $(\hat{K})_n$ is its classical right quotient ring.

A consequence of our main result (Theorem 2.3) is that the right quotient-simple rings can be determined as the class of intermediate rings of the extensions $(\hat{K})_n$ over K_n, n ranging over all natural numbers, and K ranging over all right Ore domains. Theorem 2.3 is much more precise. As a corollary we rederive a theorem of Goldie [3] on principal right ideal prime rings.

1. **General quotient rings.** If R is any ring, M_R (resp. ${}_RM$) will denote that M is a right (resp. left) R-module. If N is a submodule of M_R such that any nonzero submodule P of M has nonzero intersection with N, then M is an *essential extension of* N, or N is an *essential submodule of* M. Notation: $M \nabla N$ or $(M \nabla N)_R$. A right ideal I of R satisfying $(R \nabla I)_R$ is an *essential right ideal*.

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If A is a ring containing R, then A is a *right quotient ring of* R in case $(A \bigtriangledown R)_R$, where the module A_R is defined in the natural way. Any classical right quotient ring S of a ring R is a right quotient ring in this sense, so the results of this section are applicable to classical quotient rings.

1. THEOREM. *Let R be a semiprime ring, and let A be a right quotient ring of R. Let e be an idempotent of A such that $D = eAe$ is a field* (3). *If*

$$K = eAe \cap R \neq 0,$$

then K is a right Ore domain, and D is its right quotient field,

$$D = \{kq^{-1} | k, 0 \neq q \in K\}.$$

Proof. First note that eA is a left vector space over $D = eAe$. Thus if $0 \neq d \in D$, then $dx = 0 \Longrightarrow x = 0 \ \forall x \in eA$. Let $[d]$ denote the R-submodule of A generated by $d \in D$. If we set $d(r,n) = dr + nd \ \forall (r,n) \in R \times \mathbf{Z}$ (cartesian product), $r \in R$, $n \in \mathbf{Z}$, then $[d] = \{d(r,n) | (r,n) \in R \times \mathbf{Z}\}$. Since $(A \bigtriangledown R)_R$, we have $[d] \cap R \neq 0 \ \forall \ 0 \neq d \in D$. Hence let

$$0 \neq c = d(r,n) \in [d] \cap R.$$

Since $d = de$, clearly $e(r,n) \neq 0$, so choose $(r',n') \in R \times \mathbf{Z}$ such that

$$0 \neq e(r,n)(r',n') \in R.$$

Then setting $U = eA \cap R$, and setting $a = e(r,n)(r',n')$, we see that $a \in U$ and $b = da = c(r',n') \in U$. Since $a \neq 0$, necessarily $b \neq 0$. Since $V = Ae \cap R$ is a left ideal of $R, K = eAe \cap R = V \cap U$ is a left ideal of U. Furthermore, since $K \neq 0$ by hypothesis, since $K \subseteq D$, and since $U \subseteq eA$, U is a torsion-free left K-module. Since R is semiprime, the left annihilator ideal of R is zero, so $aR \neq 0$. Since $aR \subseteq U$, it follows that $KaR \neq 0$. Since KaR is a right ideal of R, semiprimeness of R implies that $(KaR)^2 \neq 0$, hence $aRK \neq 0$. Now choose $t \in R, q \in K$ such that $x = atq \neq 0$. Since $x \in eA$, necessarily $y = dx \neq 0$. Since $at, bt \in U$ (U is a right ideal of R and $a, b \in U$), necessarily $x = atq, y = btq \in K$ (K is a left ideal of U and $q \in K$). If x^{-1} denotes the inverse of x in D, we see that $d = yx^{-1}$, and $x, y \in K$. Thus, D is the right quotient field of K, completing the proof.

REMARK. Any quotient ring A of a semiprime ring R is semiprime. Thus (see Jacobson [1, p. 65, Proposition 1]) eAe is a field if and only if eA is a minimal right ideal.

If R is any ring, and if x^r is the right annihilator in R of $x \in R$, then

$$Z_r(R) = \{x \in R | x^r \text{ is an essential right ideal of } R\}$$

is a two-sided ideal of R (R. E. Johnson [1]), called the *right singular ideal*

(3) In our terminology, a field is not necessarily commutative.

of R. It is easy to check that $Z_r(R)$ contains no nonzero idempotents, so that $Z_r(R) = 0$ whenever R is a (von Neumann) regular ring, in particular when R is semisimple artinian.

Below we show that the vanishing of $Z_r(Q)$ is enough to insure transitivity of the relation "quotient ring of". In the proof, if x is an element in the ring Q, and if P is a subring of Q, then

$$(P\colon x) = \{p \in P \mid xp \in P\}$$

is a right ideal of P. Furthermore, if $(Q \nabla P)_P$, then $(P\colon x)$ is an essential right ideal of P, a fact which we use without proof.

2. Lemma. *Let Q be a right quotient ring of R, and let R be a right quotient ring of T.* (1) *If I is any right ideal of Q such that $I \cap R$ is an essential right ideal of R, then I is an essential right ideal of Q;* (2) $Z_r(Q) \supseteq Z_r(R) \supseteq Z_r(T)$; (3) *If $Z_r(Q) = 0$, then Q is a right quotient ring of T.*

Proof. (1) is trivial. (2) Let x^r denote the right annihilator in Q of $x \in Q$. If $x \in Z_r(R)$, then $x^r \cap R$ is an essential right ideal of R. Then (1) implies that $x \in Z_r(Q)$, proving (2).

(3) If $x \in Q$, $xR = 0$ implies by (1) that $x \in Z_r(Q)$. Since $Z_r(Q) = 0$, if $0 \neq x \in Q$, then $xR \neq 0$, so $xR \cap R \neq 0$. Let $s, r \in R$ be such that $s = xr \neq 0$. Now $(T\colon r)$ (resp. $(T\colon s)$) is an essential right ideal of T, and so is

$$(T\colon r) \cap (T\colon s).$$

Hence $s^r \supseteq (T\colon r) \cap (T\colon s)$ would imply by (1) that $s \in Z_r(R)$. But $Z_r(R) = 0$ by (2) and $s \neq 0$, so we conclude that $s^r \not\supseteq (T\colon r) \cap (T\colon s)$. Accordingly we can choose $t \in (T\colon r) \cap (T\colon s)$ such that $st \neq 0$. Then $st = x(rt) \in xT \cap T$, so $xT \cap T \neq 0$. This proves (3).

For convenience, we recall the definition of a prime ring. R is said to be prime in case any of the following three equivalent conditions are satisfied:

(a) $I^r = 0 \ \forall$ right ideals I;

(b) $I^l = 0 \ \forall$ left ideals I;

(c) $xRy = 0 \Longrightarrow x = 0$ or $y = 0 \ \forall x,\ y \in R$.

Here $I^r = \{a \in R \mid Ia = 0\}$, and $I^l = \{a \in R \mid aI = 0\}$.

If A (resp. B) is a left (resp. right) ideal of R, then $T = BA$ is defined to be the set of all finite sums of the products ba, $a \in A$, $b \in B$. It is to be observed that T is a subring of R.

3. Proposition. *Let $R \neq 0$ be a prime ring, let A be a left ideal of R whose right annihilator A^r in R is zero, and let B be a right ideal of R whose left annihilator B^l in R is zero. Then*: (1) *$T = BA$ is a prime ring*; (2) *If, in addition, B is an essential right ideal of R, then R is a right quotient ring of T.*

Proof. (1) Let $x, y \in T$ be such that $xTy = 0$. Then $AyRxB$ is an ideal of R

having square zero, so primeness of R yields $AyRxB = 0$. Then again by primeness of R, $Ay = 0$, or $xB = 0$. Since $A^r = B^l = 0$, we obtain $y = 0$ or $x = 0$, and T is therefore prime.

(2) If $0 \neq x \in R$, then $xB \neq 0$. Then $(R \nabla B)_R$ implies $xB \cap B \neq 0$. Let $b \in B$ be such that $0 \neq xb \in B$. Primeness of R implies $A^l = 0$, so $xbA \neq 0$. But $bA \subseteq T$, and $xbA \subseteq T$, so $xT \cap T \neq 0$, proving (2).

2. **Quotient-simple rings.** Before proving the main result (Theorem 2.3) we list some known properties of classical quotient rings.

1. Lemma. *Let Q be a classical right quotient ring of a ring R. Then*: (1) *If $b_1, \cdots, b_n$ are regular elements of R, there exists a regular element $c \in R$ and elements $g_i \in R$ such that $b_i^{-1} = g_i c^{-1}$, $i = 1, \cdots, n$;* (2) *If $x_1, \cdots, x_n \in Q$, there exists a regular element $c \in R$ such that $x_i c \in R$, $i = 1, \cdots, n$;* (3) *If I is a right ideal of R, then the right ideal of Q generated by I is IQ, and $IQ = \{xc^{-1} | x \in I$, regular $c \in R\}$;* (4) *If $d \in R$ is regular, then dR is an essential right ideal of R;* (5) *If R has a classical left quotient ring, then Q is a classical left quotient ring of R.*

These results occur various places in Goldie's paper [1], but none require any of the deeper results found there. For example (1) is an easy induction [1, p. 605, Lemma 4.2], (2) is an immediate consequence of (1), and (3) follows from (2) [1, p. 605, Lemma 4.3]. (4) is [1, p. 603, Theorem 10] but a shorter argument is as follows: Let $I = dR$. Since $1 = dd^{-1} \in IQ$, then $IQ = Q$. Hence if $0 \neq k \in Q$, then (3) implies that $k = xc^{-1}$ with $x \in I$, $c \in R$. Then $0 \neq kc = x \in kR \cap I$, proving (4). (5) is obvious.

We also require the following:

2. Lemma. *If R is right quotient-simple, then R is prime.*

Although this is not explicitly stated in Goldie's paper [2], it follows from [2, p. 213, Theorem 4.4] that R is semiprime. Let J be a right ideal of R such that $I = J^r \neq 0$. Then, if Q is the classical right quotient ring of R, $IQJ \cap R$ is a right ideal of R having square equal 0. Thus $IQJ \cap R = 0$ by semiprimeness of R. Since $(Q \nabla R)_R$, we obtain $IQJ = 0$. By simplicity (that is, primeness) of Q, we conclude that $J = 0$, and R is therefore prime.

If $Q = D_n$ is the complete ring of $n \times n$ matrices over a field D, then there exists a set $M = \{e_{ij} | i, j = 1, \cdots, n\}$ of matrix units of Q, and the set of elements of Q which commute with each element of M is a field isomorphic to D. Without loss of generality we can assume that this field is D; we call it the centralizer of M in Q. If x is any invertible element of Q, then $x^{-1}Mx$ is a set of matrix units in Q whose centralizer is $x^{-1}Dx$. We call any such set a complete set of matrix units of Q.

3. Theorem. *Let R be a right quotient-simple ring with quotient ring $Q = D_n$, D a field.* (1) *Then Q contains a complete set $M = \{e_{ij} | i, j = 1, \cdots, n\}$ of matrix*

units with the following property: if D is the centralizer of M in Q, then R contains a subring

$$F_n = \sum_{i,j=1}^{n} Fe_{ij},$$

where F is a right Ore domain contained in $R \cap D$ and D is the right quotient field of F. Furthermore:

$$Q = \{ak^{-1} \mid a \in F_n, 0 \neq k \in F\}.$$

(2) *If R is also left quotient-simple, then every complete set M of matrix units has the property described in* (1), *and each corresponding D is also the left quotient field of F. Finally,*

$$Q = \{q^{-1}b \mid b \in F_n, 0 \neq q \in F\}.$$

Proof. We give a proof of (1) and (2) simultaneously by showing if $M = \{e_{ij} \mid i, j = 1, \cdots, n\}$ is any complete set of matrix units of Q such that

(*) there exists a regular element $y \in R$ such that $yM \subseteq R$

then M has the property in statement (1).

Now if R is also left quotient-simple, then Q is a classical left quotient ring of R, and the right-left symmetry of (2) of 2.1 asserts that each full set M has property (*).

Next assume only that R is right quotient-simple, and let N be a complete set of matrix units of Q. Then, by 2.1, there exists a regular $y \in R$ such that $Ny \subseteq R$. Hence, $M = y^{-1}Ny$ is a complete set of matrix units of Q satisfying (*).

Accordingly let $M = \{e_{ij} \mid i, j = 1, \cdots, n\}$ be any complete set of matrix units of Q satisfying (*). Then, by 2.1 there exists regular $x \in R$ such that $Mx \subseteq R$. Hence the left ideal $A = \{r \in R \mid rM \subseteq R\}$ contains the regular element $y \in R$, and the right ideal $B = \{r \in R \mid Mr \subseteq R\}$ contains the regular element $x \in R$. Furthermore, B is an essential right ideal of R by (4) of 2.1.

Since R is a prime ring by 2.2, we apply 1.3 to conclude that $T = BA$ is a prime ring and that R is a right quotient ring of T. Since $Z_r(Q) = 0$, we deduce from (3) of 1.2 that Q is a right quotient ring of T.

Next we show that $e_{11}Qe_{11} \cap T \neq 0$. Now $0 \neq ye_{11} \in R$ and

$$ye_{11}M \subseteq yM \subseteq R$$

which shows that $ye_{11} \in A$. Since $x \in B$ and since x is regular it follows that $0 \neq xye_{11} \in T = BA$, so that $T \cap Qe_{11} \neq 0$. Since Q is a right quotient ring of T, $e_{11}Q \cap T \neq 0$. Then primeness of T implies that $(e_{11}Q \cap T)c \neq 0$, where $c = xye_{11}$. If $d \in e_{11}Q \cap T$ is such that $dc \neq 0$, then $dc \in T \cap e_{11}Qe_{11}$, proving our assertion.

Since $F_1 = e_{11}Qe_{11} \cap T \neq 0$, and since $e_{11}Qe_{11}$ is a field ($\cong D$), Theorem 1.1 implies that $e_{11}Qe_{11}$ is the right quotient field of $F_1 = e_{11}Qe_{11} \cap T$. Since D is isomorphic to $De_{11} = e_{11}Qe_{11}$ under the map ϕ: $d \to de_{11}$, $d \in D$, this shows that D is the right quotient field of $F = \phi^{-1}F_1$. Furthermore,

$$Fe_{ij} = e_{i1}F_1e_{1j} \subseteq e_{i1}Te_{1j} = (e_{i1}B)(Ae_{1j}) \subseteq RR \subseteq R,$$

$i, j = 1, \cdots, n$. Thus, R contains the subring

$$F_n = \sum_{i,j=1}^{n} Fe_{ij},$$

and $F \subseteq R \cap D$.

If $a = \sum_{i,j=1}^{n} e_{ij}d_{ij} \in Q$, $d_{ij} \in D$, $i, j = 1, \cdots, n$, then by 2.1, there exists $0 \neq k \in F$ such that $d_{ij}k = q_{ij} \in F$, $i, j = 1, \cdots, n$. Then $a = fk^{-1}$, where $f = \sum_{i,j=1}^{n} e_{ij}q_{ij} \in F_n$. This proves (1).

If R is also left quotient-simple, then Q is the classical left quotient ring of R, and Theorem 1.1 implies that $e_{11}Qe_{11}$ (resp. D) is the left quotient field of F_1 (resp. F). The computation above establishes that if $a \in Q$, then $a = q^{-1}b$, with $b \in F_n$, and $0 \neq q \in F$. This completes the proof of (2).

4. Corollary (A. W. Goldie [3]). *If R is a principal right ideal ring, and if R is prime, then $R = K_n$, where K is a right Ore domain.*

Proof. Using the notation of the theorem, we can write $B = cR$ for some $c \in R$. If $b \in Q$ is such that $bc = 0$, then $bB = 0$. But $x \in B$ is regular in R, so $x^{-1} \in Q$. Thus $bx = 0$ and $b = 0$, so c is not a right zero divisor in Q. Then, as is well known in artinian rings, $c^{-1} \in Q$.

Trivially $e_{ij}B \subseteq B$, that is, $e_{ij}cR \subseteq cR$ and $c^{-1}e_{ij}c = f_{ij} \in R$, $i, j = 1, \cdots, n$. If G is the centralizer of $N = c^{-1}Mc$ in Q, it follows that

$$R = \sum_{i,j=1}^{n} Kf_{ij},$$

where $K = G \cap R$. Since Q is the classical right quotient ring of R, an easy computation shows that G is the right quotient field of K. (This fact also follows from the theorem since the theorem states that G is the right quotient field of some integral domain contained in K.)

At present (see Goldie [3]) it is unknown whether or not K has to be a principal right ideal domain. R. Bumby has shown us that the answer is "yes" if K is commutative.

3. **Supplementary remarks.** (A) Let R be right quotient-simple with right quotient ring $Q = D_n$, $n > 1$. If f is any idempotent of Q such that fQ is a minimal right ideal, then $f = ac^{-1}$, with a, regular $c \in R$. Thus $e = c^{-1}fc$ is idempotent and $0 \neq ce \in Qe \cap R$. Thus, $Qe \cap R \neq 0$, and since $eQ \cap R \neq 0$,

it follows from primeness of R that $(eQ \cap R)(Qe \cap R) \neq 0$, so that

$$eQe \cap R \neq 0.$$

Then, Theorem 1.1 implies that $K = eQe \cap R$ is a right Ore domain, and $\hat{K} = eQe$ is its right quotient field. Since $\hat{K} \cong D$, we obtain that $Q \cong \hat{K}_n$, where K is a right Ore domain contained in R. This illustrates the precise nature of Theorem 2.3, which states much more.

(B) Next we show that (2) of Theorem 2.3 fails without the hypothesis that R is also left quotient-simple. The example below was suggested by S. U. Chase.

Let K be a right Ore domain which is not a left Ore domain (e.g., Goldie [2, p. 219]), let x, y be nonzero elements of K such that $Kx \cap Ky = 0$, and let

$$R = \begin{pmatrix} Kx & Ky \\ Kx & Ky \end{pmatrix}.$$

(R is the ring of all 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, c \in Kx$, and $b, d \in Ky$.) Since K is right Ore, if $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an arbitrary element of K_2, there exists $0 \neq q \in K$ such that $aq, bq, cq, dq \in K$, and then

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} qx & 0 \\ 0 & qy \end{pmatrix} \in R.$$

Thus, $A = BC^{-1}$, with

$$B,\ C = \begin{pmatrix} qx & 0 \\ 0 & qy \end{pmatrix} \in R.$$

Hence, $\hat{K}_2$ is the classical right quotient ring of R, that is, R is right quotient-simple.

As in Theorem 2.3, we identify $\hat{K}$ with the subring of $\hat{K}_2$ consisting of all scalar matrices $\left(\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right)$ with $k \in \hat{K}$. Now assume for the moment that R contains a subring F_2 where F is an integral domain $\subseteq \hat{K}$. The contradiction is immediately evident (even without assuming that $\hat{F} = \hat{K}$), since the form of R,

$$R = \begin{pmatrix} Kx & Ky \\ Kx & Ky \end{pmatrix},$$

where $Kx \cap Ky = 0$, precludes the possibility of its containing a nonzero scalar matrix $\left(\begin{smallmatrix} d & 0 \\ 0 & d \end{smallmatrix}\right)$ with $0 \neq d \in \hat{K}$.

(C) Theorem 2.3 implies that a right quotient-simple ring R which is not an integral domain contains nonzero nilpotent elements. However such a ring R need not contain nontrivial idempotents even if R contains an identity. Perhaps the simplest example is as follows: Let $S=Q_2$ be the ring of all 2×2 matrices over the rational number field Q, and let R be the subring consisting of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where b, c are even integers, and a, d are integers which are either both even or both odd. Then $R = (2\boldsymbol{Z})_2 + \boldsymbol{Z}$ is not an integral domain. However R is quotient-simple with quotient ring S, with an identity 1, and R does not contain idempotents $\neq 0$, 1.

(D) Now let R be a ring having a classical right quotient ring S which is semisimple artinian. If T is a simple component of S, then $T=eS$, where e is a central idempotent of S, and it can be shown that T is the classical right quotient-ring of $eR \cap R$. Thus, by the theorem, R contains a direct sum of finitely many full rings of matrices K_t over various right Ore domains K, and S is the direct sum of the corresponding simple components $(\hat{K})_t$.

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