ORDERS IN SIMPLE ARTINIAN RINGS

BY

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This note is a continuation of the preceding article [1]. The notation and terminology employed there will be used here.

A simple artinian ring Q has the form D_n, where D is a field. A subring S of Q is a right order in Q in case Q is the classical right quotient ring of S. Two right orders R, S of Q are *equivalent* in case there exist regular[2] elements $a, b, a', b' \in Q$ such that $aRb \subseteq S$ and $a'Sb' \subseteq R$. This relation is reflexive, symmetric, transitive, and we write $R \sim S$ (thereby suppressing Q).

The main result of [1] states that if R is an order in Q, then for a suitable choice of a complete set $M = \{e_{ij} | \; i, j = 1, \cdots, n\}$ of matrix units of Q, if D denotes the centralizer of M in Q, then there exists a subring F of $P = R \cap D$ such that (1) F is a right order in D, and (2) $R \supseteq F_n = \sum_{i,j=1}^{n} Fe_{ij}$. Furthermore, we indicated by example that R itself is not necessarily of the form K_n, where K is an integral domain, even when R possesses an identity element.

The main result of the present article states, in the notation of the paragraph above, that if R is a right order of Q, then $R \sim P_n$, and, in fact, there exists a right order U of D such that $R \sim U_n$ and $U_n \subseteq R$ (cf. Theorem 1). Under the additional hypothesis that R is a simple ring with identity, we show that $R \sim U_n^2$, and U^2 (resp. U_n^2) is a simple ring.

Henceforth, $R, Q, D, M = \{e_{ij} | \; i, j = 1, \cdots, n\}$, P, F are fixed as in the second paragraph above, and have the same meaning as in the proof and statement of Theorem 2.3 of [1]. Two further symbols appearing there are $A = \{r \in R | \; rM \subseteq R\}$ and $B = \{r \in R | \; Mr \subseteq R\}$. If S is any subring, and if X is a subset of Q, then $S[X]$ is the subring of Q generated by S and X. Throughout, the symbol G_n denotes that G is a subring of D, and that $G_n = \sum_{i,j=1}^{n} Ge_{ij}$. Note that $G_n = G[M]$ if and only if G contains the identity element of D.

1. THEOREM. *If R is a right order in the simple artinian ring $Q = D_n$, then*:
(1) $U = B \cap P = A \cap P$ *is an ideal of* $P = R \cap D$.
(2) $B \cap A \supseteq U_n \supseteq BA \supseteq U_n^2$.

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(2) Note that if T is the classical right quotient ring of a subring, then each regular element of T is invertible. Artinian rings with identity also have this property.

(3) *P, U, U^2 are right orders in D.*

(4) *R, U_n^2, U_n, P_n are equivalent right orders in Q.*

(5) *If $0 \neq u \in U$, then $R[u^{-1}] = P'_n$, where $P' = P[u^{-1}]$.*

Proof. (1) If $u \in U = B \cap P$, then $Mu \subseteq R$ (since $u \in B$) and $uM = Mu$ (since $u \in D$). Thus $u \in A$, that is, $U \subseteq A \cap P$. Similarly $A \cap P \subseteq U$, so $U = A \cap P$. Since B (resp. A) is a right (resp. left) ideal of R, U is a (right and left) ideal of P.

(2) Let $H = B \cap A$, and let $T = BA$. Since $e_{ij}B \subseteq B$ (resp. $Ae_{ij} \subseteq A$), $i,j = 1,\cdots,n$, it follows that $H \supseteq U_n$. Let $b \in B$, $a \in A$, let $t = ba$, and set $t_{ij} = \sum_{k=1}^n e_{ik}te_{kj}$, $i,j = 1,\cdots,n$. Since t_{ij} commutes with the elements of M, then $t_{ij} \in D$, $i,j = 1,\cdots,n$. Furthermore

$$t_{ij} = \sum_{k=1}^{n} (e_{ik}b)(ae_{kj}) \in BA = T,$$

so that

$$t_{ij} \in D \cap T \subseteq D \cap B = P \cap B = U,$$

$i,j = 1,\cdots,n$. This shows that $t = \sum_{i,j=1}^n t_{ij}e_{ij} \in U_n$. Since each element of $T = BA$ is a sum of elements of the form ba, it follows that

$$H = A \cap B \supseteq U_n \supseteq T = BA.$$

Finally, we note that

$$BA = T \supseteq H^2 \supseteq (U_n)^2 \supseteq (U^2)_n,$$

proving (2).

(3) F is a right order of D, and $F_n \subseteq R$, so clearly $F \subseteq U \subseteq P$. This shows that U and P are right orders in D. Since U^2 is an ideal of U, it follows that U^2 is a right order in D, since if $d = uv^{-1}$ with $u, v \in U$, and if $0 \neq w \in U^2$, then $d = (uw)(vw)^{-1}$, with $uw, vw \in U^2$.

(4) From (3) it follows that U_n, U_n^2, and P_n are right orders in $Q = D_n$. If $0 \neq u \in U$, then $uR \subseteq B$, so that

$$uRu \subseteq BA \subseteq U_n \subseteq P_n.$$

But U is an ideal of P, so $u^2P \subseteq U^2$, and therefore

$$u^3Ru \subseteq u^2(P_n) = (u^2P)_n \subseteq U_n^2 \subseteq U_n \subseteq P_n.$$

Conversely,

$$u^2(P_n) = (u^2P)_n \subseteq U_n^2 \subseteq U_n \subseteq R.$$

Since $u^{-1} \in Q$, the proof of (4) is complete.

(5) From the proof of (4) we have that $uRu \subseteq P_n$, so clearly $R \subseteq P'_n$, and $R[u^{-1}] \subseteq P'_n$. Conversely since $Mu \subseteq R$, then $M \subseteq Ru^{-1} \subseteq R[u^{-1}]$.

Since $P \subseteq R$, it follows that $P' = P[u^{-1}] \subseteq R[u^{-1}]$, and so $P'_n \subseteq P'[M] \subseteq R[u^{-1}]$. This proves that $R[u^{-1}] = P'_n$.

2. THEOREM. *If R in Theorem 1 is a simple ring with identity, then*:
(1) *$B, A, BA = T$, and U^2 are all simple rings.*
(2) *$T = U^2_n$.*
(3) *If $0 \neq u \in U^2$, then $P' = P[u^{-1}]$ and P'_n are simple rings.*

Proof. Let I be a nonzero ideal of $T = BA$. Then

$$I \supseteq (BA)I(BA) = B(AIB)A.$$

Since $A \cap B \supseteq U$ contains a regular element, clearly $AIB \neq 0$. Thus, simplicity of R forces $R = AIB$. Therefore $I \supseteq BRA \supseteq BA$, so BA is simple.

Already we have seen that

$$T = BA \supseteq H^2 \supseteq (U_n)^2 \supseteq T^2,$$

where $H = A \cap B$. Since T contains the integral domain U, then $T^2 \neq 0$, so simplicity of T yields $T = T^2$. It follows that

$$T = (U_n)^2 = U^2_n,$$

so simplicity of U^2 follows from that of T. If $0 \neq u \in U^2$, and if I is a nonzero ideal of $P' = P[u^{-1}]$, then $I \cap U^2$ is a nonzero ideal of U^2 (since U^2 is a right order in D). Thus simplicity of U^2 implies that $I \supseteq U^2$. Since $u \in I$ is invertible in $P' = P[u^{-1}]$, then $I = P'$, so P' (also P'_n) is simple.

Next we show that B (resp. A) is simple. Let I be a nonzero ideal of B (resp. A). If $0 \neq u \in U$, then $u^3Iu \neq 0$ and $u^3Iu \subseteq U^2_n = T$ by the proof of (4) of Theorem 1. Since $u \in B$ (resp. $u \in A$), it follows that $u^3Iu \subseteq T \cap I$, so $T \cap I$ is a nonzero ideal of T. Simplicity of T forces $I \supseteq T$. Then $I \supseteq BAB$ (resp. $I \supseteq ABA$). Since $AB = R$, then $I \supseteq B$ (resp. $I \supseteq A$), proving that B (resp. A) is simple.

3. COROLLARY. *Under the hypotheses of the theorem, $R = P_n$ if and only if P is a simple ring.*

Proof. The necessity is well known. Conversely if P is simple, then $P = U$ by (1) of Theorem 1. Consequently, $1 \in U \subseteq B$, so $M = M1 \subseteq R$, and it follows that $R = P_n$ (since $P = R \cap D$).

4. COROLLARY. *Let R be a right order of $Q = D_n$, and assume that R is a simple ring with identity element.* (1) *If z is an element of Q such that $Rz \subseteq zR$, then z is invertible in Q and $z, z^{-1} \in R$.* (2) *R contains the center of Q.*

Proof. (1) Since R is a right order of Q, $I = zR \cap R \neq 0$. Thus I is a nonzero right ideal of R, and the relation $Rz \subseteq zR$ implies that I is an ideal of R. Since R is simple, $I = R$, so $zR \supseteq R$. It follows that z is not a left zero divisor in Q, and since Q is left artinian, we conclude that $z^{-1} \in Q$. Since $R \supseteq z^{-1}R$, then $z^{-1} \in R$. Now simplicity of R implies that $Rz^{-1}R$, the ideal of R generated by z^{-1}, equals R. Since $z^{-1}R \subseteq Rz^{-1}$, we obtain that

$$R = Rz^{-1}R \subseteq Rz^{-1} \subseteq R,$$

that is, that $Rz^{-1} = R$. Thus, $(z^{-1})^{-1} = z \in R$, proving (1). (2) is an immediate consequence.

Reference

1. Carl Faith and Yuzo Utumi, *On noetherian prime rings,* Trans. Amer. Math. Soc. **114** (1965), 53-60.

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