ORDERS IN SIMPLE ARTINIAN RINGS

BY

CARL FAITH

This note is a continuation of the preceding article [1]. The notation and terminology employed there will be used here.

A simple artinian ring $Q$ has the form $D_n$, where $D$ is a field. A subring $S$ of $Q$ is a right order in $Q$ in case $Q$ is the classical right quotient ring of $S$. Two right orders $R, S$ of $Q$ are equivalent in case there exist regular elements $a, b, a', b' \in Q$ such that $aRb \subseteq S$ and $a'Sb' \subseteq R$. This relation is reflexive, symmetric, transitive, and we write $R \sim S$ (thereby suppressing $Q$).

The main result of [1] states that if $R$ is an order in $Q$, then for a suitable choice of a complete set $M = \{ e_{ij} \mid i, j = 1, \cdots, n \}$ of matrix units of $Q$, if $D$ denotes the centralizer of $M$ in $Q$, then there exists a subring $F$ of $P = R \cap D$ such that (1) $F$ is a right order in $D$, and (2) $R \supseteq F_n = \sum_{i,j=1}^{n} e_{ij}$. Furthermore, we indicated by example that $R$ itself is not necessarily of the form $K_n$, where $K$ is an integral domain, even when $R$ possesses an identity element.

The main result of the present article states, in the notation of the paragraph above, that if $R$ is a right order of $Q$, then $P \sim P_n$, and, in fact, there exists a right order $U$ of $D$ such that $P \sim U_n$ and $U_n \subseteq R$ (cf. Theorem 1). Under the additional hypothesis that $R$ is a simple ring with identity, we show that $R \sim U_n^2$ and $U_2$ (resp. $U_n^2$) is a simple ring.

Henceforth, $R, Q, D, M = \{ e_{ij} \mid i, j = 1, \cdots, n \}$, $P, F$ are fixed as in the second paragraph above, and have the same meaning as in the proof and statement of Theorem 2.3 of [1]. Two further symbols appearing there are $A = \{ r \in R \mid rM \subseteq R \}$ and $B = \{ r \in R \mid Mr \subseteq R \}$. If $S$ is any subring, and if $X$ is a subset of $Q$, then $S[X]$ is the subring of $Q$ generated by $S$ and $X$. Throughout, the symbol $G_n$ denotes that $G$ is a subring of $D$, and that $G_n = \sum_{i,j=1}^{n} e_{ij}$. Note that $G_n = G[M]$ if and only if $G$ contains the identity element of $D$.

1. Theorem. If $R$ is a right order in the simple artinian ring $Q = D_n$, then:
   (1) $U = B \cap P = A \cap P$ is an ideal of $P = R \cap D$.
   (2) $B \cap A \supseteq U_n \supseteq BA \supseteq U_n^2$.

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(2) Note that if $T$ is the classical right quotient ring of a subring, then each regular element of $T$ is invertible. Artinian rings with identity also have this property.

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(3) $P, U, U^2$ are right orders in $D$.

(4) $R, U_n, U_n, P_n$ are equivalent right orders in $Q$.

(5) If $0 \neq u \in U$, then $R[u^{-1}] = P'_n$, where $P' = P[u^{-1}]$.

**Proof.** (1) If $u \in U = B \cap P$, then $Mu \subseteq R$ (since $u \in B$) and $uM = Mu$ (since $u \in D$). Thus $u \in A$, that is, $U \subseteq A \cap P$. Similarly $A \cap P \subseteq U$, so $U = A \cap P$. Since $B$ (resp. $A$) is a right (resp. left) ideal of $R$, $U$ is a (right and left) ideal of $P$.

(2) Let $H = B \cap A$, and let $T = BA$. Since $e_iB \subseteq B$ (resp. $Ae_i \subseteq A$), $i, j = 1, \ldots, n$, it follows that $H \supseteq U_n$. Let $b \in B$, $a \in A$, let $t = ba$, and set $t_{ij} = \sum_{k=1}^n e_{ik} t_{kj}$, $i, j = 1, \ldots, n$. Since $t_{ij}$ commutes with the elements of $M$, then $t_{ij} \in D$, $i, j = 1, \ldots, n$. Furthermore

$$t_{ij} = \sum_{k=1}^n (e_{ik} b)(ae_{kj}) \in BA = T,$$

so that

$$t_{ij} \in D \cap T \subseteq D \cap B = P \cap B = U,$$

$i, j = 1, \ldots, n$. This shows that $t = \sum_{i,j=1}^n t_{ij} e_{ij} \in U_n$. Since each element of $T = BA$ is a sum of elements of the form $ba$, it follows that

$$H = A \cap B \supseteq U_n \supseteq T = BA.$$

Finally, we note that

$$BA = T \supseteq H^2 \supseteq (U_n)^2 \supseteq (U^2)_n,$$

proving (2).

(3) $F$ is a right order of $D$, and $F_n \subseteq R$, so clearly $F \subseteq U \subseteq P$. This shows that $U$ and $P$ are right orders in $D$. Since $U^2$ is an ideal of $U$, it follows that $U^2$ is a right order in $D$, since if $d = uv^{-1}$ with $u, v \in U$, and if $0 \neq w \in U^2$, then $d = (uw)(vw)^{-1}$, with $uw, vw \in U^2$.

(4) From (3) it follows that $U_n, U_n^2$, and $P_n$ are right orders in $Q = D_n$. If $0 \neq u \in U$, then $uR \subseteq B$, so that

$$uRu \subseteq BA \subseteq U_n \subseteq P_n.$$

But $U$ is an ideal of $P$, so $u^2P \subseteq U^2$, and therefore

$$u^2Ru \subseteq u^2(P_n) = (u^2P)_n \subseteq U^2_n \subseteq U_n \subseteq P_n.$$

Conversely,

$$u^2(P_n) = (u^2P)_n \subseteq U^2_n \subseteq U_n \subseteq R.$$

Since $u^{-1} \in Q$, the proof of (4) is complete.

(5) From the proof of (4) we have that $uRu \subseteq P_n$, so clearly $R \subseteq P'_n$, and $R[u^{-1}] \subseteq P'_n$. Conversely since $Mu \subseteq R$, then $M \subseteq Ru^{-1} \subseteq R[u^{-1}]$. 

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Since \( P \subseteq R \), it follows that \( P' = P[u^{-1}] \subseteq R[u^{-1}] \), and so \( P_n' \subseteq P'[M] \subseteq R[u^{-1}] \). This proves that \( R[u^{-1}] = P_n' \).

2. Theorem. If \( R \) in Theorem 1 is a simple ring with identity, then:
   (1) \( B, A, BA = T \), and \( U^2 \) are all simple rings.
   (2) \( T = U_n' \).
   (3) If \( 0 \neq u \in U^2 \), then \( P' = P[u^{-1}] \) and \( P_n' \) are simple rings.

Proof. Let \( I \) be a nonzero ideal of \( T = BA \). Then
\[
I \supseteq (BA)I(BA) = B(AIB)A.
\]
Since \( A \cap B \supseteq U \) contains a regular element, clearly \( AIB \neq 0 \). Thus, simplicity of \( R \) forces \( R = AIB \). Therefore \( I \supseteq BRA \supseteq BA \), so \( BA \) is simple.

Already we have seen that
\[
T = BA \supseteq H^2 \supseteq (U_n^2)^2 \supseteq T^2,
\]
where \( H = A \cap B \). Since \( T \) contains the integral domain \( U \), then \( T^2 \neq 0 \), so simplicity of \( T \) yields \( T = T^2 \). It follows that
\[
T = (U_n^2)^2 = U_n^2,
\]
so simplicity of \( U^2 \) follows from that of \( T \). If \( 0 \neq u \in U^2 \), and if \( I \) is a nonzero ideal of \( P' = P[u^{-1}] \), then \( I \cap U^2 \) is a nonzero ideal of \( U^2 \) (since \( U^2 \) is a right order in \( D \)). Thus simplicity of \( U^2 \) implies that \( I \supseteq U^2 \). Since \( u \in I \) is invertible in \( P' = P[u^{-1}] \), then \( I = P' \), so \( P' \) (also \( P_n' \)) is simple.

Next we show that \( B \) (resp. \( A \)) is simple. Let \( I \) be a nonzero ideal of \( B \) (resp. \( A \)). If \( 0 \neq u \in U \), then \( u^3Iu \neq 0 \) and \( u^3Iu \subseteq U_n^2 = T \) by the proof of (4) of Theorem 1. Since \( u \in B \) (resp. \( u \in A \)), it follows that \( u^3Iu \subseteq T \cap I \), so \( T \cap I \) is a nonzero ideal of \( T \). Simplicity of \( T \) forces \( I \supseteq T \). Then \( I \supseteq BAB \) (resp. \( I \supseteq ABA \)). Since \( AB = R \), then \( I \supseteq B \) (resp. \( I \supseteq A \)), proving that \( B \) (resp. \( A \)) is simple.

3. Corollary. Under the hypotheses of the theorem, \( R = P_n \) if and only if \( P \) is a simple ring.

Proof. The necessity is well known. Conversely if \( P \) is simple, then \( P = U \) by (1) of Theorem 1. Consequently, \( 1 \subseteq U \subseteq B \), so \( M = M1 \subseteq R \), and it follows that \( R = P_n \) (since \( P = R \cap D \)).

4. Corollary. Let \( R \) be a right order of \( Q = D_n \), and assume that \( R \) is a simple ring with identity element. (1) If \( z \) is an element of \( Q \) such that \( Rz \subseteq zR \), then \( z \) is invertible in \( Q \) and \( z, z^{-1} \in R \). (2) \( R \) contains the center of \( Q \).

Proof. (1) Since \( R \) is a right order of \( Q \), \( I = zR \cap R \neq 0 \). Thus \( I \) is a nonzero right ideal of \( R \), and the relation \( Rz \subseteq zR \) implies that \( I \) is an
ideal of $R$. Since $R$ is simple, $I = R$, so $zR \supseteq R$. It follows that $z$ is not a left zero divisor in $Q$, and since $Q$ is left artinian, we conclude that $z^{-1} \in Q$. Since $R \supseteq z^{-1}R$, then $z^{-1} \in R$. Now simplicity of $R$ implies that $Rz^{-1} = R$, the ideal of $R$ generated by $z^{-1}$, equals $R$. Since $z^{-1}R \subseteq Rz^{-1}$, we obtain that

$$R = Rz^{-1} \subseteq Rz^{-1} \subseteq R,$$

that is, that $Rz^{-1} = R$. Thus, $(z^{-1})^{-1} = z \in R$, proving (1). (2) is an immediate consequence.

Reference