GEOMETRY OF THE ZEROS OF THE SUMS OF LINEAR FRACTIONS

BY

J. L. WALSH([1])

The object of the present paper is to study the zeros of functions of the form

$$(1) \qquad f(z) \equiv \sum_{k=1}^{m} \frac{a_k}{z-\alpha_k} - \sum_{k=1}^{n} \frac{b_k}{z-\beta_k}, \qquad a_k > 0, \quad b_k > 0,$$

where the α_k and β_k have various geometric configurations as their loci. We investigate also functions of this form where the a_k and b_k are nonreal.

The appropriateness of this study arises from the facts that (i) Lagrange's interpolation formula for a polynomial with prescribed real values in real points α_k and β_k has a factor of precisely form (1), and a similar remark holds for nonreal values and nonreal points; (ii) Riemann sums for a Cauchy integral are of these same forms, in the respective real and nonreal cases; (iii) the logarithmic derivative of a rational function is of form (1), which enables us to study the location of the critical points. Our main theorems (Theorems 1 and 2) refer respectively to the real and nonreal cases just mentioned, where the locus of the α_k and β_k is a line segment with the a_k and b_k real, or a circular disk with the a_k and b_k not necessarily real.

Theorem 2 is a special case of a much more general theorem due to Marden [3], but is proved in detail here particularly because of the applications (i) and (ii), not mentioned by Marden. Namely, the present methods apply also to the case (Theorems 3 and 4) where the locus of the α_k, β_k, etc., is a circumference rather than a disk, a case not included in Marden's treatment yet important precisely for the study of a Cauchy or Cauchy-Stieltjes integral.

As is frequently done [2] in the study of zeros of such functions as (1), we interpret the conjugate of $f(z)$ as the force at z due to repelling particles at the α_k and attracting particles at the β_k, where each particle repels with a force equal to its mass a_k or $-b_k$ times the inverse distance; the original problem of finding the zeros of $f(z)$ is equivalent to the problem of finding the positions of equilibrium in this field of force.

THEOREM 1. *Let the conditions of* (1) *be satisfied, with* $A = \sum a_k > B = \sum b_k$, *and let all* α_k *and* β_k *lie on the interval* $-1 \leqq z \leqq +1$. *Then all nonreal zeros*

Presented to the Society, September 10, 1962; received by the editors March 22, 1963 and, in revised form, July 3, 1963.

([1]) This research was sponsored (in part) by the Air Force Office of Scientific Research.

of $f(z)$ lie in the closed interior of the ellipse

$$Bx^2 + Ay^2 = \frac{AB}{A-B}. \tag{2}$$

All real zeros of $f(z)$ lie in the interval

$$|x| \leqq \frac{A+B}{A-B}. \tag{3}$$

Indeed the sets mentioned constitute the locus of the zeros of $f(z)$ for all $f(z)$ satisfying the hypothesis.

In the field of force already introduced, if a point z_0 is considered, it is frequently convenient (cf. [2]) to replace n positive (or negative) particles α_k (or β_k) by a single *equivalent* particle whose mass is the sum of the masses of the original particles and which exerts the same force at z_0. If the particle α_k is inverted in the unit circle whose center is z_0, the corresponding force at z_0 is represented by the vector from the inverse of α_k to the point z_0 multiplied by the mass of the particle; the total force at z_0 due to all the particles α_k is represented by the vector (weighted by the total mass) from the center of gravity of the weighted inverses to z_0; the equivalent particle of the α_k is located at the inverse of this center of gravity. We often have occasion to use the fact that if a number of initial points of vectors with common terminal point and various weights are given, their center of gravity lies in their convex hull; this center of gravity is the inital point of the vector resultant, weighted by the sum of weights of the given vectors.

With the hypothesis of Theorem 1 we first choose $\mathrm{Im}(z_0) > 0$. For fixed z_0 the particle α_0 equivalent to the given α_k lies in the circular segment $S(z_0)$ bounded by the interval $-1 \leqq x \leqq 1$ and by an arc of the circle through -1, $+1$, and z_0 whose endpoints are $z = +1$ and -1; the arc lies in the closed half-plane $\mathrm{Im}(z) \leqq 0$. This remark follows from the fact that the inverse in the unit circle whose center is z_0 of the interval $-1 \leqq x \leqq +1$ is an arc of a circle through z_0; the convex hull of this arc is a certain segment of a circle whose inverse is $S(z_0)$. Moreover, $S(z_0)$ is the actual *locus* of the equivalent particle when all possible choices of the α_k are considered, not restricted in total number or in respective (positive) masses.

The locus of the particle β_0 equivalent to the β_k is also $S(z_0)$, and z_0 is a position of equilibrium if and only if α_0 and β_0 (in their respective loci) are collinear, with $|z_0 - \alpha_0| / |z_0 - \beta_0| = A/B$. If α_0' and β_0' are any two positions of α_0 and β_0 collinear with z_0 and in their proper loci, say with β_0' on the interval $-1 \leqq x \leqq 1$ and α_0' on the circular arc partially bounding $S(z_0)$, then the ratio $|z_0 - \alpha_0| / |z_0 - \beta_0|$ can be increased by rotating the line $z_0\alpha_0\beta_0$ about z_0, and by sliding β_0 from β_0' along the interval and sliding α_0 from α_0' along the circular arc in the sense so as algebraically to decrease the ordinate of α_0; this increase of the ratio is always possible as long as the abscissa of α_0 is not zero. The maximum of the ratio occurs when β_0 is on the interval, say $\beta_0 = \beta_0''$, and when the abscissa of α_0 is zero, say $\alpha_0 = \alpha_0''$. However, the ratio can take on all values and only values between unity and this maximum inclusive, for suitable choices of α_0 and β_0 in their proper loci and collinear with z_0, $|z_0 - \alpha_0| \geqq |z_0 - \beta_0|$. Thus z_0 can be a position of equilibrium if and only if we have

$$\frac{|z_0 - \alpha_0''|}{|z_0 - \beta_0''|} \geqq \frac{A}{B}. \tag{4}$$

If we set $z_0 = x_0 + iy_0$, and note that the center of the circle an arc of which bounds $S(z_0)$ in part has the ordinate $b = (x_0^2 + y_0^2 - 1)/(2y_0)$, inequality (4) can be rewritten as

$$\frac{y_0 + (1+b^2)^{1/2} - b}{y_0} \geqq \frac{A}{B},$$

which is equivalent to

$$Bx_0^2 + Ay_0^2 \leqq \frac{AB}{A-B};$$

this inequality is valid for both $y_0 > 0$ and $y_0 < 0$, so the proof of the first part of Theorem 1 is complete. It may be noted that the foci of the ellipse (2) are $z = +1$ and -1, and its eccentricity is $[(A-B)/A]^{1/2}$; thus the ellipse corresponding to an arbitrary interval as assigned locus of the α_k and β_k is found at once.

If $z_0 = x_0 + iy_0$ is real and given, say $z_0 > 1$, the maximum of the first member of (4) is $(z_0+1)/(z_0-1)$, and z_0 can be a zero of $f(x)$ if and only if we have

$$\frac{z_0+1}{z_0-1} \geqq \frac{A}{B},$$

so z_0 is a zero of some $f(z)$ if and only if we have

$$z_0 \leqq \frac{A+B}{A-B}.$$

A similar discussion applies if we have $z_0 < -1$.

On the other hand, an arbitrary point of $-1 \leqq x \leqq 1$ belongs to the locus; for instance $z = 0$ is a zero of the particular function

$$f(z) \equiv \frac{A}{z-\alpha_0} - \frac{B}{z-\beta_0}$$

provided we have merely $A\beta_0 = B\alpha_0$, so α_0 and β_0 can be chosen positive and as small as desired. Theorem 1 is established.

Under the hypothesis of Theorem 1 except that now we take $A = B$, the locus of zeros of the totality of the functions $f(z)$ consists nontrivially of the entire plane. Indeed, let z_0 be a given nonreal point of the plane and let z_1 be an interior point of the circular segment $S(z_0)$ already defined. We set

$$f(z) \equiv \frac{1}{z-(z_1+\delta)} + \frac{1}{z-(z_1-\delta)} - \frac{2}{z-(z_1+\epsilon)}, \tag{5}$$

where $|\delta|$ and $|\epsilon|$ (>0) are chosen so small that $z_1 \pm \delta$ and $z_1 + \epsilon$ lie within a circle interior to $S(z_0)$. The function $f(z)$ vanishes when $z = z_1 + \delta^2/\epsilon$, and ϵ and δ can be so chosen that this number is z_0. This discussion does not apply if z_0 is real, but in that case a slight modification of the discussion of (5) does apply, and shows that z_0 belongs to the locus of zeros of all $f(z)$.

As an application of Theorem 1 we formulate

COROLLARY 1. *Let $r(z)$ be a rational function of z whose finite zeros and poles lie on the segment $-1 \leqq z \leqq +1$, of respective total orders A and B or B and A, $A > B$. Then all finite nonreal critical points of $r(z)$ lie in the closed interior of the ellipse* (2), *and all finite real critical points lie in the closed interval* (3).

The logarithmic derivative of $r(z)$ is of form (1), where the α_k are the zeros of $r(z)$ and the β_k are the poles, each enumerated a number of times according to its multiplicity, and where all a_k and b_k are unity. The corollary follows from Theorem 1.

As a second application we have

COROLLARY 2. *Let $f(z)$ be defined by the Stieltjes integral*

$$f(z) = \int_{-1}^{1} \frac{d\sigma(t)}{t-z}, \qquad -1 \leqq t \leqq 1,$$

where the total positive variation of $\sigma(t)$ on $-1 \leqq t \leqq 1$ is A and the total negative variation is $-B$, $A > B$. Then all finite nonreal zeros of $f(z)$ lie in the closed interior of the ellipse (2), *and all real zeros lie in the closed interval* (3).

The proof of Corollary 2 follows by considering the partial sums approximating the Stieltjes integral, and by Theorem 1. If the total negative variation of $\sigma(t)$ is greater than the total positive variation it suffices to consider the zeros of $-f(z)$.

In the proof of theorems such as Theorem 1 on the geometry of zeros of functions, two methods of proof are frequently used: (i) study of the loci of particles equivalent to various categories of particles; (ii) study of the total forces due to various categories of particles. We have just employed method (i), and now proceed to use method (ii) in a different problem.

THEOREM 2 (MARDEN). *Let the function $f(z)$ be of the form*

$$f(z) \equiv \sum_{k=1}^{m} \frac{a_k}{z-\alpha_k} - \sum_{k=1}^{n} \frac{b_k}{z-\beta_k} + \sum_{k=1}^{p} \frac{ic_k}{z-\gamma_k} - \sum_{k=1}^{q} \frac{id_k}{z-\delta_k},$$

where all the $a_k, b_k, c_k,$ *and* d_k *are non-negative. We set* $A = \sum a_k, B = \sum b_k,$ $C = \sum c_k, D = \sum d_k,$ *and suppose* $(A-B)^2 + (C-D)^2 \neq 0$. *If* $\Gamma: |z| \leqq 1$ *is the simultaneous locus of the points* $\alpha_k, \beta_k, \gamma_k,$ *and* $\delta_k,$ *for all* a_k, b_k, c_k, d_k *satisfying the conditions given, then the locus of the zeros of* $f(z)$ *is the disk*

$$|z| \leqq \frac{A+B+C+D}{[(A-B)^2+(C-D)^2]^{1/2}}. \tag{6}$$

We continue to interpret the conjugate of $f(z)$ as defining a field of force in the z-plane. If z_0 is a zero of $f(z)$, then ωz_0 with $|\omega| = 1$ is a zero of $f(z)$ with the original $\alpha_k, \beta_k, \gamma_k, \delta_k$ replaced by $\omega\alpha_k, \omega\beta_k, \omega\gamma_k, \omega\delta_k,$ so it is sufficient for us to study $z_0 = a$, real; we take $a > 1$, and then we make a translation of the plane so that Γ becomes Γ_1: $|z+a| \leqq 1$ and z_0 becomes $z_1 = 0$. The inverse of Γ_1 in the unit circle whose center is z_1 is

$$\left| z + \frac{a}{a^2-1} \right| \leqq \frac{1}{a^2-1},$$

and the force exerted at z_1 due to all the particles α_k is represented by a vector with initial point z_1 and terminal point in the disk

$$C_1: \quad \left| z - \frac{aA}{a^2-1} \right| \leqq \frac{A}{a^2-1};$$

in fact C_1 is the *locus* of the terminal points of such vectors for all possible choices of the a_k and α_k, with A fixed; compare [2, p. 13]. The "disk" C_1 is represented by the formula given even if $A = 0$.

Likewise, the locus of the terminal points of the vectors with initial points in z_1 and representing the force at z_1 for all possible choices of the b_k and β_k with B fixed is the disk

$$C_2: \quad \left| z + \frac{aB}{a^2-1} \right| \leqq \frac{B}{a^2-1}.$$

The locus of the terminal points of the vectors with initial points in z_1 representing the total force at z_1 due to the particles at the α_k and β_k is the disk which is the "sum" of C_1 and C_2:

$$\left| z - \frac{a(A-B)}{a^2-1} \right| \leqq \frac{A+B}{a^2-1}, \tag{7}$$

in the sense that if C_1 and C_2 are the loci of z_1 and z_2 then (7) is the locus of $z_1 + z_2$.

By a similar method, and with the note that the conjugate of $f(z)$ defines the forces, it follows that the locus of the terminal points of the vectors with initial points in z_1 representing the total force at z_1 due to the particles at the γ_k and δ_k is the disk

$$\left| z + \frac{ia(C-D)}{a^2-1} \right| \leqq \frac{C+D}{a^2-1}. \tag{8}$$

For the two sets of forces we have vectors with initial points in z_1 $(=0)$ and terminal points whose loci are the respective disks (7) and (8). The total resultant force is represented by a vector whose initial point is z_1 and the locus of whose terminal points lies in the disk

$$\left| z - \frac{a(A-B) - ia(C-D)}{a^2-1} \right| \leqq \frac{A+B+C+D}{a^2-1}. \tag{9}$$

A necessary and sufficient condition that z_1 be a possible position of equilibrium is that the total force may be zero, or that z_1 $(=0)$ should lie in the disk (9), namely

$$\left| \frac{a(A-B) - ia(C-D)}{a^2-1} \right| \leqq \frac{A+B+C+D}{a^2-1}, \qquad a = |z_0| > 1,$$

which is essentially (6). The second member of (6) is greater than unity unless three of the four numbers A, B, C, D are zero.

If z_0 is a zero of $f(z)$ for a particular choice of the α_k etc., and if $0 < \rho < 1$, the point ρz_0 is a zero of $f(z)$ with the a_k, b_k, c_k, d_k unchanged and the α_k, $\beta_k, \gamma_k, \delta_k$ multiplied by ρ. Moreover $z_0 = 0$ is a zero of $f(z)$ with suitably chosen $\alpha_k, \beta_k, \gamma_k, \delta_k$ small in modulus, and this completes the proof of Theorem 2.

Corollary 1. *If $f(z)$ in Theorem 2 is of the form*

$$f(z) \equiv \sum_{k=1}^{m} \frac{a_k}{z-\alpha_k} + \sum_{k=1}^{n} \frac{ic_k}{z-\gamma_k}, \qquad A + C \neq 0,$$

the locus of its zeros is the disk

$$|z| \leqq \frac{A+C}{(A^2+C^2)^{1/2}}.$$

Corollary 2. *If $f(z)$ in Theorem 2 is of the form*

$$f(z) \equiv \sum_{k=1}^{m} \frac{a_k}{z-\alpha_k} - \sum_{k=1}^{n} \frac{b_k}{z-\beta_k}, \qquad A > B, \tag{10}$$

the locus of its zeros is the disk

$$|z| \leqq \frac{A+B}{A-B}. \tag{11}$$

If $r(z)$ is a rational function not identically constant, and if the exact degrees of its numerator and denominator are A and B, its logarithmic derivative is of form (10); the conclusion of Corollary 2 is essentially that all zeros of the derivative lie in the disk (11), a result [1] proved by the present author in 1918.

If $f(z)$ is multiplied by ω with $|\omega| = 1$, the zeros of the new function $f_1(z)$ $(\equiv \omega f(z))$ are unchanged, yet the second member of (6) is not unchanged by such an arbitrary transformation; indeed the denominator in the second member of (6) is precisely $|A-B+iC-iD|$, which is invariant, but the numerator is not invariant. This seeming paradox is resolved if we consider for instance the special case $B = C = D = 0$. The original theorem refers to the zeros of $\sum a_k(z-\alpha_k)^{-1}, a_k > 0$ whereas if we set $\omega = \cos\theta + i\sin\theta$, $0 < \theta < \pi/2$; the new function $f_1(z)$ is to be written $\sum a_k(\cos\theta + i\sin\theta) \cdot (z-\alpha_k)^{-1}, \sum a_k = A$, which is quite different from the function

$$\sum a_k'(z-\alpha_k)^{-1} + \sum ic_k'(z-\beta_k)^{-1}$$

for all a_k', c_k' having prescribed sums $\sum a_k' = A\cos\Theta, \sum c_k' = A\sin\Theta$.

As a consequence of the facts just discussed, we formulate the following

REMARK. *In the application of Theorem* 2 *we may replace* $f(z)$ *by* $\omega f(z)$, *where* ω *is a constant of modulus unity*; *this change may modify the second member of* (6). *In particular if* $f(z)$ *can be written so as to contain one or more terms of the form*

$$\frac{\lambda_k}{z-\zeta_k}$$

where $\arg\lambda_k$ *is independent of k, then as far as those terms are concerned it is favorable to choose* $\arg\omega = -\arg\lambda_k$.

Corollary 2 to Theorem 1 has an analogue here, concerning the integral

$$\phi(z) \equiv \int_\gamma \frac{d\alpha(t)}{t-z}.$$

If γ lies in the closed interior of the unit circle, if $\alpha(t) = \alpha_1(t) + i\alpha_2(t)$ where $\alpha_1(t)$ and $\alpha_2(t)$ are real, and if A and $-B$, and C and $-D$, are the respective total positive and negative variations of $\alpha_1(t)$ and $\alpha_2(t)$ on γ, and if $(A-B)^2 + (C-D)^2 \neq 0$, then all zeros of the approximating sums of $\phi(z)$ (which approach $\phi(z)$) lie in the closed interior of a variable disk that approaches (6), so by Hurwitz's theorem all zeros of $\phi(z)$ lie in the closed interior of (6).

The remarks just made concerning $\phi(z)$ suggest the study of the hypothesis of Theorem 2 except that now the α_k and β_k are required to lie *on* the unit circumference γ. If the locus of positive particles α_k is γ, and if z_0 lies exterior to γ, the locus of the equivalent particle is the closed interior of γ, as becomes obvious at once by inversion in the unit circle whose center is z_0. If the locus of these α_k is γ and z_0 lies interior to γ, the locus of the equivalent particle is the closed exterior of γ including the point at infinity. If z_0 lies interior to γ, we may consider the equivalent particles for each category of particles to lie at infinity, whence $f(z_0) = 0$. We have

THEOREM 3. *Let the hypothesis of Theorem* 2 *be modified so that all particles* $\alpha_k, \beta_k, \gamma_k, \delta_k$ *lie on* γ: $|z| = 1$. *As far as concerns points z not on* γ, *the locus of the zeros z of* $f(z)$ *is the disk* (6).

This proof of Theorem 3 involves essentially applying the method of proof of Theorem 2, but not applying Theorem 2 itself.

To study the points z on γ, we consider (as in the proof of Theorem 2) the actual forces at z_0 due to the various categories of particles, and the locus of the terminal points of the vectors representing these forces, when the initial points lie in z_0. We omit the assumption $(A - B)^2 + (C - D)^2 \neq 0$. As before, let us choose z_0 positive and then translate the plane, so that γ becomes $|z + 1| = 1$ and z_0 becomes $z_1 = 0$. The inverse of γ in the unit circle whose center is z_1 is the line (better, the finite points of the line) $x = -1/2$, and the locus of the terminal points of all vectors each corresponding to a set of particles α_k is the line $x = A/2$ unless $A = 0$. The locus of the terminal points of all vectors each corresponding to a set of particles β_k is the line $x = -B/2$ unless $B = 0$, and for the composition of a pair of these vectors we have as locus the line $x = (A - B)/2$; however, it is to be noted that all vectors are null vectors if we have $A = B = 0$. The loci for the vectors corresponding to the γ_k and δ_k are respectively the lines $y = -C/2$ and $y = D/2$ unless $C = 0$ or $D = 0$; for the composition of a pair of these vectors we have as locus of the terminal points the line $y = -(C - D)/2$, except if $C = D = 0$. The locus for the negatives of these last mentioned vectors having their inital points in z_1 is $y = (C - D)/2$, which always intersects the line $x = (A - B)/2$; the total sum of all vectors is null for a suitable configuration depending on given A, B, C, D, with the exceptions noted.

THEOREM 4. *With the hypothesis of Theorem* 3, *the locus of the zeros of* $f(z)$ *contains the entire circumference* γ *provided we have* $A + B \neq 0$ *and* $C + D \neq 0$. *The locus contains the entire circumference also if* $A = B, C + D = 0$, *or if* $A + B = 0, C = D$. *The locus contains no point of* γ *if* $A \neq B, C + D = 0$ *or if* $A + B = 0, C \neq D$.

The case $A = B = C = D = 0$ is of course trivial.

It is of interest to indicate how Theorems 1 and 2 apply to the study of zeros of restricted infrapolynomials; for these methods, compare [4], [5]. The category of restricted infrapolynomials on a set E as used here includes the category of similarly restricted polynomials of least norm on E, where norm is in the sense of least weighted pth powers ($p > 0$) or in the sense of (weighted) Tchebycheff.

THEOREM 5. *Let the two disjoint point sets* $\alpha_1, \alpha_2, \cdots, \alpha_m$ *and* $\beta_1, \beta_2, \cdots, \beta_n$ *consist of the distinct points indicated and lie on* $-1 \leqq z \leqq 1$. *Let the real polynomial* $P(z) \equiv Nz^{m+n-1} + \cdots$ *have the coefficient* N *prescribed, and also the* (*real*) *values* $P(\alpha_j)$, *and be a thus restricted infrapolynomial* (*i.e., have no restricted underpolynomial*) *on* E: $\{\beta_j\}$. *Set* $\sum_1^m [P(\alpha_j)/\omega'(\alpha_j)] = N_0$, *where* $\omega(z) \equiv \prod_1^m(z-\alpha_j) \cdot \prod_1^n(z-\beta_j)$. *Let* A *and* $-B$ *be the sum of the positive and negative numbers respectively among* $P(\alpha_j)/\omega'(\alpha_j), N-N_0$; *we suppose* $A > B$. *Then all nonreal zeros of* $P(z)$ *lie in the closed interior of the ellipse* (2), *and all real zeros lie in the interval* (3).

The polynomial $P(z)$ can be expressed by the Lagrange formula

$$P(z) \equiv \omega(z) \sum_1^m \frac{P(\alpha_j)}{\omega'(\alpha_j)(z-\alpha_j)} + \omega(z) \sum_1^n \frac{P(\beta_j)}{\omega'(\beta_j)(z-\beta_j)}; \tag{13}$$

here the coefficients $P(\alpha_j)$ are prescribed, and the coefficients $P(\beta_j)$ are not prescribed, but are subject to the condition

$$\sum_1^n P(\beta_j)/\omega'(\beta_j) = N - N_0. \tag{14}$$

It is then clear that for $P(z)$ thus restricted to be an infrapolynomial on E the condition

$$\mathrm{sg}[P(\beta_j)/\omega'(\beta_j)] = \mathrm{sg}(N-N_0), \qquad j = 1, 2, \cdots, n, \tag{15}$$

is necessary and sufficient. Indeed, if (15) is satisfied, there exists no restricted underpolynomial $Q(z)$ of $P(z)$ on E, for there exists no set of values $Q(\beta_j)$ with

$$\sum_1^n Q(\beta_j)/\omega'(\beta_j) = N - N_0 \tag{16}$$

such that

$$|Q(\beta_j)| < |P(\beta_j)| \quad \text{if} \quad P(\beta_j) \neq 0 \tag{17}$$

and

$$Q(\beta_j) = 0 \quad \text{if} \quad P(\beta_j) = 0. \tag{18}$$

Conversely, if $P(z)$ is a restricted infrapolynomial and if we have both (14) and $\sum_1^n |P(\beta_j)/\omega'(\beta_j)| > |N - N_0|$, then we can set

$$\frac{Q(\beta_j)}{\omega'(\beta_j)} = \frac{|P(\beta_j)/\omega'(\beta_j)| \cdot |N-N_0|}{\sum_1^n |P(\beta_j)/\omega'(\beta_j)|},$$

whence (16), (17), and (18) are valid and $Q(z)$ is an underpolynomial of $P(z)$ on E.

By virtue of (13) with (14) and (15), Theorem 5 now follows from Theorem 1. If a given polynomial $P(z)$ satisfies all the requirements of Theorem 5 except that now $A < B$, we need merely reverse the signs of the $P(\alpha_j)$ and of N to apply Theorem 5 as stated. But we draw no conclusion if $A = B$, namely if $N = 0$.

A similar application of Theorem 2, still by use of equations (13), (14), and (15), yields

Theorem 6. *Let the two disjoint point sets* $\alpha_1, \alpha_2, \cdots, \alpha_m$ *and* $\beta_1, \beta_2, \cdots, \beta_n$ *consist of the distinct points indicated, and lie in the disk* Γ: $|z| \leqq 1$. *Let the polynomial* $P(z) \equiv Nz^{m+n-1} + \cdots$ *have the coefficient* N *prescribed, and also the values* $P(\alpha_j)$, *and be a thus restricted infrapolynomial on* E: $\{\beta_j\}$. *Let* N_0 *and* $\omega(z)$ *be as defined in Theorem* 5. *Let* A, B, C, D *be respectively the sum of the positive numbers among* $\mathrm{Re}[S], \mathrm{Re}[-S], \mathrm{Re}[-iS], \mathrm{Re}[iS]$, *where* S *is the set* $\{P(\alpha_j)/\omega'(\alpha_j), N - N_0\}$, *and where we suppose* $(A-B)^2 + (C-D)^2 \neq 0$. *Then all zeros of* $P(z)$ *lie in the disk* (6).

In connection with Theorem 6, the remark following the proof of Theorem 2 is significant.

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Harvard University,
Cambridge, Massachusetts