TWO TOPOLOGICAL PROBLEMS CONCERNING INFINITE-DIMENSIONAL NORMED LINEAR SPACES(1)

BY

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Introduction. In the present work, we shall study the following two closely related topological problems concerning infinite-dimensional normed linear spaces.

Problem I. Given a closed subset K of an infinite-dimensional normed linear space E, when is $E \sim K$ homeomorphic to E?

Problem II. Given a closed subset K of an infinite-dimensional normed linear space E, when does there exist a periodic homeomorphism of E onto E with K as its set of fixed points?

Using the fact that every nonreflexive Banach space contains a decreasing sequence of nonempty bounded closed convex subsets with empty intersection, Klee [11] proved that if K is a compact subset of a nonreflexive Banach space E, then E is homeomorphic to $E \sim K$. Later [13], he showed that every infinite-dimensional normed linear space contains a decreasing sequence of unbounded but linear bounded (for the definition, see §1) nonempty closed convex subsets with empty intersection. He used this result to prove that every infinite-dimensional normed linear space E is homeomorphic to $E \sim K$ where K is an arbitrary compact subset of E. As a consequence [13], if C is the unit cell of an infinite-dimensional normed linear space E, then there exists a homeomorphism i of C onto a closed half-space J in E such that $i(\operatorname{Bd} C) = \operatorname{Bd} J$. Klee [11] also proved that if E is either a nonreflexive strictly convexifiable Banach space or an infinite-dimensional l_p-space, then Q is homeomorphic to $Q \cup K$ where K is a compact convex subset of the bounding hyperplane of an open half-space Q in E.

Concerning the set of fixed points of a periodic homeomorphism of a topological space into itself, a classical result of Smith [19] states that if M is a finite-dimensional locally compact space, acyclic mod p where p is

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a prime, then the set of fixed points of every homeomorphism of period p of M into M is acyclic mod p. For infinite-dimensional Hilbert space, the situation is quite different. A result of Klee [14] asserts that if K is a compact subset of an infinite-dimensional Hilbert space E, then for each integer $n > 1$, there exists a homeomorphism of period n of E onto E with K as its set of fixed points.

We shall continue the study in these directions and strengthen some of the known results. We first prove that in every infinite-dimensional normed linear space there exists a sequence $\{C_n\}$ of unbounded but linearly bounded closed convex bodies with empty intersection and $\text{Int.}\, C_n \supset C_{n+1}$ for each $n = 0, 1, 2, \cdots$. Then we are able to prove (Theorem 1.3) that if K is a bounded closed convex subset of an infinite-dimensional normed linear space E, then there exists a homeomorphism i of $E \times [0,1]$ onto $(E \times (0,1]) \cup ((E \sim K) \times \{0\})$ such that $i(E \times \{0\}) = (E \sim K) \times \{0\}$ and $i(x,1) = (x,1)$ for all x in E. Applying a result of Bartle and Graves [2] and Theorem 1.3, we shall show in Theorem 3.5 that for every closed subset K of a closed linear subspace F of infinite deficiency in a Banach space E, $E \sim K$ is homeomorphic to E. If, in addition, Q is an open half-space in E such that the bounding hyperplane of Q contains F, then Q is homeomorphic to $Q \cup K$. Using these properties of an infinite-dimensional Banach space E, we can show (Theorem 5.3) that for every closed subset K of a closed linear subspace of infinite deficiency in E and for each positive integer n, there exists a homeomorphism of period $2n$ of E onto E with K as its set of fixed points; in case E is a Hilbert space, then for each integer $n > 1$, there exists a homeomorphism of period n of E onto E with K as its set of fixed points. Now, for every compact subset K of an infinite-dimensional Hilbert space E, there exists a homeomorphism i of E onto E such that $i(K)$ is contained in a closed linear subspace of infinite deficiency in E [12], hence Klee's result concerning the set of fixed points of a periodic homeomorphism, mentioned in the last paragraph, is a consequence of Theorem 5.3.

The above results concerning Problems I, II deal with those sets K which are "small" in the sense that K is a closed subset of a closed linear subspace of infinite deficiency in a Banach space. Next, we shall consider those sets K which are "large" in the sense that K is a closed convex body of an infinite-dimensional closed linear subspace of a Banach space E. For a closed convex body K in an infinite-dimensional Banach space E, the characteristic cone $\{y \in E \mid x + [0, \infty)(y - x) \subset K\}$ of K relative to $x \in K$ is either a linear variety of finite deficiency, a linear variety of infinite deficiency, or not a linear variety. We shall show in Theorems 2.5, 2.6 that, if the characteristic cone of a closed convex body K of an infinite-dimensional Banach space E is not a linear variety of finite deficiency in

E, then there exists a homeomorphism i of E onto E such that $i(K)$ is a closed half-space J in E and $i(\mathrm{Bd}\,K)$ is the bounding hyperplane of J. Hence, by using Theorem 1.3, we can prove (Theorem 3.1) that if F is an infinite-dimensional closed linear subspace of a Banach space E and K is a closed convex body in F such that the characteristic cone of K is not a linear variety of finite deficiency in E, then E is homeomorphic to $E \sim K$. In case the characteristic cone of K is a linear variety of finite deficiency, E is not necessarily homeomorphic to $E \sim K$. For example, E is not homeomorphic to $E \sim H$, where H is a closed hyperplane in E. On the other hand, if K is a closed convex body on the bounding hyperplane of an open half-space Q of an infinite-dimensional Banach space E, it is unknown whether or not Q and $Q \cup K$ are homeomorphic. In particular, it is unknown whether or not Q is homeomorphic to $Q \cup \mathrm{Bd}\,Q$. Since $Q \cup \mathrm{Bd}\,Q$ is homeomorphic to the unit cell of E (Theorem 2.4), this is equivalent to the following problem: When is an infinite-dimensional Banach space homeomorphic to its unit cell? This leads to the following question: Let K be a closed convex body of a closed linear subspace of finite deficiency in an infinite-dimensional Banach space E. When is K homeomorphic to E? Klee [11] showed that every infinite-dimensional Hilbert space is homeomorphic to its unit cell and unit sphere. Later [16], he showed that if a Banach space E contains an h-compressible proper closed linear subspace, then E is homeomorphic to its unit cell. In Theorem 4.3, we show that if a Banach space E contains a closed linear subspace B of infinite deficiency such that B is homeomorphic to l_2, if K is a closed convex body of a closed linear subspace F of finite deficiency in E, then K is homeomorphic to E and $\mathrm{Bd}_F K$ is homeomorphic either to E or $E \times S_n$ for some non-negative integer n, where S_n is the n-sphere. Concerning Problem II, Theorem 5.2 states that if F is an infinite-dimensional closed linear subspace of a Banach space E and K is a closed convex body in F such that the characteristic cone of K is not a linear variety of finite deficiency in F, then, for each positive integer n, there exists a homeomorphism of period $2n$ of E onto E with $\mathrm{Bd}_F K$ as its set of fixed points; in case E is an infinite-dimensional Hilbert space, then, for each integer $n > 1$, there exists a homeomorphism of period n of E onto E with K as its set of fixed points.

Notations. All topological vector spaces considered are real vector spaces with the Hausdorff property. The field of real numbers is denoted by R, $[m,n] = \{\alpha \in R \mid m \leqq \alpha \leqq n\}$, $(m,n] = \{\alpha \in R \mid m < \alpha \leqq n\}$, and $(m,n) = \{\alpha \in R \mid m < \alpha < n\}$. For x, y in a vector space E,

$$(x,y] = \{\alpha x + (1-\alpha)y \mid 0 \leqq \alpha < 1\}.$$

The empty set is denoted by $\emptyset$. $E \sim A = \{x \in E \mid x \notin A\}$. For a subset K of a subspace F of a topological space E, $\mathrm{Int.}\,K$ will denote the interior of

K in E, $\overline{K}$ will denote the closure of K in E and $\mathrm{Bd}_F K$ will denote the boundary of K relative to F. For two topological spaces E, F, $E \approx F$ will mean that E is homeomorphic to F.

1. **Preliminary theorems.** A closed convex subset K of an Euclidean space is bounded if and only if its intersection with each line is bounded. This is not true in an infinite-dimensional normed linear space. The following terminology is introduced by Klee [12].

A subset of a linear space is said to be *linearly bounded* if its intersection with each line is a bounded subset.

THEOREM 1.1. *In every infinite-dimensional normed linear space E, there exists a sequence $\{C_n\}$ of unbounded but linearly bounded closed convex bodies such that $\bigcap_{n=0}^{\infty} C_n = \emptyset$ and* $\mathrm{Int}.C_n \supset C_{n+1}$, $C_n \neq \emptyset$, *for each* $n = 0, 1, 2, \cdots$.

Proof. *Case* 1. E is separable.

By a theorem of Klee [13], there exists a sequence of continuous linear functionals $\{f_i\}$, $\bigcap_{i=1}^{\infty} f_i^{-1}(0) = \{0\}$, $\|f_i\| = 1$ for each $i = 1, 2, \cdots$ and a sequence of positive integers $\{n_i\}$ such that the closed convex set $D_0 = \bigcap_{i=1}^{\infty} f_i^{-1}([-n_i, n_i])$ is an unbounded but linearly bounded subset of E. By the uniform boundedness principle [6], E admits a continuous linear functional f, $\|f\| = 1$, such that $\sup_{D_0} f = \infty$. Let $D_n = D_0 \cap f^{-1}[n, \infty)$ for $n = 1, 2, \cdots$. Then $\{D_n\}$ is a decreasing sequence of unbounded but linearly bounded closed convex sets with empty intersection.

For a given $\epsilon > 0$ and a subset A of E, let $N_\epsilon A = \{x \in E \mid \|x - a\| < \epsilon$ for some a in $A\}$. It is clear that if A is a convex subset of E, then $N_\epsilon A$ is also a convex subset.

We claim that for any ϵ, $0 < \epsilon < \infty$, $N_\epsilon D_n$ is a linearly bounded subset of E for each $n = 0, 1, 2, \cdots$.

For an arbitrary x in E, there is an integer i such that $f_i(x) = \delta \neq 0$. Choose m so that $m|\delta| > n_i + \epsilon$. Then for any α, $|\alpha| \geqq m$, and any $y \in f_i^{-1}([-n_i, n_i])$ we have $\epsilon < m|\delta| - n_i \leqq |\alpha||\delta| - n_i \leqq |\alpha\delta| - |f_i(y)| \leqq |f_i(\alpha x - y)| \leqq \|f_i\| \cdot \|\alpha x - y\| = \|\alpha x - y\|$. Since $D_n \subset f_i^{-1}([-n_i, n_i])$, this implies that αx is not in $N_\epsilon D_n$ for all α, $|\alpha| \geqq m$. Since $N_\epsilon D_n$ is convex, it follows easily that $N_\epsilon D_n$ is a linearly bounded subset for each $n = 0, 1, 2, \cdots$.

Let $C_n = \overline{N_{1/2^n} D_n}$, $n = 0, 1, 2, \cdots$. It is easy to prove that $C_n \subset N_{1/2^{n-1}} D_n$. We have shown that $N_\epsilon D_n$ is linearly bounded for each $n = 0, 1, 2, \cdots$ and each $\epsilon > 0$. Hence $\{C_n\}$ is a sequence of unbounded but linearly bounded closed convex bodies, $\mathrm{Int}.C_n \supset C_{n+1}$ for each $n = 0, 1, 2, \cdots$. It remains to show that $\bigcap_{n=0}^{\infty} C_n = \emptyset$.

Suppose $x \in \bigcap_{n=0}^{\infty} C_n$. For each integer $m \geqq 0$, there exist $y_{mn} \in N_{1/2^n} D_n$ such that $\|x - y_{mn}\| < 1/2^m$ for each $n = 0, 1, 2, \cdots$. $y_{mn} \in N_{1/2^n} D_n$ implies that there exists $z_{mn} \in D_n$ such that $\|y_{mn} - z_{mn}\| < 1/2^n$. Hence $\|x - z_{nn}\|$

$\leqq \|x - y_{nn}\| + \|y_{nn} - z_{nn}\| < 1/2^n + 1/2^n = 1/2^{n-1} \to 0$ as $n \to \infty$. Since $\{D_n\}$ is a decreasing sequence of closed subsets and $z_{nn} \in D_n$ for each $n = 0, 1, 2, \cdots$, this implies that $x \in D_n$ for each $n = 0, 1, 2, \cdots$, contradiction.

Case 2. E is not separable.

Let F be a separable closed linear subspace of E. By Case 1, there exists a sequence of unbounded but linearly bounded closed convex subsets $\{D_n\}$ of F with empty intersection. For each $x \in E \sim F$, by the Hahn-Banach theorem, there is a continuous linear functional g on E such that $g(F) = 0$, $g(x) = \delta \neq 0$. By an argument similiar to the argument of Case 1, it can be shown that for each ϵ, $0 \leqq \epsilon < \infty$, $N_\epsilon D_n = \{x \in E \mid \|x - a\| < \epsilon$ for some a in $D_n\}$ is a linearly bounded subset of E, for each $n = 0, 1, 2, \cdots$. Let $C_n = \overline{N_{1/2^n} D_n}$, then $\{C_n\}$ is a sequence of unbounded but linearly bounded closed convex bodies in E with empty intersection and $\mathrm{Int}.C_n \supset C_{n+1}$ for each $n = 0, 1, 2, \cdots$. Q.E.D.

The following concept introduced by Steinitz and also studied by Stoker [20] is useful in classifying the closed convex subsets in a topological vector space (see §2): Let A be a closed convex subset of a topological vector space E and $x \in A$. The *characteristic cone of A relative to x* is the set $\mathrm{cc}(A; x) = \{y \in E \mid x + [0, \infty)(y - x) \subset A\}$. It is clear that if $x, y \in A$, then $\mathrm{cc}(A; x) = \mathrm{cc}(A; y) + (x - y)$. Thus if no confusion is possible, we shall simply speak of the characteristic cone of A.

We need the following lemma of Corson and Klee [4] in proving Theorem 1.3.

LEMMA 1.2. *Suppose E_i, $i = 1, 2$, are topological vector spaces, A_i, B_i are closed convex bodies in E_i and $y_i \in E_i$ such that $y_i \in \mathrm{Int}.A_i \subset A_i \subset \mathrm{Int}.B_i$ and $\mathrm{cc}(A_i; y_i) = \mathrm{cc}(B_i; y_i)$. Then every homeomorphism h of $\mathrm{Bd}\, B_1$ onto $\mathrm{Bd}\, B_2$ can be extended to a homeomorphism k of $B_1 \sim \mathrm{Int}.A_1$ onto $B_2 \sim \mathrm{Int}.A_2$ such that $k(x) \in (y_2, h(z)]$ whenever $z \in \mathrm{Bd}\, B_1$ and $x \in (y_1, z] \sim \mathrm{Int}.A_1$.*

THEOREM 1.3. *For every bounded closed convex subset K of an infinite-dimensional normed linear space E, there exists a homeomorphism i of $E \times [0, 1]$ onto $(E \times (0, 1]) \cup ((E \sim K) \times \{0\})$ such that $i(x, 1) = (x, 1)$ for all x in E and $i(E \times \{0\}) = (E \sim K) \times \{0\}$.*

Proof. By Theorem 1.1, there exists a sequence $\{C_n\}$ of unbounded but linearly bounded closed convex bodies such that $\mathrm{Int}.C_n \supset C_{n+1}$ for each $n = 0, 1, 2, \cdots$ and $\bigcap_{n=0}^{\infty} C_n = \emptyset$. By considering $\{\alpha C_n\}$, for some $\alpha \in R$, instead of $\{C_n\}$, if necessary, we may assume that $K \subset \mathrm{Int}.C_0$.

Let $F = E \times R$ be the product space of E and R. Define

$$A_n = C_n \times \left[-\frac{1}{n+1}, \frac{1}{n+1} \right], \qquad n = 0, 1, 2, \cdots.$$

Choose $y_n \in \mathrm{Int}.C_n$, $n = 0, 1, 2, \cdots$. Then the sequence $\{A_n\}$ has the following properties:

A_n is a closed convex body in F, $(y_n, 0) \in \mathrm{Int}.A_n$ and $\mathrm{Int}.A_n \supset A_{n+1}$ for all $n = 0, 1, 2, \cdots$. $\mathrm{cc}(A_i, (y_n, 0)) = (y_n, 0)$ for each $0 \leqq i \leqq n$. $\bigcap_{n=0}^{\infty} A_n = \emptyset$.

Now, let $B_0 = A_0$. Let ϵ be a positive number such that B_0 contains the 2ϵ-neighborhood $N_{2\epsilon}(K \times \{0\})$ of $K \times \{0\}$. For $n \geqq 1$, let $B_n = \overline{N_{\epsilon/n}(K \times \{0\})}$. Let $z_n = (0, 0)$ for all $n = 0, 1, 2, \cdots$. Then the sequence $\{B_n\}$ has the following properties:

B_n is a closed convex body in F, $z_n \in \mathrm{Int}.B_n$, $\mathrm{Int}.B_n \supset B_{n+1}$ for all $n = 0, 1, 2, \cdots$. $\mathrm{cc}(B_i, z_n) = z_n$ for all $0 \leqq i \leqq n$ and $\bigcap_{n=0}^{\infty} B_n = K \times \{0\}$.

Let i_0 be the identity mapping of $F \sim \mathrm{Int}.A_0$. Then $i_0(\mathrm{Bd}\, A_0) = \mathrm{Bd}\, B_0$. By Lemma 1.2, $i_0|\mathrm{Bd}\, A_0$ can be extended to a homeomorphism i_1 of A_0 $\sim \mathrm{Int}.A_1$ onto $B_0 \sim \mathrm{Int}.B_1$ which takes points of $E \times \{0\}$ into $E \times \{0\}$. Continuing in this way, we obtain a sequence of homeomorphisms $i_0, i_1, i_2, \cdots$ such that $i_n(A_{n-1} \sim \mathrm{Int}.A_n) = B_{n-1} \sim \mathrm{Int}.B_n$ for each $n = 1, 2, 3, \cdots$. Let $i = \bigcup_{n=0}^{\infty} i_n$. Since $\bigcap_{n=0}^{\infty} A_n = \emptyset$, $\bigcap_{n=0}^{\infty} B_n = K \times \{0\}$, it follows that i is a homeomorphism of F onto $F \sim (K \times \{0\})$. The restriction of i on $E \times [0,1]$ is a homeomorphism of $E \times [0,1]$ onto $(E \times (0,1]) \cup ((E \sim K) \times \{0\})$ such that $i(x,1) = (x,1)$ for all x in E and $i(E \times \{0\}) = (E \sim K) \times \{0\}$. Q.E.D.

2. **Classification of closed convex bodies.** We begin with a result of Bartle and Graves [2], extended to the present form (Lemma 2.1) by Michael [18]. A consequence (Corollary 2.3) of the result is needed in the proofs of Theorem 2.5 and most of the results in §§3, 4, 5.

LEMMA 2.1 (BARTLE-GRAVES). *Let f be a continuous linear mapping of a Banach space E onto a Banach space F. Then there exist a positive constant m and a continuous mapping g of F into E such that $f \circ g(x) = x$, $g(\alpha x) = \alpha g(x)$, $\|g(x)\| \leqq m\|x\|$ for all x in F and α in R.*

Let $G = f^{-1}(0)$. For each $(x, y) \in F \times G$, define

$$|||(x,y)||| = \max(\|x\|, \|y\|).$$

It can be proved that $F \times G$ is a Banach space with the norm $|||\ |||$.

THEOREM 2.2. *Let f be a continuous linear mapping of a Banach space E onto a Banach space F. Then there exists a homeomorphism h of E onto $F \times G$, $G = f^{-1}(0)$, such that $h(y) = (0, y)$ for all y in G and $|||h(y)||| = \|y\|$ for all $y \in E$.*

Proof. By the previous lemma, there is a continuous mapping g of F into E such that $f \circ g(x) = x$, $g(\alpha x) = \alpha g(x)$, $\|g(x)\| \leqq m\|x\|$ for all x in F and all α in R. Define $h_1: E \to F \times G$ by $h_1(y) = (f(y),\ y - g \circ f(y))$ for all y in E. It can be proved that h_1 is a homeomorphism of E onto $F \times G$

such that $h_1(y) = (0, y)$ for all y in E. Define $h: E \to F \times G$ by $h(y) = \|y\| h_1(y) / ||| h_1(y) |||$ if $h_1(y) \neq 0$, that is, $y \neq 0$, and $h(0) = (0, 0)$. Then $\|y\| / (1 + m) \leqq ||| h_1(y) ||| \leqq ((m + 1) \|f\| + 1) \|y\|$ for all y in E and h is a homeomorphism of E onto $F \times G$ such that $h(y) = (0, y)$ for all y in F and $||| h(y) ||| = \|y\|$ for all y in E. Q.E.D.

COROLLARY 2.3. *For any closed linear subspace F of a Banach space E, there exists a homeomorphism h of E onto $(E/F) \times F$ such that $h(y) = (0, y)$ for all y in F.*

Proof. Let f be the canonical mapping of E onto the Banach space E/F. The corollary follows immediately from Theorem 2.2. Q.E.D.

Clearly, the characteristic cone of a closed convex body in an infinite-dimensional normed linear space is either a linear variety of infinite deficiency or a linear variety of finite deficiency or is not a linear variety. We shall consider each case separately.

THEOREM 2.4. *Let E be an infinite-dimensional normed linear space, $C = \{x \in E \mid \|x\| \leqq 1\}$ and $S = \{x \in E \mid \|x\| = 1\}$. Let J be a closed half-space in E with bounding hyperplane H. Then there exists a homeomorphism i of E onto E such that $i(C) = J$ and $i(S) = H$.*

Proof. By Theorem 1.3, E is homeomorphic to $E \sim \{0\}$. Using this property of E, Klee [13] proved that there exists a homeomorphism j of C onto J which maps S onto H. Without loss of generality, we may assume $0 \in J \sim H$. Define i of E onto E by $i(x) = j(x)$ if $x \in C$, $i(x) = \|x\| j(x/\|x\|)$ if $x \notin C$. It is clear that i is a homeomorphism and $i(C) = J$, $i(S) = H$. Q.E.D.

THEOREM 2.5. *Let K be a closed convex body in a Banach space E. If the characteristic cone of K is a closed linear variety L of infinite deficiency in E, then there exists a homeomorphism i of E onto E such that $i(K)$ is a closed half-space J in E and $i(\mathrm{Bd}\, K)$ is the bounding hyperplane of J.*

Proof. We may assume that $0 \in \mathrm{Int}.K$, $0 \in L$.

Case 1. Suppose $L = \{0\}$, that is, K is a linearly bounded closed convex body in E.

For any point $y \neq 0$ in E, there exists a unique point x in $\mathrm{Bd}\, K$ such that $y = \alpha x$ for some scalar $\alpha > 0$. Define $j: E \to E$ by

$$j(y) = \alpha \frac{x}{\|x\|} \quad \text{for all } y \neq 0 \text{ in } E; \qquad y = \alpha x, x \in \mathrm{Bd}\, K,$$

and

$$j(0) = 0.$$

Then j is a homeomorphism of E onto E such that $j(K) = C$ and $j(\mathrm{Bd}\, K) = S$. Let i_1 be a homeomorphism of E onto E such that $i_1(C)$ is a closed half-space J and $i_1(S)$ is the bounding hyperplane H of J obtained in Theorem 2.4. Then $i = i_1 \circ j$ is a homeomorphism of E onto E, $i(K) = J$, $i(\mathrm{Bd}\, K) = H$.

Case 2. Suppose $\dim L \geqq 1$.

We claim that $K = \bigcup_{x \in K}(x + L)$.

It is clear that $K \subset \bigcup_{x \in K}(x + L)$. Given any $x \in K$, $y \in L$, since $y/\lambda \in L$ for all real numbers $\lambda > 0$ and K is convex, $(1-\lambda)x + \lambda(y/\lambda) = (1-\lambda)x + y$ is in K for all $0 < \lambda \leqq 1$. Let $\lambda \to 0$; then we have $x + y \in K$ because K is closed.

By Corollary 2.3, the mapping h of E onto $(E/L) \times L$, defined by $h(x) = (f(x), x - g \circ f(x))$ for all x in E where f is the canonical mapping of E onto E/L, is a homeomorphism. Since $K = \bigcup_{x \in K}(x + L)$ and the characteristic cone of K is L, $f(K)$ is a linearly bounded closed convex body in E/L and $h(K) = f(K) \times L$. Let J be a closed half-space in E; we may assume that the bounding hyperplane H of J contains L. By Case 1, there exists a homeomorphism i of E/L onto E/L such that $i(f(K)) = J/L$, $i(\mathrm{Bd} f(K)) = H/L$. The mapping $i \times 1$, $(i \times 1)(x,y) = (i(x), y)$ for all (x,y) in $(E/L) \times L$, is a homeomorphism of $(E/L) \times L$ onto $(E/L) \times L$ which maps $f(K) \times L$ onto $(J/L) \times L$, $(\mathrm{Bd} f(K)) \times L = \mathrm{Bd}\, h(K)$ onto $(H/L) \times L$. Hence $j = h^{-1} \circ (i \times 1) \circ h$ is a homeomorphism of E onto E, $j(K) = J$, $j(\mathrm{Bd}\, K) = H$. Q.E.D.

REMARK. Using the argument similar to the proof of Theorem 1.3, Corson and Klee [4] proved that if F is a closed linear subspace of infinite deficiency in a normed linear space E, then E is homeomorphic to $E \sim F$. They were then able to prove Theorem 2.5 in case E is an infinite-dimensional normed linear space. However, the proof of Theorem 2.5 is simpler.

THEOREM 2.6. *Let K be a closed convex body in a normed linear space E. If the characteristic cone of K is not a linear variety then there is a homeomorphism i of E onto E such that $i(K)$ is a closed half-space J in E and $i(\mathrm{Bd}\, K)$ is the bounding hyperplane of J.*

Proof. We may assume $0 \in K$, $0 \in J \sim H$. There is a homeomorphism j of K onto J such that $j(\mathrm{Bd}\, K) = H$, $j(K) = J$ [11, p. 30]. For each element y in $E \sim K$, there exists a unique point x in $\mathrm{Bd}\, K$ such that $y = \alpha x$ for some scalar $\alpha > 0$. Define i of E onto E by $i(y) = j(y)$ if y is in K, $i(y) = \alpha j(x)$ if $y \in E \sim K$, $y = \alpha x$, $x \in \mathrm{Bd}\, K$. It can be proved that i is a homeomorphism of E onto E, $i(K) = J$, $i(\mathrm{Bd}\, K) = H$. Q.E.D.

The above two results are to be compared with the following result of Klee [11].

THEOREM 2.7. *If the characteristic cone of a closed convex body K of a normed linear space E is a linear variety L of finite deficiency n in E, then there is a homeomorphism i of K onto $L \times C_n$ where C_n is the unit cell in n-dimensional Euclidean space and $i(\mathrm{Bd}\, K) = L \times S_{n-1}$ where S_{n-1} is the $(n-1)$-sphere.*

3. **Closed subsets K of an infinite-dimensional normed linear space E such that E and $E \sim K$ are homeomorphic.** First, we consider the case when K is a closed convex body of an infinite-dimensional closed linear variety of a Banach space E. Then we shall consider the case when K is a closed subset of a closed linear variety of infinite deficiency in a Banach space E.

THEOREM 3.1. *Let F be an infinite-dimensional closed linear variety of a Banach space E. If the characteristic cone of a closed convex body K of F is not a linear variety of finite deficiency in F, then E and $E \sim K$ are homeomorphic.*

Proof. We may assume that F is a closed linear subspace. Corollary 2.3 asserts that there is a homeomorphism h of E onto $(E/F) \times F$ such that $h(x) = (0, x)$ for all x in F and $\|h(x)\| = \|x\|$ for all x in E. By Theorems 2.5, 2.6, there exists a homeomorphism k of F onto F such that $k(K) = C_F$, the unit cell of F. The mapping $i = h^{-1} \circ (1 \times k) \circ h$, $(1 \times k)(x, y) = (x, k(y))$ for all (x, y) in $(E/F) \times F$, is a homeomorphism of E onto E such that $i(K)$ is a closed convex subset of the unit cell of E. By Theorem 1.3, there exists a homeomorphism j of E onto $E \sim i(K)$. The mapping $i^{-1} \circ j \circ i$ is a homeomorphism of E onto $E \sim K$. Q.E.D.

REMARK. If the characteristic cone of K is a linear variety of finite deficiency, then E is not necessarily homeomorphic to $E \sim K$. For example, E is not homeomorphic to $E \sim H$ when H is a closed hyperplane in E.

LEMMA 3.2. *For every infinite-dimensional normed linear space E, there exists a homeomorphism j of $((E \sim \{0\}) \times (0, 1]) \cup (E \times \{0\})$ onto $E \times [0, 1]$ such that $j(x, 0) = (x, 0)$ for all x in E.*

Proof. By Theorem 1.3, there exists a homeomorphism i of $E \times [0, 1]$ onto $(E \times (0, 1]) \cup ((E \sim \{0\}) \times \{0\})$ such that $i(E \times \{0\}) = (E \sim \{0\}) \times \{0\}$. By identifying $(E \sim \{0\}) \times \{0\}$ with $E \sim \{0\}$, there is a homeomorphism i_0 of E onto $E \sim \{0\}$ defined by $i_0(x) = i(x, 0)$ for all x in E. Define

$$j\colon\ ((E \sim \{0\}) \times [0, 1]) \cup (\{0\} \times \{0\}) \to E \times [0, 1]$$

by

$$j(x, t) = \begin{cases} i(i_0^{-1}(x), t) & \text{if } t \neq 0, \\ (x, t) & \text{if } t = 0. \end{cases}$$

It can be proved that j is a homeomorphism of $((E \sim \{0\}) \times [0,1]) \cup (\{0\} \times \{0\})$ onto $E \times [0,1]$ such that $j(x,0) = (x,0)$ for all x in E. Q.E.D.

LEMMA 3.3. *For any infinite-dimensional normed linear space E, there exists a homeomorphism i of $(E \times (0,1]) \cup (0,0)$ onto $E \times (0,1]$ such that $i(x,1) = (x,1)$ for all x in E.*

Proof. By Lemma 3.2, there exists a homeomorphism j of $E \times [0,1]$ onto $((E \sim \{0\}) \times (0,1]) \cup (E \times \{0\}) = ((E \sim \{0\}) \times [0,1]) \cup (0,0)$ such that $j(x,0) = (x,0)$ for all x in E. Define i_1 of $E \times [0,1]$ onto

$$((E \sim \{0\}) \times [0,1]) \cup (\{0\} \times \{0,1\})$$

by $i_1(x,t) = j(x,2t)$ if $0 \leqq t \leqq 1/2$; $i_1(x,t) = j(x,2-2t)$ if $1/2 \leqq t \leqq 1$. It is easy to prove that i_1 is a homeomorphism and $i_1(x,1) = (x,1)$; $i_1(x,0) = (x,0)$ for all x in E.

Since E is a metric space and $\{0\}$ is conveniently situated in E [11, (2.4)], there exists a homeomorphism i_2 of $((E \sim \{0\}) \times (0,1]) \cup (\{0\} \times \{0,1\})$ onto $((E \sim \{0\}) \times (0,1]) \cup (\{0\} \times \{1/2,1\})$ such that $i_2(x,1) = (x,1)$ for all x in E. Define i_3 of $((E \sim \{0\}) \times (0,1]) \cup (\{0\} \times \{1/2,1\})$ onto $E \times (0,1]$ by $i_3(x,t) = i_1^{-1}(x,2t)$ if $0 < t \leqq 1/2$; $i_3(x,t) = i_1^{-1}(x,2t-1)$ if $1/2 \leqq t \leqq 1$. It can be proved that i_3 is a homeomorphism such that $i_3(x,1) = (x,1)$ for all x in E. Hence the mapping $i = i_3 \circ i_2 \circ i_1$ is a homeomorphism of $(E \times (0,1]) \cup (0,0)$ onto $E \times (0,1]$ such that $i(x,1) = (x,1)$ for all x in E. Q.E.D.

LEMMA 3.4. *If F is a closed linear subspace of infinite deficiency in a Banach space E, then E and $E \sim F$ are homeomorphic. If, in addition, Q is an open half-space in E whose bounding hyperplane H contains F then there exists a homeomorphism f of Q onto $Q \cup F$ such that $f(x_0 + y) = y$ for all y in F where x_0 is in Q.*

Proof. By Corollary 2.3, there is a homeomorphism h of E onto $(E/F) \times F$ such that $h(x) = (0,x)$ for all x in F. Since E/F is an infinite-dimensional Banach space, by Theorem 1.3, there exists a homeomorphism g of E/F onto $(E/F) \sim \{0\}$. Define

$$k: (E/F) \times F \to ((E/F) \times F) \sim (\{0\} \times F)$$

by

$$k(x,y) = (g(x),y) \qquad \text{for all } (x,y) \text{ in } (E/F) \times F.$$

It is clear that k is a homeomorphism of $(E/F) \times F$ onto $((E/F) \times F) \sim (\{0\} \times F)$. The mapping $h^{-1} \circ k \circ h$ is a homeomorphism of E onto $E \sim F$.

By hypothesis, F is contained in the bounding hyperplane H of an open half-space Q, hence H is homeomorphic to $(H/F) \times F$. Hence

$$E \approx H \times (-\infty, \infty) \approx (H/F) \times F \times (-\infty, \infty) \approx (H/F) \times (-\infty, \infty) \times F,$$

$$Q \approx H \times (0, \infty) \approx (H/F) \times F \times (0, \infty) \approx (H/F) \times (0, \infty) \times F,$$

$$Q \cup F \approx ((H/F) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times F).$$

To prove Q and $Q \cup F$ are homeomorphic, it suffices to show $(H/F) \times (-\infty, \infty) \times F$ is homeomorphic to

$$((H/F) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times F).$$

By Lemma 3.3, there is a homeomorphism f_0 of $(H/F) \times (0, \infty)$ onto $((H/F) \times (0, \infty)) \cup (0,0)$. Define

$$f: (H/F) \times (0, \infty) \times F \rightarrow ((H/F) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times F)$$

by

$$f(x,t,y) = (f_0(x,t),y) \quad \text{for all } (x,t,y) \text{ in } (H/F) \times (0, \infty) \times F.$$

It is easy to prove that f is a homeomorphism of $(H/F) \times (0, \infty) \times F$ onto $((H/F) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times F)$. Q.E.D.

THEOREM 3.5. *Let F be a closed linear subspace of infinite deficiency in a Banach space E. Let Q be an open half-space in E whose bounding hyperplane H contains F. If K is a closed subset in F, then Q is homeomorphic to $Q \cup K$ and E is homeomorphic to $E \sim K$.*

Proof. By the previous lemma, to prove that Q and $Q \cup K$ are homeomorphic it suffices to show that $Q \cup F$ and $Q \cup K$ are homeomorphic.

For x in F, let $\phi(x) = \inf\{ \|x - a\| \mid a \in K\}$. Clearly, ϕ is a real-valued continuous function on F. Define

$$g: ((H/F) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times F) \rightarrow ((H/F) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times K)$$

by

$$g(x,t,y) = \begin{cases} (x,t,y) & \text{if } y \in K, \\ \left(\phi(y)\, f_0^{-1}\left(\dfrac{x}{\phi(y)}, \dfrac{t}{\phi(y)}\right), y\right) & \text{if } y \in F \sim K, \end{cases}$$

where f_0 is a homeomorphism of $(H/F) \times (0, \infty)$ onto $((H/F) \times (0, \infty)) \cup (0,0)$ such that $f_0(x,t) = (x,t)$ for $t \geqq 1$ obtained by Lemma 3.3.

To show g is continuous, it suffices to show if $y_n \notin K, y_n \rightarrow y \in K$, then $g(x,t,y_n) \rightarrow g(x,t,y)$. Since $f_0(x,t) = (x,t)$ for $t \geqq 1$, for n sufficiently large,

$$g(x,t,y_n) = \left(\phi(y_n)\, f_0^{-1}\left(\frac{x}{\phi(y_n)}, \frac{t}{\phi(y_n)}\right), y_n\right) = \left(\phi(y_n)\left(\frac{x}{\phi(y_n)}, \frac{t}{\phi(y_n)}\right), y_n\right) = (x,t,y_n).$$

This implies that $g(x,t,y_n) \to g(x,t,y)$ as $y_n \to y$. Clearly, g is a one-to-one and onto mapping. Define

$$g_1: ((H/F) \times (0,\infty) \times F) \cup (\{0\} \times \{0\} \times K) \\ \to ((H/F) \times (0,\infty) \times F) \cup (\{0\} \times \{0\} \times F)$$

by

$$g_1(x,t,y) = \begin{cases} (x,t,y) & \text{if } y \in K, \\ \left(\phi(y) f_0\left(\dfrac{x}{\phi(y)}, \dfrac{t}{\phi(y)}\right), y\right) & \text{if } y \in F \sim K. \end{cases}$$

Then if $\phi(y) > 0$, we have $g \circ g_1(x,t,y) = g(\phi(y) f_0(x/\phi(y), t/\phi(y)), y)$ $= (\phi(y) f_0^{-1}(\phi(y) f_0^{-1}(x/\phi(y), t/\phi(y))/\phi(y)), y) = (\phi(y) f_0^{-1} \circ f_0(x/\phi(y), t/\phi(y)), y)$ $= (x,t,y)$. This implies that g_1 is the inverse of g. By similar argument as g, it can be proved that g_1 is continuous. Hence g is a homeomorphism. We have shown that $Q \cup F$ is homeomorphic to $Q \cup K$.

By Lemma 3.4, the mapping f of $(H/F) \times (0,\infty) \times F$ onto

$$((H/F) \times (0,\infty) \times F) \cup (\{0\} \times \{0\} \times F)$$

defined by $f(x,t,y) = (f_0(x,t), y)$ for (x,t,y) in $(H/F) \times (0,\infty) \times F$ is a homeomorphism. Hence $g \circ f$ is a homeomorphism of $(H/F) \times (0,\infty) \times F$ onto $((H/F) \times (0,\infty) \times F) \cup (\{0\} \times \{0\} \times K)$ and $g \circ f(x_0, t_0, y) = (0,0,y)$ for all y in K where $f_0(x_0, t_0) = (0,0)$. The restriction of $(g \circ f)^{-1}$ on $(H/F) \times (0,\infty) \times F$ is a homeomorphism of $(H/F) \times (0,\infty) \times F$ onto

$$((H/F) \times (0,\infty) \times F) \cup (\{x_0\} \times \{t_0\} \times K).$$

This can be used to define a homeomorphism of E onto $E \sim K$. Q.E.D.

REMARK. The restriction of g on

$$((H/F) \times (0,\infty) \times F) \cup (\{0\} \times \{0\} \times (F \sim K))$$

is a homeomorphism onto $(H/F) \times (0,\infty) \times F$. Since K is a closed subset of F, $F \sim K$ is an open subset in F. This implies that supposing F is a closed linear subspace of infinite deficiency in a Banach space E and Q is an open half-space in E such that the bounding hyperplane of Q contains F, then for any open subset G in F, Q is homeomorphic to $Q \cup G$.

REMARK. In case E is an infinite-dimensional Hilbert space, Theorem 3.5 had been proved by Klee [14] implicitly.

Corollary 2.3 is not available in the proof of Theorems 3.1, 3.5 when E is not complete [15]. But if there exists a projection of E onto F, then $E = G \times F$ for some closed linear subspace G in E. Using the property that $E = G \times F$ in place of Corollary 2.3 we can prove the following results.

THEOREM 3.6. *Let F be a closed linear subspace of finite deficiency in an infinite-dimensional normed linear space E. Let K be a closed convex body in F.*

(a) *If the characteristic cone of K is not a linear variety, then E is homeomorphic to $E \sim K$.*

(b) *If the characteristic cone of K is a linear variety L of infinite deficiency in F and if there exists a continuous projection of F onto L, then E is homeomorphic to $E \sim K$.*

THEOREM 3.7. *Let F be a closed linear subspace of infinite deficiency in a normed linear space E such that there exists a continuous projection of E onto F. If K is a closed subset of F then E and $E \sim K$ are homeomorphic. If, in addition, Q is an open half-space in E such that the bounding hyperplane of Q contains F, then Q and $Q \cup K$ are homeomorphic.*

4. **Topological equivalence of a Banach space with its closed convex subsets.** The problem whether an open half-space Q of an infinite-dimensional Banach space E is homeomorphic to $Q \cup \operatorname{Bd} Q$ is equivalent to whether E is homeomorphic to its unit cell (Theorem 2.4). In this section, we shall consider the problem that if K is a closed convex body of a closed linear subspace of finite deficiency in a Banach space E, when is K homeomorphic to E? We shall first prove two lemmas.

LEMMA 4.1. *Let B be a Banach space containing a proper closed linear subspace F which is homeomorphic to an infinite-dimensional Hilbert space. If a Banach space E admits a continuous linear mapping f onto B, then E, $C = \{x \in E \mid \|x\| \leqq 1\}$ and $S = \{x \in E \mid \|x\| = 1\}$ are homeomorphic.*

Proof. By Theorem 2.2, there exists a homeomorphism h of E onto $P = B \times G$, $G = f^{-1}(0)$, such that $|||h(x)||| = \|x\|$ for all x in E. To prove the theorem, it suffices to show that P, $C_P = \{x \in P \mid |||x||| \leqq 1\}$ and $S_P = \{x \in P \mid |||x||| = 1\}$ are homeomorphic. Let $Y = F \times \{0\}$. Y is a proper closed linear subspace of P and Y is homeomorphic to an infinite-dimensional Hilbert space X. A theorem of Klee [11] assures that X, $X \times (0, \infty)$ and $X \times [0, \infty)$ are homeomorphic. Hence Y, $Y \times (0, \infty)$ and $Y \times [0, \infty)$ are homeomorphic. Since Y is a proper closed linear subspace of P, there is a closed hyperplane H in P containing Y. Using Corollary 2.3 and the properties of Y, we have

$$\begin{aligned} P &\approx H \times (0, \infty) \approx (H/Y) \times Y \times (0, \infty) \approx (H/Y) \times Y \times [0, \infty) \\ &\approx H \times [0, \infty) \approx C_P, \\ P &\approx H \times (0, \infty) \approx (H/Y) \times Y \times (0, \infty) \\ &\approx (H/Y) \times Y \approx H \approx S_P. \end{aligned}$$

Q.E.D.

REMARK 1. This lemma is a consequence of a result of Klee [16]. But the proof is simpler.

REMARK 2. Klee [13] showed that for every infinite-dimensional normed linear space E, the unit cell C of E is homeomorphic to $E \sim \mathrm{Int}.C$. Hence the problem whether an infinite-dimensional normed linear space E is homeomorphic to its unit cell C is equivalent to the problem whether E is homeomorphic to $E \sim \mathrm{Int}.C$. Notice that in Theorem 1.3, we have shown that every infinite-dimensional normed linear space E is homeomorphic to $E \sim C$.

REMARK 3. We have shown that the unit sphere S of an infinite-dimensional Banach space E is homeomorphic to a closed hyperplane of E. Hence the problem whether every infinite-dimensional Banach space is homeomorphic to its unit sphere is equivalent to the problem whether every infinite-dimensional Banach space is homeomorphic to a closed hyperplane.

Let F be a separable infinite-dimensional closed linear subspace of a Banach space E. Let H be a closed hyperplane in E containing F. Let H_F be a closed hyperplane in F. By Corollary 2.3, $H \approx (H/H_F) \times H_F$ and $E \approx (E/H_F) \times H_F \approx (H/H_F) \times (-\infty, \infty) \times H_F \approx H/H_F \times F$. Hence if F is homeomorphic to H_F, then E and H are homeomorphic. Now suppose E is a separable infinite-dimensional Banach space, then there exists a continuous linear mapping f of l_1 onto E [6]. Let G be the kernel of f, then G is a closed linear subspace of l_1. E is isomorphic to l_1/G and the closed hyperplane H of E is isomorphic to $f^{-1}(H)/G$. $f^{-1}(H)$ is a closed hyperplane in l_1. So the problem is reduced to the following: If G is a closed linear subspace of l_1 and H is a closed hyperplane in l_1 containing G, does there exist a homeomorphism of l_1 onto H which maps G onto G? A recent result of Bessaga and Pełczyński [3] showed that every infinite-dimensional closed linear subspace of l_1 is homeomorphic to l_1.

LEMMA 4.2. *Let E be a Banach space containing a closed linear subspace F of infinite deficiency which is homeomorphic to an infinite-dimensional Hilbert space. If L is a closed linear subspace of finite deficiency m in E, then for each integer $n \geqq 0$, E and $L \times [0,1]^n$ are homeomorphic.*

Proof. Since F is a closed linear subspace of infinite deficiency in E, there exists a closed linear subspace M of deficiency m in E such that F is contained in M. It is easy to define a linear homeomorphism of E onto E which maps L onto M. Hence we may assume that F is contained in L. Since F is homeomorphic to an infinite-dimensional Hilbert space, F is homeomorphic to $F \times R^m$ [11]. Hence we have

$$E \approx L \times R^m \approx (L/F) \times F \times R^m \approx (L/F) \times F \approx L.$$

For $n > 0$, to show that E and $L \times [0,1]^n$ are homeomorphic, it suffices to show that E and $H \times [0,1]$ are homeomorphic where H is a closed hyperplane in E. We may assume that H contains F. By Lemma 4.1, H and C_H, the unit cell of H, are homeomorphic. Since the unit cell C of E is homeomorphic to $C_H \times [0,1]$, we have

$$E \approx C \approx C_H \times [0,1] \approx H \times [0,1]. \qquad \text{Q.E.D.}$$

THEOREM 4.3. *Let E be a Banach space containing a closed linear subspace B of infinite deficiency such that B is homeomorphic to an infinite-dimensional Hilbert space. Let F be a closed linear variety of finite deficiency in E, and K a closed convex body in F.*

(a) *If the characteristic cone of K is not a linear variety of finite deficiency in F, then E, K and $\mathrm{Bd}_F K$ are homeomorphic.*

(b) *If the characteristic cone of K is a linear variety of finite deficiency n in F, then K is homeomorphic to E and $\mathrm{Bd}_F K$ is homeomorphic to $E \times S_{n-1}$, where S_{n-1} is the $(n-1)$-sphere.*

Proof. Without loss of generality, we may assume that F contains B.

(a) From Theorems 2.5, 2.6, K is homeomorphic to $C_F = \{x \in F \mid \|x\| \leqq 1\}$ and $\mathrm{Bd}_F K$ is homeomorphic to $S_F = \{x \in F \mid \|x\| = 1\}$. Lemma 4.1 implies that C_F, S_F and F are homeomorphic. Hence K, $\mathrm{Bd}_F K$ and F are homeomorphic. But F is homeomorphic to E by Lemma 4.2. This shows that K and $\mathrm{Bd}_F K$ are homeomorphic to E.

(b) By Theorem 2.7, K is homeomorphic to $L \times C_n$ and $\mathrm{Bd}_F K$ is homeomorphic to $L \times S_{n-1}$. Lemma 4.2 assures that F, L and $L \times [0,1]^n$ are homeomorphic. Hence K is homeomorphic to F and $\mathrm{Bd}_F K$ is homeomorphic to $F \times S_{n-1}$. But F is homeomorphic to E. The proof of the theorem is completed. Q.E.D.

REMARK. Lemma 4.1 and Theorem 4.3 should be compared with the following result of Corson and Klee [4]: Suppose the normed linear space E admits (for each finite n) a closed linear subspace of deficiency n which is homeomorphic with its own unit cell. Then E is homeomorphic with all its closed convex bodies.

REMARK 2. Every infinite-dimensional Banach space clearly contains a separable infinite-dimensional proper closed linear subspace. If Banach's conjecture that all separable infinite-dimensional Banach spaces are homeomorphic is true, then every infinite-dimensional Banach space would contain a proper closed linear subspace homeomorphic to l_2. Hence if Banach's conjecture is true, Theorem 4.3 would imply that if K is a closed convex body in a closed linear subspace of finite deficiency in an infinite-dimensional Banach space E, then K is homeomorphic to E and $\mathrm{Bd}_F K$ is homeomorphic to E or $E \times S_n$ for some integer $n \geqq 0$.

REMARK 3. If E is an infinite-dimensional Hilbert space, then every infinite-dimensional closed linear subspace F of E is isomorphic to E. Let K be a closed convex body in F. By Theorem 4.3, K is homeomorphic to F, hence to E. Klee [12] had proved that every locally compact closed convex subset of an infinite-dimensional normed linear space is homeomorphic either to $[0,1]^m \times (0,1)^n$ or $[0,1]^m \times [0,1)$, where m, n are cardinal numbers, $0 \leqq m \leqq \aleph_0$, $0 \leqq n < \aleph_0$. Hence it is natural to ask whether in an infinite-dimensional Hilbert space E, there exists a closed convex but not locally compact subset K, which is not contained in any proper closed linear subspace of E and every point of K is a boundary point. The answer is positive. Let $K = \{x \in l_2 \mid x = (x_1, x_2, \cdots),\ x_i \geqq 0 \text{ for all } i = 1, 2, \cdots\}$. It is clear that K is closed convex and is not contained in any proper closed linear subspace of l_2. Klee [10] proved that every point of K is a boundary point. It can be proved that every neighborhood of $(0, 0, \cdots)$ in K is not compact. The topological classification of all closed convex subsets of an infinite-dimensional Hilbert space is far from complete.

The hypothesis of Theorem 4.3 lead us to consider the problem: When is an infinite-dimensional separable Banach space homeomorphic to l_2? Each of the following separable infinite-dimensional Banach spaces E is homeomorphic to l_2.

(a) E is a w^*-closed linear subspace of the dual space of a normed linear space.

(b) E is (c_0), the space of all sequences converging to 0 with supremum norm.

(c) E admits an unconditional basis.

(d) E is a closed linear subspace of $L(X, \mu)$ of all real-valued μ-absolutely summable functions on a compact metric space X and μ is a Borel measure on X.

5. **Periodic homeomorphisms with preassigned set of fixed points.** This section is devoted to Problem II. The main results are Theorems 5.2, 5.3.

LEMMA 5.1. *Let E be a normed linear space. For every positive integer n, there exists a homeomorphism of period $2n$ of E onto E with 0 as the only fixed point.*

Proof. Let F be a two-dimensional closed linear subspace of E. Then there exist a closed linear subspace G of E and a homeomorphism h_1 of E onto $G \times F$ such that $h_1(0) = (0,0)$ and $h_1(x) = (0, x)$ for all x in F. For each positive integer n, there exists a homeomorphism of period n of the Euclidean plane R_2 onto itself with 0 as the only fixed point. Since F is topologically equivalent to R_2, there exists a homeomorphism k of period n of F onto F with 0 as the only fixed point. Define

$$h_2\colon\ G\times F\to G\times F$$

by

$$h_2(x,y) = (-x, k(y)) \quad \text{for } (x, y) \text{ in } G\times F.$$

Then h_2 is a homeomorphism of period $2n$ of $G\times F$ onto $G\times F$ with $(0,0)$ as the only fixed point. Hence $h_1^{-1}\circ h_2\circ h_1$ is a homeomorphism of period $2n$ of E onto E with 0 as the only fixed point. Q.E.D.

THEOREM 5.2. *Let F be an infinite-dimensional closed linear subspace of a Banach space E. If the characteristic cone of a closed convex body K of F is not a linear variety of finite deficiency in F then for each positive integer n, there exists a homeomorphism of period $2n$ of E onto E with $\mathrm{Bd}_F K$ as its set of fixed points. In case E is an infinite-dimensional Hilbert space, then for each integer $n > 1$, there exists a homeomorphism of period n of E onto E with K as its set of fixed points.*

Proof. Let H be a closed hyperplane in E containing F (in case $F = E$, let H be the space E). Let H_F be a closed hyperplane in F. H_F is a closed linear subspace of H. Corollary 2.3 implies that there is a homeomorphism h_1 of E onto $(H/H_F)\times H_F\times(-\infty,\infty)$. By Theorems 2.5, 2.6, there is a homeomorphism h_2 of $H_F\times(-\infty,\infty)$ onto $H_F\times(-\infty,\infty)$ such that $h_2(K) = H_F\times[0,\infty)$, $h_2(\mathrm{Bd}_F K) = H_F\times\{0\}$. Define h_3 of $(H/H_F)\times H_F\times(-\infty,\infty)$ onto itself by $h_3(x,y,t) = (r(x), y, -t)$ for all (x,y,t) in $(H/H_F)\times H_F \times(-\infty,\infty)$, where r is a homeomorphism of period $2n$ of H/H_F onto itself with 0 as the only fixed point. h_3 is a homeomorphism of period $2n$ of $(H/H_F)\times H_F\times(-\infty,\infty)$ onto itself with $\{0\}\times H_F\times\{0\}$ as its set of fixed points. Hence $h_1^{-1}\circ(1\times h_2)^{-1}\circ h_3\circ(1\times h_2)\circ h_1$ is a homeomorphism of period $2n$ of E onto E with $\mathrm{Bd}_F K$ as its set of fixed points.

Suppose E is an infinite-dimensional Hilbert space. Since F is an infinite-dimensional closed linear subspace of E, F is isomorphic to E. By a theorem of Klee [11, p. 33], for each integer $n > 1$, there exists a homeomorphism k_1 of period n of F onto F with $(F\sim \mathrm{Int}.C_F) = \{x\in F \mid \|x\|\geqq 1\}$ as its set of fixed points. On the other hand, there is a homeomorphism k_2 of F onto F such that $k_2(C_F) = F\sim \mathrm{Int}.C_F$ [13]. By Theorems 2.5, 2.6, there is a homeomorphism k_3 of F onto F such that $k_3(K) = C_F$. Hence $k = k_3^{-1}\circ k_2^{-1}\circ k_1\circ k_2\circ k_3$ is a homeomorphism of period n of F onto F with K as its set of fixed points. It is clear that k can be used to define a homeomorphism of period n of E onto E with K as its set of fixed points. Q.E.D.

THEOREM 5.3. *Let F be a closed linear subspace of infinite deficiency in a Banach space E. If K is a closed subset in F then, for each positive integer n, there exists a homeomorphism of period $2n$ of E onto E with K as its set of fixed points. In case E is an infinite-dimensional Hilbert space, for each integer*

$n > 1$, *there exists a homeomorphism of period n of E onto E with K as its set of fixed points.*

Proof. Let Q be an open half-space in E such that the bounding hyperplane H of Q contains F. By Theorem 3.5, there is a homeomorphism h_2 of Q onto $Q \cup K$ such that $h_2(x_0 + y) = y$ for all y in K where $x_0 \in Q$. Let h_1 be a homeomorphism of E onto Q such that $h_1(y) = x_0 + y$ for all y in H. Corollary 2.3 asserts that there is a homeomorphism h_3 of $Q \cup K$ onto $((H/F) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times K)$. Since H/F is an infinite-dimensional Banach space, by Lemma 3.2, there is a homeomorphism j of $(H/F) \times [0, \infty)$ onto $(((H/F) \sim \{0\}) \times (0, \infty)) \cup ((H/F) \times \{0\})$ such that $j(x, 0) = (x, 0)$ for all x in H/F. Define

$$h_4\colon ((H/F) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times K)$$
$$\rightarrow (((H/F) \sim \{0\}) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times K)$$

by $h_4(x, t, y) = (j(x, t), y)$ for $x \in H/F$, $t \in [0, \infty)$, $y \in F$. It can be proved that h_4 is a homeomorphism of $((H/F) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times K)$ onto $(((H/F) \sim \{0\}) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times K)$ such that $h_4(0, 0, y) = (0, 0, y)$ for all y in K. Define h_5 of $(((H/F) \sim \{0\}) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times K)$ onto itself by $h_5(x, t, y) = (r(x), t, y)$ where r is a homeomorphism of period $2n$ of H/F onto itself with 0 as the only fixed point obtained by Lemma 5.1. It is clear that h_5 is a homeomorphism of period $2n$ with $\{0\} \times \{0\} \times K$ as its fixed-point set. Then the mapping $h_1^{-1} \circ h_2^{-1} \circ h_3^{-1} \circ h_4^{-1} \circ h_5 \circ h_4 \circ h_3 \circ h_2 \circ h_1$ is a homeomorphism of period $2n$ of E onto E with K as its set of fixed points.

If E is an infinite-dimensional Hilbert space, then H/F is isomorphic to E. Hence for each integer $n > 1$, there exists a rotation r of angle $2\pi/n$ of H/F onto H/F with 0 as the only fixed point (see [1] or [11]). Define h_5' of $(((H/F) \sim \{0\}) \times (0, \infty) \times F) \cup (\{0\} \times \{0\} \times K)$ onto itself by $h_5'(x, t, y) = (r(x), t, y)$. Then $h_1^{-1} \circ h_2^{-1} \circ h_3^{-1} \circ h_4^{-1} \circ h_5' \circ h_4 \circ h_3 \circ h_2 \circ h_1$ is a homeomorphism of period n of E onto E with K as its fixed-point set. Q.E.D.

Remark. In the case when E is an infinite-dimensional Hilbert space, Theorem 5.3 had been proved implicitly by Klee [14].

Corollary 2.3 is not available in the proof of Theorems 5.2, 5.3, when E is not complete [15]. But if there exists a continuous projection of E onto F, then $E = G \times F$ for some closed linear subspace G in E. Using the property that $E = G \times F$ in place of Corollary 2.3 we can prove the following results.

Theorem 5.4. *Let F be a closed linear subspace of finite deficiency in an infinite-dimensional normed linear space E. Let K be a closed convex body in F.*

(a) *If the characteristic cone of K is not a linear variety then for each positive integer n, there exists a homeomorphism of period $2n$ of E onto E with* $\mathrm{Bd}_F K$ *as its fixed-point set.*

(b) *If the characteristic cone of K is a linear variety L of infinite deficiency in F and if there exists a continuous projection of F onto L, then for each positive integer n, there exists a homeomorphism of period $2n$ of E onto E with* $\mathrm{Bd}_F K$ *as its set of fixed points.*

Theorem 5.5. *Let F be a closed linear subspace of infinite deficiency in a normed linear space E such that there exists a continuous projection of E onto F. For every closed subset K of F and for each positive integer n, there exists a homeomorphism of period $2n$ of E onto E with K as its set of fixed points.*

Theorem 5.3 leads us to consider the following problem: Given a compact subset K of an infinite-dimensional topological vector space E, does there exist a homeomorphism i of E onto E such that $i(K)$ is contained in a closed linear subspace of infinite deficiency in E? Unfortunately, we only have the following partial result due to Klee [12]. For any given nonnegative integer n and compact subset K of an infinite-dimensional Banach space E, there exist closed linear subspaces L_1, L_2 of E, $E = L_1 \oplus L_2$, $\dim L_2 = n$, and an isotopy f on E such that f_1 maps K linearly into L_1 and f_0 is the identity mapping on E. It can be proved that the result is true even when E is an infinite-dimensional complete metrizable locally convex topological vector space.

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