

ON THE MINKOWSKI UNIT OF SLICE LINKS

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1. **Introduction.** Let l be an oriented polygonal link of multiplicity μ in 3-space R^3 and let L be a diagram of l , i.e., an image of a normed regular projection of l onto a plane R^2 [9]. The orientation of L is inherited from that of l . To L is associated a symmetric integral matrix M , called the *symmetric link matrix* of l (with respect to L). It is the symmetrized matrix of the L -principal minor of the matrix of l [8, §3]. If two links are of the same type, then their symmetric link matrices are transformed by a finite sequence of the following two operations and their inverses [8], [10]:

$$Q_1 : A \rightarrow RAR', \text{ with } R \text{ integral and unimodular,}$$

$$Q_2 : A \rightarrow \left(\begin{array}{c|cc} A & & 0 \\ \hline & 0 & 1 \\ 0 & \hline & 1 & 0 \end{array} \right),$$

R' denoting the transposed matrix of R .

Now the symmetric link matrix M of a link l may be singular. Then, to M is associated a nonsingular integral matrix B of the highest rank, unless M is a zero matrix (Lemma 2.1). Thus the Minkowski unit $C_p(B)$ ⁽¹⁾ of B can be defined for any prime integer p , including $p = \infty$ (see §2), and is in fact an invariant of the link type. It will be denoted by $C_p(l)$.

The Minkowski unit for a *knot*, i.e., a link of multiplicity one, was defined by Goeritz only for odd prime integers [4]. Our Minkowski unit for a link is regarded as a generalization of Goeritz's, although the underlying quadratic form is quite different from the one used by him. The particular importance of our quadratic form is that its *signature*⁽²⁾ is an invariant of the link type. This enables us to define the Minkowski unit for $p = 2$ and $p = \infty$ [5].

Now the concept of a slice link was first introduced by Fox and Milnor for a knot [2] and later Fox generalized it to the case of links [3]. A link l is called a *slice link* if it is a cross section of a locally flat 2-sphere in 4-space R^4 . If l bounds a locally flat surface of genus 0 in the *half-space* H^4 ⁽³⁾ of R^4 , then we say that l is a *slice link in the weak sense*. If l is a slice link, then it is also a slice link in the weak sense, but the converse is not true. Cf. the

Received by the editors November 20, 1963.

⁽¹⁾ By $C_p(B)$ is meant the Minkowski unit $C_p(f)$ of the quadratic form f associated to B .

⁽²⁾ For definition, see §2.

⁽³⁾ H^4 is a set of all points in R^4 whose x_4 -coordinate ≥ 0 .

remark in §3. In the case of knots, there are no distinctions between them.

Up to the present there are three necessary conditions for a knot to be a slice knot. They are the following:

- (1) The Alexander polynomial $\Delta(t)$ must be of the form [2]: $\Delta(t) = f(t)f(t^{-1})$.
- (2) The Minkowski unit defined by Goeritz must be $+1$ for any odd prime integer p [1].
- (3) The signature of a slice knot must be zero [8].

The main purpose of this paper is to obtain conditions similar to (1) and (2) for slice links.

2. Minkowski unit. In this section some definitions, propositions and lemmas will be given without proofs. For details, for example, see [6] or [7].

Two $n \times n$ symmetric rational matrices A_1 and A_2 are said to be *S-equivalent* if one can be obtained from the other by a finite sequence of the operations Q_1 and $Q_2^{\pm 1}$. If they are transformed by a finite sequence of Q'_1 , instead of Q_1 , and $Q_2^{\pm 1}$, then they are said to be *R-equivalent*, where

$$Q'_1 : A \rightarrow RAR', \text{ with a nonsingular rational matrix } R.$$

If two matrices are *S-equivalent* then they are *R-equivalent*.

LEMMA 2.1. *Any $n \times n$ nonzero symmetric rational matrix A may be transformed by Q'_1 into a matrix of the form*

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

where B is nonsingular. In particular, if A is an integral matrix, then A may be transformed by Q_1 into the same form.

B is called a *nonsingular matrix associated to A* . If A is an integral matrix, we may assume that so is B . We define the *nullity*, $n(A)$, of A as n minus the rank of B and $\delta(A)$ as the square free part of $|\det B|$.

(2.1) *If A_1 and A_2 are R-equivalent, then $n(A_1) = n(A_2)$ and $\delta(A_1) = \delta(A_2)$.*

Let A be an $n \times n$ symmetric integral matrix of rank r and B a nonsingular integral matrix associated to A .

LEMMA 2.2. *There is a sequence B_1, B_2, \dots, B_r , called the σ -series, of principal minors of B such that*

- (1) B_i is of order i and is a principal minor of B_{i+1} ,
- (2) no consecutive matrices B_i and B_{i+1} are both singular, for $i = 1, 2, \dots, r - 1$ [6, p. 9].

Let us denote $D_i = \det B_i$. Then for any prime integer p , we define [6, p. 31]

$$c_p(B) = (-1, -D_r)_p \prod_{i=1}^{r-1} (D_i, -D_{i+1})_p,$$

where $(a, b)_p$ denotes the Hilbert symbol. If $D_{i+1} = 0$, then $(D_i, -D_{i+1})_p \times (D_{i+1}, -D_{i+2})_p$ is interpreted to be $(D_i, -h)_p(h, -D_{i+2})_p$, h being an arbitrary nonzero integer.

(2.2) $c_p(B)$ is independent of the choice of σ -series of B .

(2.3) If two nonsingular rational matrices B_1 and B_2 can be transformed by Q_1 from each other, then $c_p(B_1) = c_p(B_2)$ for any prime integer p ⁽⁴⁾.

Now following Hasse [5] the Minkowski unit $C_p(B)$ for a nonsingular integral matrix will be introduced as follows, r being the rank of B .

DEFINITION 2.1. For any odd prime integer p ,

$$C_p(B) = c_p(B) (\det B, p)_p^\alpha$$

and

$$C_2(B) = c_2(B) (-1)^\beta$$

where α denotes the exponent of p occurring in $\det B$, and

$$\beta = [r/4] + \{1 + [r/2]\}(d + 1)/2 + (d^2 - 1)m/8,$$

$[]$ denoting the Gauss symbol m the exponent of 2 occurring in $\det B$ and $d = 2^{-m} \det B$.

Finally, for $p = \infty$,

$$C_\infty(B) = \prod C_p(B),$$

product extending over all prime integers.

Let σ be the signature of B . σ is defined by means of the σ -series of B as follows.

$$\sigma = \sum_{i=0}^{r-1} (\text{sign } D_i \cdot \text{sign } D_{i+1}),$$

where $\text{sign } D_i = D_i/|D_i|$ for $D_i \neq 0$, $\text{sign } 0 = 0$ and $D_0 = 1$. Then we can prove [5, p. 223]

$$(2.4) \quad C_\infty(B) = (-1)^\gamma,$$

where $\gamma = [(\sigma - 2\nu)/2] + [(\sigma - 2\nu)/4]$ and ν denotes the number of odd primes of the form $4s + 3$ occurring with odd powers in the prime factor decomposition of $\det B$.

(2.5) For any odd prime integer p and

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix},$$

B_1 and B_2 being integral and nonsingular,

⁽⁴⁾ $c_p(B)$ is not invariant under an operation $Q_2^{\pm 1}$.

$$C_p(B) = C_p(B_1)C_p(B_2)(\det B_1, p)^{\alpha_1}(\det B_2, p)^{\alpha_2}(\det B_1, \det B_2)_p,$$

where α_1 and α_2 denote the powers of p occurring in $\det B_1$ and $\det B_2$, respectively.

Let A be a symmetric integral matrix.

LEMMA 2.3. *Let B_1 and B_2 be nonsingular integral matrices associated to A . Then $C_p(B_1) = C_p(B_2)$ for any prime integer, including $p = \infty$.*

Consequently, $C_p(A)$ is defined as $C_p(B_1)$.

LEMMA 2.4. *If two symmetric integral matrices A_1 and A_2 are R -equivalent, then $C_p(A_1) = C_p(A_2)$ for any prime integer p , including $p = \infty$.*

Now let l be an oriented link and let L be its diagram. Then as is stated in §1, the S -equivalent class of symmetric link matrix M of l is a link invariant. Thus from Lemma 2.4, we can define the Minkowski unit $C_p(l)$ for l as $C_p(M)$ for any prime integer p , including $p = \infty$. Moreover, from (2.1), $n(M)$ and $\delta(M)$ are also link invariants, which will be denoted by $n(l)$ and $\delta(l)$ respectively.

3. Slice links. Let $R_{[a,b]}$, $-\infty < a \leq b < \infty$, be the subspace in 4-space R^4 defined as follows:

$$R_{[a,b]} = \{ (x_1, x_2, x_3, x_4) \mid a \leq x_4 \leq b \}.$$

In particular R_a^3 means $R_{[a,a]}$.

Now consider a polyhedral and locally flat orientable surface F without singularity in general position in R^4 . Cut it by the family of hyperplanes R_t^3 , $-\infty < t < \infty$. We can assume without loss of generality that there are only a finite number of t -values that are singular. A singular hyperplane may intersect F in an isolated point, which may be either a maximum or minimum for the height, called an *extreme point*, or it may intersect F in a graph with just one node, which may be assumed of order four. These nodes are called *saddle points*.

Let us denote $F_t = F \cap R_t^3$. F_t is a link if R_t^3 is not singular.

Let P be a *singular point*, i.e., an extreme point or a saddle point in F . Let t be the *height* of P , i.e., $P \in F_t$. Take a sufficiently small $\epsilon > 0$ so that $R_{[t-\epsilon, t+\epsilon]} \cap F$ contains no singular points other than P . We may assume, hereafter, that none of $F_{t-\epsilon}$ and $F_{t+\epsilon}$ are empty. Thus they are both links.

LEMMA 3.1. *If $n(F_{t-\epsilon}) - n(F_{t+\epsilon}) = \pm 1$, then*

$$\delta(F_{t-\epsilon}) = \delta(F_{t+\epsilon}) \quad \text{and} \quad C_p(F_{t-\epsilon}) = C_p(F_{t+\epsilon}),$$

for any prime integer p , including $p = \infty$.

Proof. The multiplicity of $F_{t-\epsilon}$ differs by one from that of $F_{t+\epsilon}$. We need only prove the case where the multiplicity of $F_{t-\epsilon}$ is greater than that of $F_{t+\epsilon}$. Now the symmetric link matrices M and M' of $F_{t-\epsilon}$ and $F_{t+\epsilon}$ can be chosen as follows. (Cf. Lemma 7.1 in [8].)

$$M' = \left(\begin{array}{c|c} b & M_{12} \\ \hline M_{21} & M \end{array} \right), \quad b \text{ being an integer.}$$

Let B and B' be nonsingular integral matrices associated to M and M' . Hence there exist integral unimodular matrices R and S such that

$$RMR^t = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad SM'S^t = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} M' \begin{pmatrix} 1 & 0 \\ 0 & R^t \end{pmatrix} = \left(\begin{array}{c|c} b & M_{12} \\ \hline M_{21} & B & 0 \\ & 0 & 0 \end{array} \right)$$

and since B is not singular, we can find a nonsingular rational matrix T such that

$$TM'T^t = \left(\begin{array}{c|c|c} B & 0 & 0 \\ \hline 0 & c & d_1 \cdots d_s \\ \hline 0 & d_1 \\ & \vdots \\ & d_s & 0 \end{array} \right),$$

d_i being integers and $s = n(M)$.

Case 1. Suppose that $n(F_{t-}) = n(M) = s$ and $n(F_{t+d}) = n(M') = s + 1$.

Since the order of M' is equal to that of M plus one, the rank of M' and that of M are equal. Thus c, d_1, \dots, d_s are all zero. Therefore it follows that

$$\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} = (TS^{-1}) \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix} (TS^{-1})^t.$$

Since the order of B and that of B' are equal, there is a rational matrix U such that $B = UB'U^t$ (cf. [7, Lemma 2]). Hence B and B' are R -equivalent. Thus from Lemma 2.4 it follows that $C_p(B') = C_p(B)$, i.e., $C_p(M') = C_p(M)$, for any prime integer p , including $p = \infty$. Since $\det B' = (\det U)^2 \det B$, it is evident that $\delta(M') = \delta(B') = \delta(B) = \delta(M)$.

Case 2. Suppose that $n(F_{t-}) = s$ and $n(F_{t+d}) = s - 1$.

Since the rank of M' is equal to that of M plus two, at least one of d_1, \dots, d_s , say d_1 , is not zero. Thus there exists a rational matrix V such that

$$VM'V^t = \left(\begin{array}{c|c|c|c} B & 0 & 0 & 0 \\ \hline 0 & 0 & d_1 & 0 \\ \hline 0 & d_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right).$$

Since the order of B' is equal to that of B plus two, and since B and B' are not singular, we can find a rational matrix W such that

$$WB'W^t = \left(\begin{array}{c|cc} B & 0 & 0 \\ \hline 0 & 0 & d_1 \\ 0 & d_1 & 0 \end{array} \right).$$

Hence B and B' are R -equivalent. Thus from Lemma 2.4 it follows that $C_p(B') = C_p(B)$, i.e., $C_p(M') = C_p(M)$, for any prime integer p , including $p = \infty$. It is evident that $\delta(M') = \delta(B') = \delta(B) = \delta(M)$, q.e.d.

If P is an extreme point, then $n(F_{t-}) = n(F_{t+}) \pm 1$. Thus we obtain

COROLLARY 3.2. *If $P \in R_i^3$ is an extreme point, then*

$$\delta(F_{t-}) = \delta(F_{t+}) \quad \text{and} \quad C_p(F_{t-}) = C_p(F_{t+})$$

for any prime integer p , including $p = \infty$.

LEMMA 3.3. *Let F be a locally flat 2-sphere in R^4 and let P be a singular point in F_t . Then*

$$\delta(F_{t-}) = \delta(F_{t+}) \quad \text{and} \quad C_p(F_{t-}) = C_p(F_{t+}),$$

for any prime integer p , including $p = \infty$.

Proof. From (9.16) in [8], it follows that $\sigma(F_{t-}) = \sigma(F_{t+})$. Thus Lemma 7.1 (2) in [8] implies that $n(F_{t-}) = n(F_{t+}) \pm 1$. Therefore the Lemma follows from Lemma 3.1.

From this lemma, we obtain the following

THEOREM 3.4. *If l is a slice link, then*

$$\delta(l) = 1 \quad \text{and} \quad C_p(l) = 1,$$

for any prime integer p , including $p = \infty$.

Proof. From Lemma 3.3 it follows that $\delta(l) = \delta(\iota)$ and $C_p(l) = C_p(\iota)$ for any prime integer p , including $p = \infty$, where ι denotes a trivial knot. Since $\delta(\iota) = 1$ and $C_p(\iota) = 1$, the theorem follows immediately.

COROLLARY 3.5. *If k is a slice knot, then $\Delta(-1)$ is a square and $C_p(k) = 1$ for any prime integer p , including $p = \infty$.*

REMARK. *There is a slice link in the weak sense such that $C_2 = -1$.*

Consider the torus link l of the type $(2, 2p)$, p being a prime integer of the form $8s \pm 3$. Give an orientation to each component so that the symmetric link matrix M of l is given by an 1×1 matrix $(2p)$. Then

$$C_q = (-1, -2p)_q(2p, p)_q = (-1, -1)_q(2p, -p)_q = (-1, -1)_q(2, -p)_q.$$

Therefore $C_2 = -1$ and $C_p = -1$.

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