FACTORIZATION OF DIFFERENTIABLE MAPS WITH BRANCH SET DIMENSION AT MOST \( n - 3 \)

BY

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1. **Introduction.** Let \( M^n \) and \( N^n \) be separable \( n \)-manifolds (without boundary, unless otherwise specified). The branch set \( B_f \) of a map \( f: M^n \to N^n \) is the set of points in \( M^n \) at which \( f \) fails to be a local homeomorphism. The map \( f \) is monotone if, for each \( y \in N^n \), \( f^{-1}(y) \) is a continuum, and \( f \) is proper if, for each compact set \( X \subseteq N^n \), \( f^{-1}(X) \) is compact.

1.1. **Theorem.** Let \( M^n \) and \( N^n \) be connected \( C^m \) manifolds (\( m \geq 3 \)), and let \( f: M^n \to N^n \) be \( C^m \) and proper with \( \dim(B_f) \leq n - 3 \). Then there is a factorization \( f = hg \) such that

1. \( g: M^n \to K^n \) is a \( C^m \) monotone map onto the \( C^m \) \( n \)-manifold \( K^n \); and
2. \( h: K^n \to N^n \) is a \( k \)-to-1 \( C^m \) diffeo-covering map. Moreover,
3. if \( \overline{hg} \) is another such factorization with intermediate space \( L^n \), then there is a \( C^m \) diffeomorphism \( \alpha \) of \( K^n \) onto \( L^n \) such that \( \overline{g} = \alpha g \) and \( \overline{h} = h\alpha^{-1} \).

The differentiability condition \( C^m \) may be replaced by \( C^n \) or real analytic. If \( M^n \) and \( N^n \) are compact oriented manifolds, then \( k \) is the absolute value of the degree of \( f \).

1.2. **Corollary.** Let \( M^n \) and \( N^n \) be compact connected oriented \( C^3 \) manifolds, and let \( f: M^n \to N^n \) be \( C^3 \). If \( N^n \) is simply connected and degree \( f \neq \pm 1 \), then \( \dim(B_f) \geq n - 2 \).

1.3. **The outline of the proof of (1.1).** In (2.1) it is shown that the existence of the desired factorization is equivalent to two topological properties ((1) and (2)), and the remainder of the paper is devoted to showing that if \( f \) satisfies the hypotheses of (1.1), then it satisfies these properties. In (2.4), conclusion (1) is proved in a very special case, and (2.4) is used in (2.8) to deduce property (1) in case \( n \geq 4 \) and \( f(B_f) \subseteq f(R_{n-2}(f)) \) (definition below). In (3.5) it is shown that, given any map \( f \) satisfying the hypotheses of (1.1) (with \( n \geq 4 \)), there is another map \( h \) such that, for each \( y \in N^n \), \( h^{-1}(y) \) and \( f^{-1}(y) \) have the same number of components and \( h \) satisfies the hypothesis of (2.8); it follows that \( f \) satisfies condition (1). That it satisfies condition (2) is shown in (3.2). The cases \( n \leq 3 \) are treated separately in (3.6), and the uniqueness in (1.1) is also given in (3.6).

The set of points \( x \in M^n \) at which the Jacobian matrix of \( f \) has rank at most \( q \) is denoted by \( R_q(f) \) or, if there is no ambiguity, by \( R_q \).

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nates are written up, e.g., $x^i, f^i$, and $f|X$ is the restriction of $f$ to the set $X$. The interior of $X$ is denoted by $\text{int } X$, the boundary by $\text{bdy } X$, and closure by $\text{Cl } [X]$ or $\overline{X}$. The distance between $x$ and $y$ is $d(x, y)$, and $S(x, \delta)$ is the sphere about $x$ of radius $\delta$; a map is a continuous function. Čech homology and cohomology are consistently used.

2. The proof of (1.1) in case $f(B) \subset f(R_{n-2})$.

2.1. Lemma. Let $M^n$ and $N^n$ be connected $C^m$ $n$-manifolds, let $f : M^n \to N^n$ be a proper $C^m$ map, and let $k = 1, 2, \ldots$. Suppose that, for each point $y \in N^n$,

(1) $f^{-1}(y)$ has exactly $k$ components, and
(2) for each region $U$ and each component $V$ of $f^{-1}(U)$, $f(V) = U$.

Then $f$ has the factorization of Theorem (1.1).

For $M^n$ compact, a map $f$ satisfying (2) is quasi-monotone [24, pp. 151, 152, (8.1)].

Proof. The manifold $N^n = \bigcup_{i=1}^\infty X_i$, where $X_i$ is compact and $X_i \subset X_{i+1}$ $(i = 1, 2, \ldots)$. Since $f$ is proper, $f|f^{-1}(X_i)$ has a unique monotone-light factorization [24, p. 141]; it follows that $f$ also is $hg$, where $g$ is monotone and $h$ is light [24, p. 130, (4.4)]. Given any point $y \in N^n$, there is a connected open set $U$ containing $y$ such that the $k$ components of $f^{-1}(y)$ are in different components of $f^{-1}(U)$. By (1) and (2), $f^{-1}(U)$ has exactly $k$ components. Thus $h^{-1}(U)$ also has $k$ components [24, p. 138, (2.2)] and $h$ is 1-to-1 on each. By the local compactness of $M^n$ and the Theorem on Invariance of Domain, $h$ is a local homeomorphism. Hence $h$ is a $k$-to-1 covering map [18, p. 128]. A natural $C^m$ structure is thus induced by $h$ on $K^n = g(M^n)$, so that $g$ is $C^m$ and $h$ is a $C^m$ diffeo-covering map.

2.2. Lemma. Let $f : M^2 \to N^2$ be a nonconstant map, where $M^2$ and $N^2$ are connected compact 2-manifolds, possibly with boundary. Suppose that

$$\dim (f(B_i)) \leq 0, \quad f(B_i) \cap \text{bdy } (N^2) = \emptyset, \quad f| [M^2 - f^{-1}(f(B_i))]$$

is a $k$-to-1 covering map, and $H^1(f^{-1}(y); Z_2) = 0$ for each $y \in N^2$. Then there exists a unique factorization $f = hg$, where $g$ is a monotone map of $M^2$ onto $M^2$ and $h$ is a light open map of $M^2$ onto $N^2$.

Proof. Since $f$ is nonconstant and $\dim (f(B_i)) \leq 0$, $f(M^2)$ meets $N^2 - f(B_i)$; from the covering property of $f$, it follows that $f$ is quasi-monotone [24, p. 152, (8.1)]. Let $hg$ be the monotone-light open factorization [24, p. 153, (8.4)] of $f$; it is unique [24, p. 141, (4.1)]. Since $h(g(M^2))$ is both open in $N^2$ and compact, and since $N^2$ is connected, $h$ is onto. It suffices to prove that $g(M^2)$ is (homeomorphic to) $M^2$.

Each component $K$ of $\text{bdy } (M^2)$ is a simple closed curve; by the acyclic condition, $f(K)$ is a nondegenerate continuum. Since $\dim (f(B_i)) \leq 0$, it follows from the covering property that $f(K) \subset \text{bdy } (N^2)$; thus $\text{int } (M^2) \supset f^{-1}(\text{int } (N^2))$. Since $f(B_i) \cap \text{bdy } (N^2) = \emptyset$, $\text{int } (M^2) = f^{-1}(\text{int } (N^2))$. Hence,
for each \( y \in g(\text{int}(M^2)) \), \( g^{-1}(y) \subseteq \text{int}(M^2) \), so that \( g|\text{int}(M^2) \) is acyclic mod 2. Since \( h \) is light, \( \dim(g(M^2)) \leq 2 \) [10, pp. 91-92, Theorem VI. 7]. Since any manifold is an orientable gm (generalized manifold) mod 2, it now follows that \( g(\text{int}(M^2)) \) is a 2-gm [25, p. 22], and thus is a 2-manifold [26, pp. 271-280]. Hence, \( g(M^2) \) is a 2-manifold with boundary (and, in fact, \( H^i(M^2; \mathbb{Z}_2) \approx H^i(g(M^2); \mathbb{Z}_2) \) \( (i = 0, 1, 2) \) by the Vietoris Mapping Theorem [2]).

Since \( B_h \) consists of a finite set of points [24, p. 198, (5.1)], there exists a finite set of mutually disjoint closed 2-cells \( E_j \) \( (j = 1, 2, \ldots, m) \) such that \( \partial E_j \cap f(B_j) = \emptyset \), \( f(B_j) \subseteq \bigcup_{j=1}^m \text{int}(E_j) \), and each component of \( h^{-1}(E_j) \) is again a closed 2-cell containing at most one point of \( B_h \) [24, p. 198]. If \( L \) is a component of \( h^{-1}(E_j) \), then \( g^{-1}(L) \) is a 2-manifold with boundary \( g|\text{int}(L) \) is acyclic mod 2. By [2], \( H^i(g^{-1}(L); \mathbb{Z}_2) = 0 \), so that \( g^{-1}(L) \) is a closed 2-cell. Thus \( M^2 \) is homeomorphic to \( g(M^2) \).

2.3. Lemma. Let \( M^n \) and \( N^n \) be compact connected \( n \)-manifolds, possibly with boundary, and let \( n \geq 2 \). Let \( f: M^n \rightarrow N^n \) be a map with \( \dim(f(B_j)) \leq 0 \), \( f(B_j) \subseteq \text{int}(N^n) \), and \( f^{-1}(y) \) having at most \( k \) components for each \( y \in N^n \). Then for all but at most \( k + r \) points \( y \), where \( r = \dim(H^{n-1}(M^n; \mathbb{Z}_p)) \) and \( p \) is any prime, each component \( A \) of \( f^{-1}(y) \) has \( H^{n-1}(A; \mathbb{Z}_p) = 0 \) (and, if \( n = 2 \), is acyclic mod \( p \)). If \( \partial y(N^n) \neq \emptyset \) and \( f \) is not constant, then there are at most \( r \) such exceptional points.

Proof. Suppose that there are (at least) \( m \) points \( y \) such that

\[
H^{n-1}(f^{-1}(y); \mathbb{Z}_p) \neq 0.
\]

If \( \Gamma \) is the union of these sets \( f^{-1}(y) \), then \( \dim(H^{n-1}(\Gamma; \mathbb{Z}_p)) \geq m \). From the exactness of the cohomology sequence, it follows that

\[
\dim(H^n(M^n, \Gamma; \mathbb{Z}_p)) \geq m - r.
\]

Now \( f(\Gamma) \subseteq f(B_j) \), and the hypotheses imply that \( f|f^{-1}(N^n - f(B_j)) \) is a covering map [18, p. 128] of degree at most \( k \) (\( \Gamma \) may be \( M^n \) itself). Suppose that \( f \) is not constant; since \( \dim(f(B_j)) \leq 0 \), each component \( L \) of \( f^{-1}(N^n - f(\Gamma)) \) has \( f(L) \cap (N^n - f(B_j)) = \emptyset \). Hence each component of \( f^{-1}(N^n - f(\Gamma)) \) is mapped by \( f \) onto \( N^n - f(\Gamma) \), so that \( M^n - \Gamma \) has at most \( k \) components and \( \dim(H^n(M^n, \Gamma; \mathbb{Z}_p)) \leq k \). Thus \( m \leq k + r \). The same conclusion is immediate if \( f \) is constant.

If \( \partial y(N^n) \neq \emptyset \) and \( f \) is not constant, then (since \( f(B_j) \subseteq \text{int}(N^n) \)) each component of \( M^n - \Gamma \) meets the (nonempty) set \( \partial y(M^n) \) so that \( H^n(M^n, \Gamma; \mathbb{Z}_p) = 0 \). Thus \( m \leq r \).

2.4. Lemma. Let \( F: M^2 \times E^{n-2} \rightarrow N^2 \times E^{n-2} \) be a \( C^2 \) map such that \( F^2(x, t) = t \), where \( M^2 \) and \( N^2 \) are compact, connected 2-manifolds (possibly with boundary), \( n \geq 3 \), \( \dim(B_E) \leq n - 3 \), and

\[
F(B_E) \subseteq F(R_{n-2}(E)) \cap (\text{int}(N^2) \times E^{n-2}).
\]
Suppose further that $F[[(M^2 \times E^{n-2}) - F^{-1}(F(B_F))]$ is a $k$-to-1 covering map. If $\pi_1(N^2) = 0$, then $F$ is acyclic mod 2.

(The expression $F^2$ is the second coordinate of $F$.)

The lemma is first proved under the additional hypothesis that $H^1(F^{-1}(y, t); Z_2) = 0$ for all $(y, t) \in N^2 \times E^{n-2}$ (yielding $F$ monotone). Then, from this case and (2.3), which shows that “most” points $(y, t)$ satisfy this condition, the general result is finally obtained.

**Proof.** It follows from [5, (1.1)] that if $F_t$ is the map $F|((M^2 \times \{t\})$ then

$$F(R_{n-2}(F)) \cap (N^2 \times \{t\}) = F_t(R_0(F_t))$$

for all $t \in E^{n-2}$, and thus [3, (1.3)] has dimension at most 0. Hence

(1) $\dim(F(B_F) \cap (N^2 \times \{t\})) \leq 0$ for each $t \in E^{n-2}$.

Note that the branch set of $F_t$ is contained in $B_F$. From (1) and the covering property of $F$ ($k \geq 1$) it follows that

(2) each $F_t$ is onto.

Suppose that $D$ is a closed 2-cell in $N^2$ and, for some $s \in E^{n-2}$ and $\nu > 0$, $F(B_D) \cap ((\text{bdy} D) \times S(s, \nu)) = \emptyset$; let $K$ be a component of $F^{-1}(D \times S(s, \nu))$.

From the covering property of $F$ it follows that each component of $F^{-1}(\text{bdy} D \times S(s, \nu))$ is homeomorphic to $S^1 \times E^{n-2}$, where $S^1$ is a circle; and from (2) that $F(K) = D \times S(s, \nu)$. As a result

(3) there is a connected 2-manifold-with-boundary $L^2$ and a homeomorphism $\mu$ of $K$ onto $L^2 \times S(s, \nu)$ such that the restriction

$$\mu|(K \cap (M^2 \times \{t\})) = L^2 \times \{t\}$$

for each $t \in S(s, \nu)$.

**First case.** Suppose that $H^1(F^{-1}(y, t); Z_2) = 0$ for each $(y, t) \in N^2 \times E^{n-2}$. Thus, each component of $F^{-1}(y, t)$ is acyclic mod 2. In this case it suffices to prove that $F$ is monotone.

Let $H_t G_t$ be the factorization of $F_t$ given by (2.2) (cf. (2)). Suppose that, for some $t \in E^{n-2}$, $H_t$ has no branch point. Since $H_t$ is a covering map [18, p. 128] and $\pi_1(N^2) = 0$, $H_t$ is a homeomorphism; thus $F_t$ is monotone and $k = 1$. From (1) and the covering property of $F$, it follows that $F_s$ is also monotone for each $s \in E^{n-2}$. Thus,

(4) if, for some $t \in E^{n-2}$, $H_t$ has no branch point, then $F$ is monotone, the desired conclusion.

It follows from [24, p. 198] that $H_t$ is topologically equivalent to a simplicial map. By [22] the number $\lambda(t)$ of branch points of $H_t$ is at most $k \chi(N^2) - \chi(M^2)$, where $\chi$ is the Euler characteristic. Choose $t \in E^{n-2}$ such that $\lambda(t)$ is a maximum. By (1) there exists a mutually disjoint finite family of closed 2-cells $D_j \subset \text{int}(N^2)$ ($j = 1, 2, \ldots, J$) such that

(a) $F(B_D) \cap (\text{bdy}(D) \times \{t\}) = \emptyset$,

(b) $F(B_D) \cap (N^2 \times \{t\}) \subset \bigcup_{j=1}^J(\text{int}(D_j) \times \{t\})$, and
(c) each component of $H^{-1}_{t}(D_{j} \times \{t\})$ is a closed 2-cell containing at most one branch point of $H_{t}$ [22, p. 198]. (Note that $F_{n}$ and thus $H_{n}$ is locally a homeomorphism on a neighborhood of $F_{t}^{-1}(\text{bdy}(N^{2}) \times \{t\})$.) From (c) and (2.2) applied to $F_{i}|F_{t}^{-1}(D_{j} \times \{t\})$, each component of $F_{t}^{-1}(D_{j} \times \{t\})$ is a closed 2-cell.

There exists a $\delta > 0$ such that each $t$ in $S(\overline{t}, \delta)$ also satisfies condition (a) and (b). Let $K_{m}$ $(m = 1, 2, \ldots, \lambda(\overline{t}))$ be those components of $F_{t}^{-1}(D_{j} \times \{t, \delta\})$ $(j = 1, 2, \ldots, J)$ such that $G_{t}(K_{m} \cap (M^{2} \times \{t\}))$ has a branch point of $H_{t}$ (and only one by (3) and condition (c)). By (4) we may suppose that $H_{t}$ has a branch point, i.e., $\lambda(\overline{t}) \geq 1$. Since $\pi_{1}(D_{j}) = 0$, the argument of (4) may be applied to the restriction $F|K_{m}$. Since $F|K_{m}$ is not monotone, $G_{t}(K_{m} \cap (M^{2} \times \{t\}))$ has at least one branch point of $H_{t}$ for each $t \in S(\overline{t}, \delta)$; by the maximality of $\lambda(\overline{t})$, it has precisely one $(m = 1, 2, \ldots)$. Let $D_{j}$ be the 2-cell such that $F(K_{m}) = D_{j} \times S(\overline{t}, \delta)$, and for each $t \in S(\overline{t}, \delta)$ let $\alpha_{i}(t)$ be the image under $H_{t}$ of the (unique) branch point of $H_{t}$ in $G_{t}(K_{m} \cap (M^{2} \times \{t\}))$. The function

$$\alpha : S(\overline{t}, \delta) \rightarrow \text{int}(D_{i}) \times S(\overline{t}, \delta)$$

is 1-to-1, and (since the union of the branch sets of the $H_{i}$ is closed) $\alpha$ is continuous. Thus the image of $\alpha$ is a (tame) $(n-2)$-cell $\Omega$. Let $\Psi$ be $F_{t}^{-1}(\Omega) \cap K_{i}$, let $\hat{\Omega}$ and $\hat{\Psi}$ be the one-point compactifications of these spaces, and let $\hat{\alpha} : \hat{\Psi} \rightarrow \hat{\Omega}$ be the natural extension of $F|\Psi$. By our assumption, $\hat{\alpha}$ is acyclic mod 2, and, by the Vietoris Mapping Theorem [2], $H_{n-2}(\hat{\Psi}; Z_{2}) = Z_{2}$. Thus $\dim \hat{\Psi} \geq n - 2$ [10, p. 137, (F)], so that $\dim(B_{\Omega}) \geq n - 2$, yielding a contradiction. Thus $F$ is monotone, and hence acyclic mod 2.

Second case. Thus, we may suppose that there exists a point $(y, t) \in N^{2} \times E^{n-2}$ such that $H^{1}(F_{t}^{-1}(y, t); Z_{2}) \neq 0$. By (2.3), for each $t \in E^{n-2}$, there are at most $k + r$ such points $y \in N^{2}$, where $r$ is the dimension of $H^{1}(M^{2}; Z_{2})$. Choose $\overline{t}$ such that the number of such points is maximal, and call the points $y_{i}$ $(i = 1, 2, \ldots, m; m \geq 1)$. There exist mutually disjoint closed 2-cells $D_{i}$ such that $y_{i} \in \text{int}(D_{i})$ and (by (1))

$$F(B_{\Omega}) \cap (\text{bdy}(D_{i}) \times \{t\}) = \emptyset.$$ 

Choose $\delta > 0$ such that

$$F(B_{\Omega}) \cap (\text{bdy}(D_{i}) \times S(\overline{t}, \delta)) = \emptyset \quad (i = 1, 2, \ldots, m).$$

Let $K_{i}$ be a component of $F_{t}^{-1}(D_{i} \times S(\overline{t}, \delta))$ such that

$$H^{1}(F_{t}^{-1}(y_{i}, \overline{t}) \cap K_{i}; Z_{2}) \neq 0,$$

and let $K_{i}(t) = K_{i} \cap F_{t}^{-1}(D_{i} \times \{t\})$. Suppose that for some $i$ and $s \in S(\overline{t}, \delta)$, each $y \in \text{int}(D_{i})$ has $H^{1}((f|F_{i})^{-1}(y, s); Z_{2}) = 0$. Let $f$ be the restriction $F|K_{i}(s)$; by (2), $f$ is nonconstant. Let $hg$ be the factorization given by (2.2).

Suppose first that $h$ has no branch points. It follows as in (4) that $f$ is
monotone, and (by (2.2)) $K_i(s)$ is a closed 2-cell. From (3), $K_i(\tilde{t})$ is also a closed 2-cell, and from (2.3) applied to the restriction map $F\mid K_i(\tilde{t})$, a contradiction of the choice of $K_i$ results.

Thus we may suppose that there exists $x \in B_h$. There exists a closed 2-cell $E \subset D_i$ such that $h(x) \in E \times \{s\}$ and the component $E'$ of $h^{-1}(E \times \{s\})$ containing $x$ is a closed 2-cell containing no other point of $B_h$ [24, p. 198]; by (1), we may suppose that $F(B_h) \cap ((\text{bdy } E) \times \{s\}) = \emptyset$. By (2.2) applied to $F|g^{-1}(E')$, $g^{-1}(E')$ is a closed 2-cell. Choose $\xi > 0$ such that $S(s, \xi) \subset S(\tilde{t}, \delta)$ and $F(B_h) \cap ((\text{bdy } E) \times S(s, \xi)) = \emptyset$. Let $U$ be the component of $f^{-1}(E \times S(s, \xi))$ containing $g^{-1}(E')$; then $U \cap (M^2 \times \{t\})$ is a closed 2-cell for each $t \in S(s, \xi)$ (by (2)). It follows from (2.3) that each component in $U$ of $f^{-1}(y, t)$ is acyclic mod 2 for $y \in E$. By the first part of this proof, $F\mid U$ is monotone, contradicting the choice of $E$ and $E'$.

Thus, there is no such $s$; hence for each $t \in S(\tilde{t}, \delta)$, there is at least one point $\alpha(t) \in \text{int}(D_i) \times \{t\}$ with $H^i(K_i \cap f^{-1}(\alpha(t)); Z_\delta) \neq 0$.

By the choice of $\tilde{t}$, there is precisely one. The function

$$\alpha_t : S(\tilde{t}, \delta) \to \text{int}(D_i) \times S(t, \delta)$$

is 1-to-1 and continuous. Thus, if $D_i$ is assumed to be the closed unit disk, then there is a homeomorphism $\phi$ of $D_i \times S(\tilde{t}, \delta)$ onto itself with $\phi^*(x, t) = t$ and $\phi(\alpha(S(\tilde{t}, \delta))) = \{0\} \times S(\tilde{t}, \delta)$.

Let $P$ and $Q$ be the one-point compactifications of $K_i$ and $K_i \cap (\phi F)^{-1}([0] \times S(\tilde{t}, \delta))$, respectively. It is immediate that

$$H^{n-1}(P, \{p\}; Z_\delta) \approx H^{n-1}(P, \{p\}; Z_\delta),$$

where $p$ is the added point; since $K_i$ is homeomorphic to $L^2 \times S^{n-2}$ (by (3)), $H^{n-1}(P, \{p\}; Z_\delta)$ is isomorphic to $H^{n-1}(L^2 \times Z^{n-2}, L^2 \times \{z\}; Z_\delta)$, where $z$ is any point of $S^{n-2}$. By the exactness of the cohomology sequence ($n \geq 3$), this is mapped onto $H^{n-1}(L^2 \times Z^{n-2}; Z_\delta)$, which, by the K"unneth Formula, is isomorphic to $H^1(L^2; Z_\delta)$. From (5) and (2.3) it follows that $H^1(L^2; Z_\delta) \neq 0$; thus $H^{n-1}(P, \{p\}; Z_\delta) \neq 0$.

Given any closed 2-cell $E \subset D_i - \{0\}$ with $[(\text{bdy } E) \times \{t\}] \cap \phi F(B_{h}) = \emptyset$, there exists $\xi > 0$ such that, for each $s \in S(t, \xi)$, $[(\text{bdy } E) \times \{s\}] \cap \phi F(B_{h}) = \emptyset$ also. Since $H^i((\phi F)^{-1}(y, s); Z_\delta) = 0$ for each $y \in D_i - \{0\}$, it follows from the first part of this proof that

$$F$$

is monotone on each component of $K_i \cap (\phi F)^{-1}(E \times S(t, \xi))$.

Let $V$ be a closed 2-cell such that $V \subset \text{int}(D_i)$, $0 \in \text{int} V$, and

$$[(\text{bdy } V) \times \{t\}] \cap \phi F(B_{h}) = \emptyset,$$

and let $W$ be a component of $K_i \cap (\phi F)^{-1}((D_i - \text{int} V) \times \{t\})$. In the factorization given by (2.2) for $f = \phi F|W$, $h$ has no branch points (from (6)), and so is a finite-to-one covering map [16, p. 128]. Since each finite covering space of the annular region $(D_i - \text{int} V) \times \{t\}$ is itself, (2.2)
implies that \( W \) is homeomorphic to \((D_i - \text{int} V) \times \{t\}\). Since \( V \) may be chosen arbitrarily small, each component of \( K_i \cap (\phi F)^{-1}(D_i - \{0\}) \times \{t\} \) is homeomorphic to \( D_i - \{0\} \). It follows that \( P - Q \) is homeomorphic to \((D_i - \{0\}) \times S(t, \delta)\). Since \( H^{k - 1}(P; Z_2) \neq 0 \), \( H^{k - 1}(Q; Z_2) \neq 0 \) from the continuity of Čech cohomology.

Thus, \( \dim Q \geq n - 1 \) [10, p. 137, (F)], and \( K_i \cap (\phi F)^{-1}(\{0\} \times S(t, \delta)) \) has dimension at least \( n - 1 \). On the other hand, since

\[
H^k(K_i \cap (\phi F)^{-1}(0, t) ; Z_2) \neq 0
\]

for all \( t \in S(t, \delta) \) (by (5) and the definition of \( \phi \)), \( K_i \cap (\phi F)^{-1}(\{0\} \times S(t, \delta)) \subset B_h \), yielding a contradiction.

2.5. Lemma. Let \( N^p \) be a \( C^\infty \) manifold, and let \( \Gamma^p_q \) be a \( C^r \) submanifold of \( N^p \) (\( r = 1, 2, \cdots \)). Then there is a \( C^r \) diffeomorphism \( \psi \) of \( N^p \) onto itself such that \( \psi(\Gamma^p_q) \) is a \( C^r \) submanifold of \( N^p \) and \( \psi \) is arbitrarily near (in the fine \( C^r \) topology) the identity on \( N^p \).

Proof. The fine \( C^r \) topology is defined in [15, p. 25 and p. 28]. (The \( C^r \) topology of [16] is the coarse \( C^r \) topology of [15].) In [15, p. 35, (4.1)] a slight modification of the proof shows that \( f_j \) may be chosen to approximate \( f \) in the \( C^r \) topology, i.e., the partial derivatives of \( f_j \) of orders at most \( r \) approximate those of \( f \).

In the proof of [15, p. 40, (4.7)] let

\[
\psi(x) = x + \tilde{g}(x) - g(x)
\]

for \( x \in \pi^{-1}(\partial E) \), and \( \psi(x) = x \) elsewhere. If \( \delta \) is sufficiently small, then \( \psi \) is a \( C^r \) diffeomorphism near the identity [15, p. 29, (3.10)]; since \( \psi g = \tilde{g} \), it follows that the \( \psi f = h \).

The remainder of the proof is an analog of that of [15, pp. 41, 42, (4.8) and Exercise (a)], where \( f_j = \psi_{j-1} f_j \) (\( j = 1, 2, \cdots \)) and the diffeomorphism \( \psi \) is the (well-defined) limit, as \( j \to \infty \), of the composition \( \psi_j \psi_{j-1} \cdots \psi_2 \psi_1 \).

Morse in [14] proved (2.5) in the special case that \( N^p = E^p \), \( \Gamma^p_q \) is compact, and \( q = 1 \). Assuming only that \( N^p \) is a \( C^r \) manifold, Cerf [3, p. 260] proved that there exist a \( C^m \) manifold \( Q^p \) and a \( C^r \) diffeomorphism \( \lambda \) of \( N^p \) onto \( Q^p \) such that \( \lambda(\Gamma^p_q) \) is a \( C^m \) submanifold of \( Q^p \).

2.6. Remark. The differentiability hypothesis in [22, p. 26, Theorem I. 5] can be changed from \( C^m \) to \( C^{\max(n-q+1,1)} \). (In the proof, replace the theorem of A. P. Morse [22, p. 240] by Sard's Theorem [20].) Moreover, given any \( m \) (\( m = 1, 2, \cdots \)) the diffeomorphism \( A \) in [22, p. 26] may be chosen \( C^m \).

In [22, p. 27] the hypothesis \( C^1 \) suffices, and \( V^m \) need not be compact if \( f \) is proper.

The following remark follows from the proof of [22, p. 27].

2.7. Remark. Let \( f: M^m \to N^n \) be \( C^m \) and proper, let \( K^p \) be a compact \( C^m \)
p-manifold, and let $\sigma$ be a $C^m$ diffeomorphism of a region in $N^n$ onto $K^p \times E^{n-p}$ ($m, p = 1, 2, \ldots$). If $f$ is transverse regular \[22, pp. 22-23\] on $\rho^{-1}(K^p \times \{0\})$, then there are a $C^m$ p-manifold $L^p, \epsilon > 0$, and a $C^m$ diffeomorphism $\sigma$ of $L^p \times S(0, \epsilon)$ onto $f^{-1}(\rho^{-1}(K^p \times S(0, \epsilon)))$ such that $(\rho \sigma)^2(z, r) = r$ for each $r \in S(0, \epsilon)$.

**Proof.** Since $f$ is transverse regular on $\rho^{-1}(K^p \times \{0\})$, $f^{-1}(\rho^{-1}(K^p \times \{0\}))$ is a $C^m$ p-manifold in $M^n$ \[22, p. 23\] (the hypothesis $C^1$ suffices). By \[15, p. 42, (4.9)\] and \[3, p. 260\] there is a $C^m$ diffeomorphism $\psi$ of $M^n$ onto a $C^m$ p-manifold $Q^n$ such that $\psi(f^{-1}(\rho^{-1}(K^p \times \{0\})))$ is a $C^m$ submanifold, call it $L^p$, of $Q^n$. Choose $\epsilon > 0$ so that $f \psi^{-1}$ is transverse regular on $\rho^{-1}(K^p \times \{r\})$ for each $r \in S(0, \epsilon)$ \[22, p. 27\]; thus \[20, p. 26\] $\psi f^{-1}(\rho^{-1}(K^p \times \{r\}))$ is a $C^m$ p-manifold. By the proof of \[22, p. 27\], for each point $x$ of $L^p$, the normal $(n - p)$-plane to $L^p$ at $x$ meets $\psi(f^{-1}(\rho^{-1}(K^p \times \{r\})))$ in precisely one point, call it $\tau(x, r)$; thus $\tau$ is a $C^m$ diffeomorphism of $L^p \times S(0, \epsilon)$ onto $\psi f^{-1}(\rho^{-1}(K \times S(0, \epsilon)))$. Let $\sigma = \psi^{-1} \tau$. (For each $\bar{x} \in f^{-1}(\rho^{-1}(K \times S(0, \epsilon)))$, there is a $C^m$ diffeomorphism $\mu$ of a neighborhood $V$ of $f(\bar{x})$ onto $E^n$ such that $\mu(V \cap \rho^{-1}(K \times \{r\}))$, for each $r \in E^{n-p}$, is defined by $x^i$ constant ($i = 1, 2, \ldots, n - p$). Perhaps it is clearer that both $\sigma$ and $\sigma^{-1}$ are $C^m$ if we observe \[5, (1.1)\] and \[6, (2.3)\] that there is a neighborhood $U$ of $\bar{x}$ and a $C^m$ diffeomorphism $\lambda$ of $U$ onto $E^n$ such that the map $g = \mu \lambda^{-1}$ satisfies $g^i(x^1, x^2, \ldots, x^n) = x^i$ ($i = 1, 2, \ldots, n - p$).

2.8. **Lemma.** Let $M^n$ and $N^n$ be connected $C^3$ n-manifolds, and let $f: M^n \rightarrow N^n$ be $C^3$ and proper, with $n \geq 4$, dim$(B_i) \leq n - 3$ and $f(B_i) \subset f(R_{n-2}(f))$. Then $f$ satisfies hypotheses (1) and (2) of (2.1).

**Proof.** By \[19, §5\], dim$(f(R_{n-2})) \leq n - 2$, so that dim$(f^{-1}(f(R_{n-2}))) \leq n - 2$. Since $f$ is proper, and $R_{n-2}$ is locally compact, $f(R_{n-2})$ is locally compact. The restriction map $f| [M^n - f^{-1}(f(R_{n-2}))]$ is a $k$-to-1 covering map \[18, p. 128\] with connected domain, so that $f$ satisfies (2). Suppose that, for some point $y$ in $N^n$, $f^{-1}(y)$ has at least $k + 1$ components $Y_i$ ($i = 1, 2, \ldots, k + 1$). There is a connected open neighborhood $W$ about $y$ such that the components $Y_i$ are contained in different components of $f^{-1}(W)$. A contradiction results from (2) and the covering property. Thus, for each $y \in N^n$, $f^{-1}(y)$ has at most $k$ components.

If, for each $y$ in $N^n$, there is a Euclidean coordinate neighborhood $E$ about $y$ such that $f^{-1}(E)$ has exactly $k$ components, then, by (2), condition (1) is satisfied. Thus we may suppose that $N^n = E$ and $k \neq 1$, and deduce a contradiction. By \[15, p. 41\] we may suppose that $M^n$ and $N^n$ are $C^\infty$ manifolds, and thus that $E^n = E (= N^n)$.

The homomorphism $f_*$ of $\pi_1(M^n - f^{-1}(f(R_{n-2})))$ into $\pi_1(E^n - f(R_{n-2}))$ induced by $f| [M^n - f^{-1}(f(R_{n-2}))]$ is one-to-one but not onto \[9, pp. 93 and 96\]. Since the semi-linear maps of the circle $S^1$ into $E^n - f(R_{n-2})$
generate $\pi_1(E^n - f(R_{n-2}))$ and $n \geq 3$, the (polyhedral) embeddings of $S^1$ also generate $\pi_1(E^n - f(R_{n-2}))$. Let $\gamma$ be an embedding such that homotopy class of $\gamma$ is not in $f_*(\pi_1(M^n - f^{-1}(f(R_{n-2}))))$; each component of $f^{-1}(\gamma(S^1))$ is a topological circle (by the covering property), and each is mapped by $f$ onto $\gamma(S^1)$ nonhomeomorphically. Moreover, there exists $\epsilon > 0$ such that, if $\lambda$ is any embedding of $S^1$ in $E^n$ with (uniform) distance $d(\lambda, \gamma) < \epsilon$, then $\lambda(S^1)$ has the same property.

By [8, p. 111, Theorem 2a], there is a $C^\omega$ embedding $\lambda$ of $S^1$ in $E^n$ ($n \geq 4$) such that $d(\lambda, \gamma) < \epsilon/2$. By [8, p. 110, Corollary ] there is a $C^\omega$ embedding $\mu$ of the unit 2-disk $D^2$ in $E^n$ which extends $\lambda$. If $D_j$ denotes the unit $j$-disk in $E^j \subset E^n$, $D^2 \subset D^3 \subset D^n$, there is [15, p. 275] a $C^\omega$ embedding $\nu$ of $D^n$ into $E^n$ which extends $\mu$. Since $\text{bdy}(D^2) \subset \text{bdy}(D^3) = S^2$, $\xi = \nu|S^2$ is a $C^\omega$ embedding which extends $\lambda$.

Let $A$ be the ($C^\omega$) map given by [22, p. 26] (cf. 2.6), with (uniform) distance from the identity map less than $\epsilon/2$, for $f$ and $N^{p-q} = \xi(S^2)$. By the proof of the theorem, $f$ is transverse regular [22, p. 22-23] on $A^{-1}(\xi(S^2))$, call it $X^2$. Because $\xi$ is the restriction of $\nu$, $X^2$ has a tubular neighborhood $Z$ and a $C^\omega$ diffeomorphism $\rho$ of $Z$ onto $X^2 \times E^{n-2}$.

By 2.7, if $Z$ is a sufficiently small tubular neighborhood, the restriction of $f$ to each component $U$ of $f^{-1}(Z)$ satisfies the hypotheses of 2.4; thus the restriction $f|U$ is monotone. Because of the choice of $A^{-1}(\xi(S^2))$, each component of $f^{-1}(A^{-1}(\lambda(S^1)))$ is a topological circle mapped by $f$ onto the topological circle $A^{-1}(\lambda(S^1))$ nonhomeomorphically; from condition (2), $f(U) = Z$, and a contradiction results. Thus $f$ does satisfy hypothesis (1) of 2.1.

3. The proof of (1.1). Given $U \subset E^n$ and a $C^1$ map $h: U \to E^1$, $D_j h$ denotes the first partial derivative of $h$ with respect to the $j$th independent variable. If $n = 1$ and $h$ is $C^j$, then $D^{(j)} h$ is the derivative of order $j$ ($D^{(0)} h = h$).

3.1. Lemma. Let $f: M^n \to N^n$ be $C^m$ and proper, with dim$(B_j) \leq n - 2$ and $m = 2, 3, \ldots$. Suppose that the Jacobian matrix (derivative map) has rank at least $n-1$ at every point of $M^n$. Then,

1. for each point $y$ in $N^n$, $f^{-1}(y)$ has a finite number of components, each of which is either a point or a $C^m$ embedding of a closed interval. Moreover, for each open neighborhood $U$ of $y$, there is a region $W$ such that $y \in W \subset U$ and each component of $f^{-1}(W)$ which meets $f^{-1}(y)$ is mapped by $f$ onto $W$.

Proof. We may suppose that $n \geq 2$ [18, p. 128], $M^n$ and $N^n$ are $C^\omega$ Riemannian manifolds [15, p. 42, (4.9)], and, in fact, that $N^n = E^n$. For each point $x \in f^{-1}(y)$, there are $[5, (1.1)]$ $C^m$ diffeomorphisms $\lambda$ of a Euclidean neighborhood $U(x)$ of $x$ onto $E^n$ and $\mu$ of $E^n$ onto itself (the latter merely interchanging dependent variables $[5, (2.3)]$) such that the map $g = \mu^\lambda^{-1}$ has $g^i(x^1, x^2, \ldots, x^n) = x^i$ ($i = 1, 2, \ldots, n-1$). Since dim$(B_j) \leq n - 2$,
the Jacobian determinant of \( g \) is either non-negative or nonpositive. Thus \( D_vg^n(x_1, x_2, \ldots, x_n) \geq 0 \) (say) on all of \( \lambda(U(\bar{x})) \), so that the map \( h: E^1 \rightarrow E^1 \) defined by \( g^n(x_1, x_2, \ldots, x_n) = h(x^n) \) is monotone, for all \( (x_1, x_2, \ldots, x_n) \in E^n \). As a result, since \( f \) is proper, \( f^{-1}(y) \) has a finite number of components, each a point, a \( C^m \) embedding of a closed interval, or a \( C^m \) embedding of the circle \( S^1 \).

Suppose that \( \Omega \) is any component of \( f^{-1}(y) \) and that \( U \) is any open set containing \( y \). There is an open \( n \)-cell \( Y, y \in Y \) and \( \overline{Y} \subset U \), such that each component of \( f^{-1}(\overline{Y}) \) meets at most one component of \( f^{-1}(y) \); let \( X \) be the component of \( f^{-1}(Y) \) which contains \( \Omega \). There is a \( C^m \) embedding \( \alpha \) of a 1-manifold \( M^1 \) in \( X \) such that either \( M^1 = S^1 \) and \( \Omega = \alpha(M^1) \) or \( M^1 = E^1 \) and \( \Omega \subset \alpha(M^1) \). By 2.5 we may suppose that \( \alpha(M^1) \) is a \( C^\infty \) submanifold of \( X \).

Let \( N(\delta) \) consist of the vectors with length at most \( \delta \) in the normal bundle to \( \alpha(M^1) \). For \( \delta \) sufficiently small the restriction of the exponential map \( E \) to \( N(\delta) \) is a \( C^\infty \) diffeomorphism onto a (tubular) neighborhood of \( \alpha(M^1) \). For each \( x \in \Omega \), let \( D(x) \) be the image under \( E \) of the \( (n-1) \)-disk of vectors normal to \( \alpha(M^1) \) at \( x \). From the coordinate representation at \( x \) given above, observe that any set of \( (n-1) \)-vectors spanning an \( (n-1) \)-plane transversal to \( \alpha(M^1) \) at \( x \) is mapped by the Jacobian matrix of \( f \) onto a set of \( n-1 \) independent vectors. Thus the Jacobian matrix of the restriction map \( f|D(x) \) has maximal rank at \( x \). Moreover, given any \( \bar{x} \in \Omega \), if an interval from \( x \) to \( \bar{x} \) in \( \Omega \) is contained in the coordinate neighborhood \( U(\bar{x}) \), then it follows from the form of \( f \) on \( U(\bar{x}) \) (the fact that \( D_\bar{x}g^n \geq 0 \)) that \( f(D(\bar{x})) \) and \( f(D(x)) \) have the same tangent \( (n-1) \)-plane at \( y \). Thus, for all \( x \in \Omega \), the sets \( f(D(x)) \) have the same tangent \( (n-1) \)-plane at \( y \), which we may suppose is that spanned by the first \( n-1 \) coordinate vectors of \( E^n \).

There is a \( C^\infty \) embedding \( \beta \) of \( S^1 \) in \( Y \) such that (a) \( y \in \beta(S^1) \) and (b) \( \beta(S^1) \) is normal to that plane at \( y \). We have observed above that, for any such embedding \( \beta, f \) is transverse regular \([22, pp. 22-23]\) to \( \beta(S^1) \) at each point of \( \Omega \). In \([22, pp. 24-26]\) (cf. 2.6), the space \( H \) of \( C^m \) maps \( A \) may be replaced by that subspace such that \( A(y) = y \) and the first partial derivatives of \( A \) agree with those of the identity map at \( y \). As a result, there is a \( C^m \) embedding \( \gamma \) of \( S^1 \) in \( Y \) such that it satisfies conditions (a) and (b), and \( f \) is transverse regular at each point of \( f^{-1}(\gamma(S^1)) - \{y\} \). Thus \( f|X \) is transverse regular at each point of \( f^{-1}(\gamma(S^1)) \). By 2.5 and \([22, p. 27]\), we may suppose that \( \gamma(S^1) \) is a \( C^\infty \) submanifold of \( Y \).

Define, as above, a tubular neighborhood \( T \) of \( \gamma(S^1) \), \( T \subset Y \). Then there is a \( C^\infty \) diffeomorphism \( \rho \) of \( T \) onto \( \gamma(S^1) \times E^{n-1} \) such that \( \rho(z) = (z, 0) \) for each \( z \in \gamma(S^1) \) (for \( n = 2 \), the Moebius Band cannot be embedded in \( E^2 \); for \( n \geq 3 \), by the proof of \([11, (1.2)]\)). Let \( L^1, \epsilon > 0 \), and \( \sigma \) be as given
by 2.7 for \( f'(X \cap f^{-1}(T)) \) and \( K^n = \gamma(S^i) \), and let \( R \) be the component of \( X \cap f^{-1}(\gamma(S^i) \times S(0, \epsilon)) \) containing \( \Omega \); we will still denote \( R \cap f^{-1}(\gamma(S^i) \times \{0\}) \) by \( L^1 \), a 1-sphere by 2.7.

Suppose that \( \Omega \) is a 1-sphere, i.e., that \( f(L^1) = \{y\} \). Given any closed interval \( \Gamma \) of \( \gamma(S^i) \), \( y \in \text{int} \Gamma \), there exists a sufficiently small neighborhood \( S(0, \xi) \subset S(0, \epsilon) \) such that \( \rho f\sigma(L^1 \times S(0, \xi)) \subset \Gamma \times S(0, \epsilon) \). Because of the form of \( f \) on the neighborhoods \( U(x) \) \( (D_n g^\alpha \geq 0) \), it follows that \( \rho f\sigma(L^1 \times \{r\}) \) is a single point, for each \( r \in S(0, \xi) \), contradicting the fact that dim(\( B \)) < \( n \). Thus \( \Omega \) is not a 1-sphere, yielding conclusion (1).

From the local form of \( f \) \( (D_n g^\alpha \geq 0) \) and the fact that no point inverse is a 1-sphere, \( \rho f\sigma |(L^1 \times \{r\}) \) is the composition of a monotone map and a finite-to-one covering map, and \( \rho f\sigma(L^1 \times \{r\}) = \gamma(S^i) \times \{r\} \). Since \( \Omega \) \( (= f^{-1}(y) \cap R) \) is connected, each map \( \rho f\sigma |(L^1 \times \{r\}) \) is monotone; thus \( f|R \) is monotone, and \( f(R) = T \).

For each \( \bar{U} \), in the finite set of components of \( f^{-1}(y) \), there is an open tubular neighborhood \( T_i \) of \( y \) satisfying the conclusion of 2.7, \( T_i \subset Y \subset U \); let \( R_i \) be the component of \( f^{-1}(T_i) \) containing \( \Omega \). Since \( T_i \subset Y \), the sets \( R_i \) are mutually disjoint. By the above paragraph, \( f|R_i \) is monotone and \( f(R_i) = T_i \). Let \( W \) be the component of \( \bigcap_i T_i \) containing \( y \). Then \( f(R_i \cap f^{-1}(W)) = W \), and \( f(R_i \cap f^{-1}(W)) \) is monotone. By [24, p. 138, (2.2)] (it is sufficient that \( f \) be proper), \( R_i \cap f^{-1}(W) \) is connected, and conclusion (2) follows.

3.2. Corollary. Let \( f : M^n \to N^n \) be a \( C^2 \) proper map with \( \text{dim}(B) \leq n - 2 \). Then \( f \) satisfies condition (2) of (2.1).

Proof. Given a region \( U \subset N^n - f(R_{n-2}) \), let \( V \) be a component of \( f^{-1}(U) \). Suppose that \( f(V) \neq U \). Since \( V \subset \overline{V} \cap f^{-1}(U) \subset V \), \( \overline{V} \cap f^{-1}(U) \) is connected and thus is contained in \( V \). Since \( f \) is proper it follows that \( f(V) = C[f(V)] \cap U \). Thus there exists \( y \in U \cap \text{bdy}(f(V)) \). Let \( W \) be the region given by 3.1, conclusion (2), for \( U \) and \( y \). Since \( y \in f(V) \), \( W \subset f(V) \), and a contradiction results; thus \( f(V) = U \).

Since \( \text{dim}(f(R_{n-2})) \leq n - 2 \) [19, §5] and \( \text{dim}(B) \leq n - 2 \), \( \text{dim}(f^{-1}(f(R_{n-2}))) \leq n - 2 \); that \( f \) satisfies conclusion (2) of 2.1 for arbitrary regions \( U \subset N^p \) follows.

3.3. Lemma. Let \( E^*_\alpha \), \( E^\mu \), and \( \Delta \) be the sets in \( E^n \) defined by \( x^\alpha > 0 \), \( x^\mu < 0 \), and \( \lambda < x^i < \lambda \) \((i = 1, 2, \ldots, n - 1) \) and \( -\mu < x^\nu < \nu \) \((\lambda, \mu, \text{ and } \nu > 0) \), respectively. Let \( U \) be an open set with \( \Delta \subset U \subset E^n \), let \( \epsilon > 0 \), and let \( f : U \to E^n \) be \( C^m \) \((m = 1, 2, \ldots) \) with \( f = x^i \) \((i = 1, 2, \ldots, n - 1) \) on \( \Delta \), \( D_n f^\alpha \geq 0 \) on \( \Delta \), and \( D_n f^\alpha > 0 \) on \( \Delta \cap E^*_\alpha \). Then there is a \( C^m \) map \( g : U \to E^n \) such that

(i) \( f = g \) off \( \Delta \),

(ii) the (uniform) distance between each partial derivative of \( f \) with order \( 0, 1, \ldots, m \) and the corresponding partial of \( g \) is at most \( \epsilon \),

(iii) the Jacobian determinant \( J(g) > 0 \) on \( \Delta \),
(iv) $B_q \subset B_p$ and
(v) $g(\Delta) = f(\Delta)$. Moreover,
(vi) for each $y$ in $f(\Delta)$, $\Delta \cap f^{-1}(y)$ and $\Delta \cap g^{-1}(y)$ each have one component.

**Proof.** We may suppose that $\Delta$ is defined by $0 < x^i < 1$ $(i = 1, 2, \ldots, n-1)$ and $-1 < x^n < 1$, that $\epsilon < 1$, and that $\epsilon$ is less than the minimum of $D_n f^n$ on $\Delta \cap \overline{E^n_+}$. There exists a $C^m$ map $\alpha : E^n \to E^n$ such that $\alpha \geq 0$, $\alpha > 0$ precisely on $(0,1)$, and $D^0 \alpha < \epsilon$ $(j = 0, 1, \ldots, m)$. Let $\beta_k : E^n \to E^n$ be defined by

$$\beta_k(x^1, x^2, \ldots, x^n) = \alpha((-1)^k x^n) \prod_{i=1}^{n-1} \alpha(x^i) \quad (k = 1, 2),$$

let $\gamma = \beta_1$ for $x^n \leq 0$ and $\gamma = -\beta_2$ for $x^n \geq 0$, and let

$$\delta = \int_{-1}^{x^n} \gamma(x^1, x^2, \ldots, x^{n-1}, t) \, dt.$$

Let $g^i = f^i$ $(i = 1, 2, \ldots, n-1)$, and $g^n = f^n + \delta$.

Since $\gamma > 0$ on $\Delta \cap E^n_+$ and $D_n \delta = \gamma$, $D_n g^n > 0$ on $\Delta \cap E^n_+$; since $|\gamma| < \epsilon$ on $\Delta \cap E^n_+$ and $\epsilon < D_n f^n$ on $\Delta \cap \overline{E^n_+}$, $D_n g^n > 0$ on $\Delta \cap \overline{E^n_+}$. Conclusion (iii) follows. Since

$$f^n(x^1, x^2, \ldots, x^{n-1}, 1) = g^n(x^1, x^2, \ldots, x^{n-1}, 1)$$

and

$$f^n(x^1, x^2, \ldots, x^{n-1}, 1) = g^n(x^1, x^2, \ldots, x^{n-1}, 1);$$

since $f = g = x^i$ $(i = 1, 2, \ldots, n-1)$, and since $D_n f^n \geq 0$ and $D_n g^n \geq 0$ on $\Delta$, $g(\Delta) = f(\Delta)$. The reader may verify that $g$ satisfies the remaining conditions.

3.4 **Lemma.** Let $M^n$ and $N^p$ be $C^\omega$ manifolds, and let $\mathcal{C}^m$ be the set of all $C^m$ maps $g : M^n \to N^p$ $(m = 1, 2, \ldots)$. Then there are an embedding of $N^p$ as a closed submanifold of a Euclidean space, and a complete metric $\rho$ on $\mathcal{C}^m$ such that:

1. $\rho(g, g_0) \to 0$ implies that $g_0 \to g$ in the (coarse) $C^m$ topology; and

2. if $d$ is the metric induced on $N^p$ by the embedding and $\rho(g, h) < 1$, then $d(g(x), h(x)) \leq \rho(g, h)$ for all $x \in M^n$.

**Proof.** The coarse $C^m$ topology is defined in [16] (cf [15, pp. 25-28]). Let $T_j(M^n)$ be the tangent bundle of $M^n$, let $T_j(M^n) = T_j(T_{j-1}(M^n))$, and let $d_m g : T_m(M_n) \to T_m(N^p)$ be the $m$th derivative map. Since $T_m(M^n)$ and $T_m(N^p)$ are $C^\omega$ manifolds, each may be embedded as a closed $C^\omega$ submanifold of some Euclidean space [15, p. 20]. For each $x \in M^n$, replace the fiber $F_x$ over $x$ by the unit ball $B_x$ (using the Euclidean metric), $B_x$
$\subset F \subset T_m(M^n)$, defining the space $B(M^n)$. For maps $G$ and $H$ of $B(M^n)$ into $T_m(N^n)$, let $\sigma(G, H)$ be the minimum of 1 and the least upper bound of \{d(G(x), H(x)) : x \in B(M^n)\}, where $d$ is the Euclidean metric; then $\sigma$ is a complete metric. Let $\rho(g, h) = \sigma(d_m g, d_m h)$. From the natural embedding of $N^n$ in $T_m(N^n)$, conclusion (2) follows.

The coarse $C^m$ topology on $\mathbb{C}^n$ is, similarly, the compact open topology of the $m$th derivative maps of $B(M^n)$ (in fact, $T_m(M^n)$) into $T_m(N^n)$. Since the compact open topology is the topology of uniform convergence on compact sets [1, p. 485], $\rho(g, g_j) \to 0$ implies that $g_j \to g$ in the coarse $C^m$ topology. (In fact, if $M^n$ is compact, the topology of $\rho$ is the (coarse = fine) $C^m$ topology.)

Lastly we wish to show that $\rho$ is complete. Suppose that $g_j$ is a $\rho$-Cauchy sequence of $\mathbb{C}^n$; then $d_m g_j$ is a $\sigma$-Cauchy sequence, and thus has a limit $D$. Since the restrictions $d_m g_j | M^n \to D | M^n$ uniformly, i.e., $g_j \to D$ uniformly, it suffices to prove that $d_m(D | M^n) = D$. In case $m = 1$ and $M^n = N^n = E^1$, the result is a straightforward consequence of the Mean-Value Theorem, and the general result follows from this special case.

The topology defined above is intermediate between the fine and coarse $C$ topologies. In [3, p. 272], Cerf observes that the fine $C$ topology (his $C^r$ [3, p. 269]) is not metrizable, but has a complete uniform structure. The coarse $C$ topology is metrizable with a complete metric (use [1, p. 490, Theorem 10] and the metrization theorem for uniform spaces), but, in general, it does not have a metric satisfying condition (2).

3.5. LEMMA. Let $M^n$ and $N^n$ be connected $C^3$ manifolds, and let $f : M^n \to N^n$ be a $C^3$ proper map with $\dim(B_j) \leq n - 3$ and $n \geq 4$. Then $f$ satisfies condition (1) of (2.1).

Proof. By [15, p. 42] we may suppose that $M^n$ and $N^n$ are $C^\infty$ manifolds and that $\mathbb{G}^3$, $d$, and $\rho$ are as in 3.4. By [19, §5], $\dim(f(R_{n-2}(f))) \leq n - 2$, and, since $f$ is proper, $f(R_{n-2}(f))$ is locally compact. There exist compact sets $Y_j$ and open coordinate neighborhoods $U_j$ and $V_j$ such that $Y_j \subset V_j$, $V_j \subset U_j$, $\overline{U}_j$ is compact, the $\overline{U}_j$ are locally finite in $N^n - f(R_{n-2}(f))$, and $N^n - f(R_{n-2}(f)) = \bigcup_{j=1}^\infty Y_j = \bigcup_{j=1}^\infty \overline{U}_j$. Let $F_j$ be the set of $x \in M^n$ such that $d(x, B_j) < 1/j$.

Given $h \in \mathbb{G}^3$ and $x \in M^n$, let $r(h, x)$ be the rank of (the Jacobian matrix of) $h$ at $x$; let $0 < \epsilon < 1$. A sequence of proper maps $f_i \in \mathbb{G}^3$ ($i = 1, 2, \ldots$; $f_0 = f$) satisfies property $\mathbb{P}$ if

(a) $f_i$ agrees with $f_{i-1}$ off $F_i \cap f^{-1}V_j$;
(b) $f_i(f^{-1}(V_j)) \subset U_j$ ($j = 1, 2, \ldots$);
(c) $\bigcup_{j=1}^\infty f^{-1}(Y_j) \cap B(f_i) = \emptyset$, where $B(f_i)$ is the branch set of $f_i$;
(d) for each $x \in M^n$, $r(f_i, x) \geq r(f_{i-1}, x)$, and either $f_i(x) = f(x)$ or $r(f_i, x) = n$. 

(e) $B(f) \subset B(f_{i-1})$;
(f) $\rho(f_i, f_{i-1}) < 2^{-i};$ and
(g) for each $y \in N^n$, $f_i^{-1}(y)$ and $f_{i-1}^{-1}(y)$ have the same number of components.

We wish to construct a sequence $\{f_i\}$ satisfying $\mathfrak{F}$. Suppose that $f_1, f_2, \ldots, f_{i-1}$ have been defined; by (e) $\dim(B(f_{i-1})) \leq n - 3$. There exists $\eta$, $0 < \eta < 2^{-i}$, such that, if $f_i$ is any $C^3$ map satisfying (a) and $\rho(f_i, f_{i-1}) < \eta$, then $f_i$ satisfies (b) (and (f)).

Since $r(f_{i-1}, x) \geq n - 1$ for $x \in f^{-1}(\bar{V})$ (by (d)) there exists $\xi$, $0 < \xi < \eta$, such that $h \in \mathcal{G}^3$, $x \in f^{-1}(\bar{V})$, and $\rho(h, f_{i-1}) < \xi$ implies that $r(h, x) \geq n - 1$. For each $\bar{h} \in F_i \cap f^{-1}(V)$, let $U[h, \bar{x}]$ and $V[h, \bar{x}]$ be the neighborhoods given by [5, (1.1)], $U[h, \bar{x}] \subset F_i \cap f^{-1}(V)$, with diffeomorphisms $k_1[h, \bar{x}]$ of $E^n$ onto $U[h, \bar{x}]$ and $k_2[h, \bar{x}]$ of $V[h, \bar{x}]$ onto $E^n$ such that $k_2h k_1$, call it $H$, has the form $H^j = x^j$ ($j = 1, 2, \ldots, n - 1$). We may suppose that the image of $k_1[h, \bar{x}]$ evaluated at 0 is $\bar{x}$. Let $\Delta_{h, x, \eta, \mu, \nu}$ be the $k_1[h, \bar{x}]$ image of $\Delta_{h, x, \eta, \mu, \nu}$, and let the top of $\Delta_{h, x, \eta, \mu, \nu}$ (respectively, bottom; center line; $\Delta_{h, x, \eta, \mu, \nu}$) be the $k_1[h, \bar{x}]$ image of the subset of $\Delta_{h, x, \eta, \mu, \nu}$ defined by $x^u = \nu$ (respectively, $x^u = -\mu$; $x^u = 0$, $i = 1, 2, \ldots, n - 1$; $x^n \geq 0$).

The compact set $\overline{F_{i+1}} \cap f^{-1}(Y)$ is contained in the open set $F_i \cap f^{-1}(V)$. Given $y \in N^n$ and a component $\Gamma$ of $f_{i-1}^{-1}(y)$ such that $\Gamma \cap \overline{F_{i+1}} \cap f^{-1}(Y) \neq \emptyset$, we will now prove that $\Gamma \subset F_{i+1} \cap f^{-1}(Y)$. We may suppose that $\Gamma$ is nondegenerate; thus $\Gamma \subset B(f_{i-1}) \subset B_i$ (by (e) for $f_{i-1}$), so that $\Gamma \subset F_{i+1}$. By (d) for $f_{i-1}$, $f|\Gamma = f_{i-1}|\Gamma$, so that $f(\Gamma) = \{y\}$. Since $\Gamma \cap f^{-1}(Y) \neq \emptyset$, $\Gamma \subset f^{-1}(Y)$, yielding the desired conclusion.

Since $\dim(B(f_{i-1})) \leq n - 3$ by (e), $r(f_{i-1}, x) \geq n - 1$ for each $x \in f^{-1}(V)$ (by (d)), and $f_{i-1}$ is proper (by (a)), it follows from 3.1 that for each $y \in N^n$, each component $\Gamma$ of $f^{-1}(y) \cap \overline{F_{i+1}} \cap f^{-1}(Y)$ is a point or a $C^3$ embedding of an interval. For each $\bar{x} \in \overline{F_{i+1}} \cap f^{-1}(Y)$, there exist $\lambda(\bar{x}) > 0$, $\mu(\bar{x}) > 0$, and $\nu(\bar{x}) > 0$ such that

$$\Delta[f_{i-1}, \bar{x}, \lambda(\bar{x}), \mu(\bar{x}), \nu(\bar{x})] \subset F_i \cap f^{-1}(V).$$

For each $h \in \mathcal{G}^3$, let $\Delta[h, \bar{x}, \lambda(\bar{x}), \mu(\bar{x}), \nu(\bar{x})]$ be denoted by $\Delta(h, \bar{x})$.

Let $\Gamma$ be the component of $f_{i-1}^{-1}(f_{i-1}(\bar{x}))$ containing $\bar{x}$, then $\Gamma \cap \Delta(f_{i-1}, \bar{x})$ is contained in the center line of $\Delta(f_{i-1}, \bar{x})$. If $\Gamma$ is an arc and $\bar{x} \in \text{int} \Gamma$, we may as well suppose, by replacing $\mu(\bar{x})$ and $\nu(\bar{x})$ by small positive numbers, that $\Gamma \cap \Delta(f_{i-1}, \bar{x})$ is that center line. Similarly, if $\Gamma = \{\bar{x}\}$ or if $\Gamma$ is an arc and $\bar{x}$ is one of the two endpoints of $\Gamma$, then $\Gamma$ is disjoint from either the top or bottom of $\Delta(f_{i-1}, \bar{x})$, say the top, and we may suppose that $f_{i-1}$ has rank $n$ on (a neighborhood of) the top of $\Delta(f_{i-1}, \bar{x})$.

For each such arc component $\Gamma$, there exists a finite number of points $x_m$ ($m = 1, 2, \ldots, L(\Gamma)$) such that the sets $\Delta(f_{i-1}, x_m)$ are a minimal cover
of \( \Gamma \). By renumbering the points \( x_m \), replacing \( \lambda(x_m) \), \( \mu(x_m) \), and \( \nu(x_m) \) by smaller numbers, and by (possibly) interchanging tops and bottoms (by a reflection of \( E \)), we may suppose that

(i) \( f_{i-1} \) has rank \( n \) on top of \( \Delta(f_{i-1}, x_i) \), and that

(ii) the top of \( \Delta(f_{i-1}, x_{m+1}) \) is contained in \( \Delta(f_{i-1}, x_m) \) \((m = 1, 2, \ldots, L(\Gamma) - 1)\).

For \( \Gamma \) an arc component, let \( \Omega(\Gamma) \) be the union of these sets \( \Delta(f_{i-1}, x_m) \); for \( \Gamma \) the single point \( \overline{x} \), let \( \Omega(\Gamma) = \Delta(f_{i-1}, \overline{x}) \). The sets \( \Omega(\Gamma) \) constitute and open cover of \( F_{i+1} \cap f^{-1}(Y) \), and thus there is a finite subcover. The corresponding sets \( \Delta(f_{i-1}, x_j) \) \((j = 1, 2, \ldots, J)\) thus also constitute a finite subcover, and, for each \( \Gamma \) of the finite subcover, may be ordered consistent with the ordering on \( \Gamma \). If \( \mu' + \nu' = \mu + \nu \) \((\mu, \mu', \nu, \nu' > 0)\), then \( \Delta_{\mu,\nu} \) is homeomorphic to \( \Delta_{\mu',\nu'} \) by a translation in the \( n \)th coordinate direction.

Thus we may suppose that either

(i) \( f_{i-1} \) has rank \( n \) on (a neighborhood of) \( \Delta^+(f_{i-1}, x_i) \) \((j = 1, 2, \ldots, J)\) or

(ii) \( \Delta^+(f_{i-1}, x_j) \subset \Delta(f_{i-1}, x_{j-1}) \) \((j = 2, 3, \ldots, J)\).

The reader may verify from the proofs of [4, (1.1)] and the rank theorem [12, p. 7, (1.8)] that, if \( \rho(h_j, f_{i-j}) \to 0 \) as \( j \to \infty \) and \( \overline{x} \in F_{i+1} \cap f^{-1}(Y) \), then we may suppose that \( k_i[h_j, \overline{x}]|\overline{\Delta}_{\mu,\nu} \) approaches \( k_i[f_{i-j}, \overline{x}]|\overline{\Delta}_{\mu,\nu} \) in the \( C^1 \) topology.

Thus there exists \( \xi, 0 < \xi < \xi \) such that \( h_j \in C^3 \) and \( \rho(h_j, f_{i-j}) < \xi \) \((j = 1, 2, \ldots, J)\) implies that

(1) the sets \( \Delta(h_j, x_j) \) cover \( F_{i+1} \cap f^{-1}(Y) \) and each set \( \Delta(h_j, x_j) \subset F_i \cap f^{-1}(V) \), and

(2) for each \( j \), either

(i) \( h_j \) has rank \( n \) on \( \Delta^+(h_j, x_j) \), or

(ii) \( \Delta^+(h_j, x_j) \subset \Delta(h_{j-1}, x_{j-1}) \).

Let \( h_1 = f_{i-1} \). Let \( h_2 \) be the map given by 3.3 such that \( \rho(h_1, h_2) < \xi/2, h_2 \) agrees with \( h_1 \) off \( \Delta(h_1, x_1) \), and \( h_2 \) has rank \( n \) on \( \Delta(h_1, x_1) \). Suppose that \( h_{j+1} \) \((j = 1, 2, \ldots, m - 1)\) have been given by 3.3 for \( h_j \) with \( \rho(h_j, h_{j+1}) < 2^{-j} \xi, h_{j+1} \) agrees with \( h_j \) off \( \Delta(h_j, x_j) \), and \( h_{j+1} \) has rank \( n \) on \( \Delta(h_j, x_j) \). By (2), either (i) \( h_m \) has rank \( n \) on \( \Delta^+(h_m, x_m) \) or (ii) \( \Delta^+(h_m, x_m) \subset \Delta(h_{m-1}, x_{m-1}) \); but, by the inductive hypothesis, \( h_m \) has rank \( n \) on \( \Delta(h_{m-1}, x_{m-1}) \). Thus, in either case, \( h_m \) has rank \( n \) on \( \Delta^+(h_m, x_m) \). Let \( h_{m+1} \) be the map given by 3.3 such that \( \rho(h_{m+1}, h_m) < 2^{-m} \xi, h_{m+1} \) has rank \( n \) on \( \Delta(h_m, x_m) \), and \( h_{m+1} \) agrees with \( h_m \) off that set.

Let \( f_i \) be the map \( h_{i+1} \) thus defined. Condition \( \mathcal{P}(a) \) follows from (1) and 3.3(i); \( \mathcal{P}(c) \) from (1) and 3.3(ii); \( \mathcal{P}(b) \) and (f) from the fact that \( \xi < \eta; \mathcal{P}(d) \) from 3.3(i) and (ii); \( \mathcal{P}(e) \) from 3.3(iv); and \( \mathcal{P}(g) \) from 3.3(i) and (vi). Thus there exists a sequence \( \{f_i\} \) satisfying \( \mathcal{P} \).

Since \( \rho \) is complete, a limit map \( h \in C^3 \) exists, and \( \rho(f, h) < \epsilon \). Let \( X \subset Y \) be any compact set, and let \( Y \) consist of those points \( y \in N^p \) such
that \(d(y, X) \leq \epsilon\). Then \(Y\) is closed and bounded, and, since \(N^n\) is a closed subset of a Euclidean space (3.4), \(Y\) is compact. If \(x \in M^n - f^{-1}(Y)\), then \(d(f(x), X) > \epsilon\), so that (3.4(2)) \(h(x) \notin X\). Thus \(h^{-1}(X) \subset f^{-1}(Y)\), and hence is compact. Since \(X\) is an arbitrary compact set in \(N^n\), \(h\) is proper.

Given \(x \in B_h\), choose \(j\) such that \(x \in \overline{F}_j\). By \(\Psi(a)\), \(f_{j-1} = h\) on a neighborhood of \(x\); by \(\Psi(e)\), \(x \in B_h\). Thus \(B_h \subset B_j\). Given \(z \in f^{-1}(f(R_{n-2}(f)))\), there exists \(i\) such that \(z \in f^{-1}(Y_i)\); by \(\Psi(c)\), \(z \in B(j)\). There exists \(J\) such that \(\bigcup_{j \in J} \overline{U}_j = \emptyset\) for all \(j \geq J\). By \(\Psi(a)\) and \(b\) \(h\) agrees with \(f_j\) on a neighborhood of \(z\). By \(\Psi(e)\), \(z \in B_h\). Thus \(B_h \subset B_j \cap f^{-1}(f(R_{n-2}(f)))\).

From \(\Psi(e)\) and 3.4(1), for all \(x \in M^n\) \(d(f(x), f_{i-1}(x)) < 2^{-i} \epsilon\) \((0 < \epsilon < 1)\); thus \(d(f(x), h(x)) < 2^{-i} \epsilon\). From \(\Psi(a)\) and \(\Psi(b)\), there exists a neighborhood \(W\) of \(f^{-1}(f(R_{n-2}(f)))\) such that, for all \(x \in W\), \(d(f(x), h(x)) < 2^{-i} \epsilon\). Since \(i\) is arbitrary, the restriction map \(h|f^{-1}(f(R_{n-2}(f))) = f|f^{-1}(f(R_{n-2}(f)))\).

Similarly, all partial derivatives with order at most 3 of \(f\) and \(h\) agree on \(f^{-1}(f(R_{n-2}(f)))\). Thus \(f^{-1}(f(R_{n-2}(f))) \subset h^{-1}(h(R_{n-2}(h)))\), so that \(B_h \subset B_j \cap h^{-1}(h(R_{n-2}(h)))\).

Since \(\dim B_h \leq n - 3\) and \(B_h \subset h^{-1}(h(R_{n-2}(h)))\), there exists (2.8) \(k\) such that, for each \(y \in N^n\), \(h^{-1}(y)\) has precisely \(k\) components. Since \(h|f^{-1}(f(R_{n-2}(f))) = f|f^{-1}(f(R_{n-2}(f)))\) and \(h(M^n - f^{-1}(f(R_{n-2}(f)))) \subset N^n - f(R_{n-2}(f))\) (by \(\Psi(a)\) and \(\Psi(b)\), for each \(y \in f(R_{n-2}(f))\), \(f^{-1}(y)\) has precisely \(k\) components also. For each \(y \in N^n - f(R_{n-2}(f))\), there exists \(i\) such that \(y \in V_i\); there exists \(J\) such that \(\bigcup_{j \in J} \overline{U}_j = \emptyset\), and thus \((\Psi(a)\) and \(\Psi(b)\)) \(h^{-1}(\overline{V}_i) = f_j^{-1}(\overline{V}_i)\) and \(f_j\) agrees with \(h\) on \(h^{-1}(\overline{V}_i)\). As a result, \(h^{-1}(y)\) has exactly \(k\) components. By \(\Psi(g)\), \(f_j^{-1}(y)\) and \(f^{-1}(y)\) have the same number of components. Thus \(f\) satisfies conclusion (1) of 2.1.

3.6. Proof of Theorem (1.1). First, for the uniqueness, suppose that \(hg\) and \(\tilde{h}g\) are two factorizations with intermediate manifolds \(K^n\) and \(L^n\). For each point \(y\) in \(K^n\), define \(\alpha(y) \in L^n\) as the single point \(\tilde{g}(g^{-1}(y))\). That \(\alpha\) is one-to-one and onto is immediate. Since \(\tilde{h} \alpha = h\), \(\alpha\) is locally a diffeomorphism and thus is a diffeomorphism.

For \(n = 1\) and \(2\), \(f\) is, by definition, a local homeomorphism; for \(n = 3\), \(f\) is a local homeomorphism by [4, p. 469] and [5, p. 91]. Thus \(f\) is a \(k\)-to-1 \(C^m\) covering map [18, p. 128], and the existence of the factoring follows from (2.1). For \(n \geq 4\) it follows from (2.1), (3.2) and (3.5).

3.7. Remarks. The manifold \(K^n\) need not be \(C^m\) diffeomorphic to \(M^n\). Let \(\tilde{M}^n = S'\), \(N^n\) be a \(C^n\) homotopy 7-sphere other than \(S'\), and let \(f\) be a homeomorphism of \(M^n\) onto \(N^n\) which is a \(C^n\) diffeomorphism except at one point [13]; we may suppose that \(f\) itself is \(C^n\) everywhere (e.g., by the argument of [5, p. 95, (3.3)]). In any factorization \(K^n\) is \(C^n\) diffeomorphic to \(N^n\), and thus is not diffeomorphic to \(M^n\).

In the special case that \(f(B_1) \subset f(R_{n-3})\), \(\pi_1(N^n) = 0\) and \(f\) is \(C^n\), it follows from [6, (1.1)] for \(p = n, m = 1\), and \(k = n - 3\) that \(\pi_1(N^n - f(R_{n-3})) = 0\).
Since \( \dim(f(R_{n-3})) \leq n - 3 \) \([5, (1.3)]\) and \( \dim(B_i) \leq n - 3 \), \( \dim(f^{-1}(f(R_{n-3}))) \leq n - 3 \); thus \( M^n - f^{-1}(f(R_{n-3})) \) is connected and the restriction map \( f| \left[ M^n - f^{-1}(f(R_{n-3})) \right] \) is a homeomorphism. Hence \( f \) is monotone, and (1.1) is immediate in this case.

A map \( f \) satisfying the hypothesis of (1.1) is not a branched covering in the sense of Fox \([7, \text{p. 250}]\) unless it is actually a covering map.

In the case the degree is 0, the conclusion of (1.2) can be improved.

3.8. Remark. If \( M^n \) and \( N^n \) are compact connected oriented \( C^1 \) \( n \)-manifolds, and \( f: M^n \to N^n \) is \( C^1 \) with degree 0, then \( \dim(B_i) \geq n - 1 \).

Proof. Suppose the contrary, i.e., that \( f \) has degree 0 and \( \dim(B_i) \leq n - 2 \). Since \( B_i \) does not separate \( M^n \), the Jacobian of \( f \) is either non-negative or nonpositive (well defined by \([21, \text{p. 341, Lemma 3}]\)). By \([19, \text{§5}]\) \( \dim(f(R_{n-1})) \leq n - 1 \), so there exists \( y \) with \( f^{-1}(y) \subset R_n - R_{n-1} \); let \( x_k \) \((k = 1, 2, \ldots, m)\) be the points of \( f^{-1}(y) \). The result follows from \([21, \text{Lemma 2 and Theorem 2}]\).

If \( f \) is not differentiable but \( f(B_i) \neq N^n \), essentially the same proof is valid. The author is grateful to R. F. Williams for suggesting this proof, which is simpler than the author’s version.

References

13. ______, On manifolds homeomorphic to the \( \mathbb{S} \)-sphere, Ann. of Math. (2) 64 (1956), 399-405.

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