1. Introduction. In a recent paper Heerema [1] has shown that if $K^{*}$ is a $p$-adic field, i.e., a field which is complete and unramified with respect to a discrete rank 1 valuation, with ring of integers $R^{*}$ and natural place $H^{*}$: $K^{*} \to \{ \mathbb{k}, \infty \}$, then for every derivation $d: k \to k$ there exists a derivation $D^{*}$: $K^{*} \to K^{*}$ such that $D(R^{*}) \subset R^{*}$ and $d(H^{*}(a)) = H^{*}(D^{*}(a))$ for every $a \in R^{*}$.

It follows from this, for example, that the inertial automorphisms of $K^{*}$ are Taylor-series-like expressions in powers of $p$ using integral derivations, i.e., derivations $D^{*}: K^{*} \to K^{*}$ such that $D^{*}(R^{*}) \subset R^{*}$. The fact that every derivation on $k$ is induced by one on $K^{*}$ also yields a simple way of constructing an example to show that if $K$ is a ramified $\overline{p}$-adic field and $K^{*}$ is unramified having the same residue field, then $K^{*}$ is uniquely embedded in $K$ if and only if $k$ is perfect [4].

In this paper we study ramified $\overline{p}$-adic fields $K$ with ring of integers $R$, residue field $k$ and natural place $H$: $K \to \{ k, \infty \}$ with the property that for every derivation $d: k \to k$ there exists a derivation $D: K \to K$ such that $D(R) \subset R$, $D(\pi) \in (\pi)$ for a prime element $\pi$ of $R$ and $a \in R$ implies $d(H(a)) = H(D(a))$. For convenience of discussion we will call this property of $\overline{p}$-adic fields property (H).

We derive several characterizations of $\overline{p}$-adic fields with property (H). These characterizations are essentially of two kinds. The first characterization gives a condition on the Eisenstein equation of the $\overline{p}$-adic field with respect to a given fixed $p$-adic subfield with the same residue field in the restricted valuation, while the other characterizations are intrinsic and yield properties this class of $\overline{p}$-adic fields must have.

The first characterization mentioned above makes use of the valuation topology of the fixed $p$-adic subfield and depends on the "distance" arbitrary elements can be moved by derivations in the metric generating the valuation topology. We have found a theoretically simple way of determining this distance which allows us to assign a numerical value to the Eisenstein polynomial with the property that the given $\overline{p}$-adic field has property (H) if and only if this numerical value is greater than zero.

The other characterizations are in terms of the derivations on the $\overline{p}$-adic field $K$ and automorphisms on this field. We show that $K$ has property (H)
if and only if every integral derivation on $K$ is an inducing derivation, i.e., if and only if $D(R) \subseteq R$ implies $D((x)) \subseteq (x)$.

Finally, after establishing some connections between integral derivations on an arbitrary $\overline{p}$-adic field $K$ and inertial automorphisms we are able to obtain some information on the structure of the pseudo-ramification groups of $K$. We are also able to give a third characterization of property (H), namely, if $p$ is an odd prime and if $n \geq (e+p)/(p-1)$, where $e$ is the ramification of $K$, then $K$ has property (H) if and only if for every automorphism $T: K \rightarrow K$ such that $T(a) - a \in (\pi)^n$ for all $a \in R$, it is true that $T(\pi) - \pi \in (\pi)^{n+1}$. This third result is a corollary to an extension of a result of MacLane [2] from the case of $p$-adic fields to the $\overline{p}$-adic field situation.

2. The index of inertia. As in the introduction, we assume that $K^*$ is a $p$-adic field, $R^*$ its valuation ring and $k = R^*/(p)$ its residue field. We assume that $H^*: K^* \rightarrow \{k, \infty\}$ is the natural place. For $a \in R^*$, let $\Delta(a) = \min \{ V(D^*(a)) \}$, where $D^*$ ranges over all integral derivations on $K^*$. The symbol $\Delta(a)$ is called the index of inertia of $a$. If we assume $V(p) = 1$, then $\Delta(a)$ will have value a non-negative integer. If $D^*(a) = 0$ for all integral derivations $D^*$, then we will assign $\Delta(a)$ the value $\infty$. If $k_0$ is the maximum perfect subfield of $k$, then $R^*$ contains a unique subring $R^*_0$ such that $H^*(R^*_0) = k_0$ [2]. It is easily seen that if $a \in R^*_0$ then $\Delta(a) = \infty$. We will show below that if $\Delta(a) = \infty$, then $a \in R^*_0$.

For notational convenience we will introduce the symbol $a{[b]}$ meaning $a^b$.

If $a \in R^*$, then one can easily show that $a \in R^*_0$ or $a = \sum_{i=0}^{\infty} p[ia_i][p[n_i]]$ and for some $i$, $H^*(a_i) \in k\{p\}$. Let $\mathcal{F}$ denote the set of all such subscripts.

**Theorem 1.** $\Delta(a) = \min_{i \in \mathcal{F}} (i + n_i)$ if $\mathcal{F} \neq \emptyset$, $\Delta(a) = \infty$ otherwise.

**Proof.** Let $a \in R^*$ and decompose

$$a = \sum_{i < j \in \mathcal{F}} p[ia_i][p[n_i]] + \sum_{j \in \mathcal{F}} p[ja_j].$$

Let $n = \min_{i \in \mathcal{F}} (i + n_i) = N(a)$. Let $i_0 < i_1 < \cdots < i_q$ be the collection of indices such that $i_0 + n_{i_0} = \cdots = i_q + n_{i_q} = n$. Let $a' = \sum_{i=0}^{\infty} p[ia_i][p[n_i]]$. We have $a = a' + a''$, where $N(a') = N(a)$ and $N(a'') \geq N(a) + 1$. Thus if $\Delta(a) = N(a)$, then $\Delta(a') = N(a')$ and conversely. Hence, it suffices to show that $\Delta(a') = N(a')$. Now $a' = p[i_0]a^*$, where $a^*$ is a unit. Also, $H^*(a^*) = \gamma_{i_0}[p[n_{i_0}]]$, where $H^*(a_{i_0}) = \gamma_{i_0} \in k\{k^p\}$. Let $\gamma_{i_0} \in \mathcal{F}$ be a $p$-basis for $k$. Then $\gamma_{i_0}[p[n_{i_0}]]$ is a $p$-basis for $k^* = [p[n_{i_0}]](m)$ and $[k:k^*] = p[n_{i_0}] = p[n_0]$ with $k = k^*(\gamma_{i_0})$, where the minimum polynomial of $\gamma_{i_0}$ is $X[p[n_{i_0}]] - \gamma_{i_0}[p[n_{i_0}]]$. Let $m^*$ be a set of representatives in $R^*$ of $m$. Let $K^*_m$ be the complete closure of $K^*_m(m^*)$, where $K^*_m$ is the unique subfield of $K^*$ having residue field $k_0$. Let $R^*_m$: be its ring of integers and construct $R' = R^*_m[a^*, R^*[p[n_{i_0}]]]$. Then $R'$ is an integral domain and $H^*(R') = k^*$. 


Let \( K' \) be the complete closure of the quotient field of \( R' \), then \( K' \) is the quotient field of \( R' \), since \( K^* \) is complete and unramified. Now let \( c \) be any representative for \( \gamma_0^* \). Then \( K^* = K'(c) \) and \( [K^*:K] = p[n_0] \), since \( K^* \) is unramified. Thus the minimum polynomial of \( c \) is of the form

\[
X[p[n_0]] + \alpha_1X[p[n_0] - 1] + \ldots = f(X).
\]

Now \( f(c) = 0 \) implies \( H^*(f(c)) = f''(\gamma_0^*) = 0 \), where

\[
X[p[n_0]] + H^*(\alpha_1)X[p[n_0] - 1] + \ldots = f''(X).
\]

Thus \( X[p[n_0]] - \gamma_0[p[n_0]]|f''(X) \), but since they have the same degree, \( f''(X) = X[p[n_0]] - \gamma_0[p[n_0]] \). Hence

\[
\begin{align*}
& (i) \quad H^*(\alpha_i) \in (p) \quad \text{for } 1 \leq i \leq p[n_0] - 1, \\
& (ii) \quad H^*(f(0)) = \gamma_0[p[n_0]].
\end{align*}
\]

Now \( \{a^*, m^*\} \) is a collection of representatives of a \( p \)-basis of \( k^* \). Thus, as Heerema has shown \([1]\), we determine a unique integral derivation

\( D: K' \rightarrow K' \) by letting

\[
\begin{align*}
& (i) \quad D(a) = 0, \quad a \in K^*_0, \quad a \in m^*, \\
& (ii) \quad D(a^*) = p[n_0]u, \quad u \text{ some unit in } R'.
\end{align*}
\]

Now consider the extension of \( D \) to \( K^* \). By the construction of the fields and a straightforward argument on the valuation, \( R^* = R'[c] \). Hence, if we can show that \( V(D(c)) \geq 0 \), then \( D: K^* \rightarrow K^* \) is an integral derivation. We know that \( D(c) \) is uniquely determined and \( D(c) = -f''(c)/f'(c) \). We observe that \( a \in R' \) implies \( V(D(a)) \geq n_0 \). Thus \( V(D(a_i)) \geq 1 + n_0 \) for all \( i, 1 \leq i \leq p[n_0] - 1 \) by condition (C), (i). Next observe that \( f(0) = a^* + p \cdot v \) and \( D(f(0)) = D(a^*) + pD(v) = p[n_0]u + p \cdot p[n_0]v^* = p[n_0](u + pv^*) = p[n_0]u^* \), where \( u^* \) is a unit. Thus \( V(f''(c)) = n_0 \).

Observe that \( f'(c) \) has degree \( \leq p[n_0] - 1 \). Thus from the fact that \( R^* = R'[c], V(f'(c)) \) is equal to the minimum of the values of the coefficients. Now the coefficient of \( p[n_0] - 1 \) in \( f'(c) \) is \( p[n_0] \). Thus the minimum value of the coefficients is \( \leq n_0 \). Thus \( V(f'(c)) \leq n_0 \). Hence \( V(D(c)) \geq 0 \). Thus \( D: K^* \rightarrow K^* \) is an integral derivation. Now \( D(a') = D(p[i_0]a^*) = p[i_0]D(a^*) = p[i_0 + n_0]u \) and \( V(D(a')) = i_0 + n_0 = n \). Since \( V(D'(a')) \geq n \) for all \( D' \), it follows that \( \Delta(a') = N(a') = n \) and thus \( \Delta(a) = N(a) = n \) and the theorem follows.

3. \( p \)-adic fields and property (H). In this section we shall be concerned with \( \overline{p} \)-adic fields \( K \), with ring of integers \( R \), maximal ideal \( (\pi) \), residue field \( k = R/(\pi) \) and natural place \( H: K \rightarrow \{k, \infty\} \). If \( V \) is the valuation on \( K \) and if \( V(\pi) = 1, V(p) = e \), then \( K = K^*(\pi) \), where \( K^* \) is a \( p \)-adic field in the restricted valuation \( V^* = V/K^* \), \( [K:K^*] = e \) and \( \pi \) is the root of an Eisen-
stein polynomial \( f(X) = X^e + p \sum_{i=0}^{e-1} f_i X^i, f_i \in R^* = R \cap K^* \) and \( f_0 \in R^* \setminus (p), (p) = R^* \cap pR [3] \).

It is also true that if \( H^* = H/K^* \), then \( H^*(R^*) = k \). Throughout the discussion we will assume that \( K \) is given and that \( K^* \) and \( \pi \) have been fixed once chosen.

Let \( \Delta_{K|K^*} = \min \{ (\Delta(f) + 1)e + i + V(f'(\pi)) \} \), where \( f'(X) \) is the derivative with respect to \( X \) of \( f(X) \). Notice that since \( 0 \leq i \leq e - 1 \), then by the properties of \( V \), \( \min \{ (\Delta(f) + 1)e + i \} \) is uniquely determined and equal to \( (\Delta(f_0) + 1)e + i_0 \) for some index \( i_0, 0 \leq i_0 \leq e - 1 \). Hence \( \Delta_{K|K^*} = (\Delta(f_0) + 1)e + i_0 - V(f'(\pi)) \). Since \( V(f'(\pi)) \) is fixed, \( \Delta_{K|K^*} \) depends only on the coefficients of \( f(X) \) once \( K^* \) and \( \pi \) have been chosen.

Let an integral derivation \( D \) on \( K \) be an integral derivation if \( D(\pi) \in (\pi)^n \).

**Theorem 2.** \( K \) has the property that every integral derivation is an integral derivation, \( n \geq 1 \), if and only if \( \Delta_{K|K^*} \geq n \). If \( n \geq 1 \), then it is also true that for any such \( D \), \( D((\pi)) \subset (\pi)^n \).

**Proof.** Suppose \( \Delta_{K|K^*} = n \geq 0 \). Let \( D^* \) be an integral derivation on \( K^* \). Then \( D^* \) has a unique extension \( D: K \to K \) which is completely determined by \( D(\pi) \). Now \( D(\pi) = -f'^\nu(\pi)/f(\pi) \). Hence \( V(D(\pi)) = V(f'^\nu(\pi)) - V(f(\pi)) \).

Since \( a \in R \) implies \( a = g_a(\pi) \), where \( g_a(X) \in R^*[X] \), it follows that \( D(a) = g_a'^\nu(\pi) + g_a(\pi)D(\pi) \subset R \). Hence \( D(R) \subset R \).

Notice that \( n \geq 1 \) implies \( D(\pi) = \pi D(a) + aD(\pi) \in (\pi) \) if \( a \in R \), hence \( D((\pi)) \subset (\pi)^n \).

Now suppose that \( D \) is an integral derivation on \( K \), then \( a = g_a(\pi) \) implies \( D(a) = g_a'^\nu(\pi) + g_a(\pi)D(\pi) \). In particular, if \( a \in R^* \) then \( D(a) = g_a'^\nu(\pi) = g_a(\pi)D(\pi) \).

Consider \( g_{a,b}(X) \). Since \( [K:K^*] = e \), we may choose \( g_{a,b}(X) \) of degree at most \( e - 1 \). Hence, if we do this, then

\[ g_{a,b}(X) + g_{b,b'}(X) = g_{a+b,b'}(X) \]

and

\[ ag_{b,b}(X) + bg_{a,b}(X) = g_{ab,b}(X) \]

for all \( a, b \in R^* \).

Thus we may write \( D' = \sum_{i=0}^{e-1} \pi_i D_i^* \), where \( D_i^*(a) \) is the coefficient of \( X^i \) in \( g_{a,b}(X) \), and so \( D_i^*: K^* \to K^* \) is an integral derivation on \( K^* \). The fact that the polynomials \( g_{a,b}(X) \) are uniquely determined implies that the representation \( D' = \sum_{i=0}^{e-1} \pi_i D_i^* \) is unique.

Since \( \Delta_{K|K^*} = n \geq 0 \), each derivation \( D_i^* \) has a unique extension \( D_i \) to \( K \) such that \( D_i(R) \subset R \), \( D_i((\pi)) \subset (\pi)^n \), and \( D_i((\pi)) \subset (\pi) \) if \( n \geq 1 \). Hence since
D' has unique extension D to K, it follows that \( D = \sum_{i=0}^{\epsilon-1} \pi^i D_i \). Thus \( D(\pi) \in (\pi)^n \) and \( D((\pi)) \subset (\pi) \) if \( n \geq 1 \).

Conversely, suppose that every integral derivation D on K is an integral derivation, \( n \geq 1 \). Suppose also that \( \Delta_{K|K'} = m < n \). Suppose \( \Delta_{K|K'} = (\Delta(f_0) + 1)e + i_0 - V(f'(\pi)) \) and suppose that \( D^* \) is an integral derivation on \( K'^* \) such that \( V^*(D^*(f_0)) = \Delta(f_0)e \). Then if we extend \( D^* \) to a derivation \( D \) on K, \( V(D(\pi)) = m < n \).

If \( m \geq 0 \), then \( D(R) \subset R \) and hence \( V(D(\pi)) = n > m \), a contradiction. If \( m < 0 \), then \( \pi^{-m}D(R) \subset R \) and hence \( V(\pi^{-n}D(\pi)) = 0 \geq n \geq 1 \), a contradiction. Thus \( m \geq n \) and the theorem follows.

**Corollary.** Every integral derivation \( D^* \) on \( K'^* \) has an integral extension \( D \) to K if and only if \( \Delta_{K|K'} \geq 0 \).

**Theorem 3.** K has property (H) if and only if \( \Delta_{K|K'} \geq 1 \).

**Proof.** Suppose \( \Delta_{K|K'} \geq 1 \). Let \( d: k \rightarrow k \) be any derivation.

Then there is an integral derivation \( D^* \) on \( K'^* \) such that \( a \in R'^* \) implies \( H^*(D^*(a)) = d(H^*(a)) = d(H(a)) \). Since \( \Delta_{K|K'} \geq 1 \), then \( D^* \) has a unique extension \( D \) to K which is at least an integral derivation. Hence \( D \) induces a derivation on \( k \). Since \( a \in R'^* \) implies \( D(a) = D^*(a) \), it follows that \( D \) induces \( d: k \rightarrow k \). Hence \( K \) has property (H).

Conversely, suppose that \( \Delta_{K|K'} = m \leq 0 \). Then there is an integral derivation \( D^* \) on \( K'^* \) such that if \( D \) is the unique extension of \( D^* \) to K, then \( V(D(\pi)) = m \leq 0 \). If \( D_0^* \) is any derivation such that \( D^* \) and \( D_0^* \) induce the same derivation on \( k \), then \( D_0^* = D^* + pD^* \) and thus if \( D_0 \) is the unique extension of \( D_0^* \) to \( K \), then \( V(D_0(\pi)) = V(D(\pi)) = m \). Now suppose \( d: k \rightarrow k \) is induced by \( D^*: K'^* \rightarrow K'^* \) and suppose that \( D: K \rightarrow K \) also induces \( d: k \rightarrow k \).

Then if \( D' = D|K' \), \( D' = \sum_{i=0}^{\epsilon-1} \pi^i D_i \), where \( D_i(R'^*) \subset R'^* \). For \( a \in R'^* \), \( H^*(D^*(a)) = H(D(a)) = H(D'(a)) = H^*(D_0^*(a)) \) and thus \( D_0^* \) induces \( d: k \rightarrow k \).

Now suppose \( D \) is the unique extension of \( D^* \) to \( K \), then \( V(\pi^i D_i(\pi)) \geq m + i > V(D_0(\pi)) \) and thus \( V(\sum_{i=0}^{\epsilon-1} \pi^i D_i(\pi)) = V(D_0(\pi)) = m \leq 0 \). But

\[
D = \sum_{i=0}^{\epsilon-1} \pi^i D_i
\]

and thus \( V(D(\pi)) = V(D_0(\pi)) = m \leq 0 \). However, \( D \) is an inducing derivation and so \( V(D(\pi)) \geq 1 \). Hence \( d: k \rightarrow k \) is not induced by any \( D: K \rightarrow K \) and \( K \) does not have property (H).

**Corollary 1.** K has property (H) if and only if every integral derivation D on K is an inducing derivation.

**Corollary 2.** If \( K \) is a \( \overline{p} \)-adic field of ramification \( e \) and if \( (e, p) = 1 \), then \( K \) has property (H).
4. Automorphisms on \( \overline{p} \)-adic fields. In this section we shall be concerned with establishing a connection between derivations and inertial automorphisms on \( \overline{p} \)-adic fields \( K \). Let \( G = \{ T \mid T \text{ is an automorphism on } K \} \); for \( n \geq 1 \), let \( G_n = \{ T \in G \mid T(a) - a \in (\pi)^n \text{ for all } a \in R \} \);
\[
\overline{G}_n = \{ T \in G_n \mid T(\pi) - \pi \in (\pi)^{n+1} \}.
\]

As in §2, \( a[b] \) means \( a^b \).

**Lemma 1.** Let \( n \geq 1 \) and let \( T \in G_n \). Then if \( Z = T - I \), \( Z[q]: R \rightarrow (\pi)[q(n-1)+1] \).

**Proof.** \( Z(ab) = T(ab) - ab = aZ(b) + bZ(a) + Z(a)Z(b) \). Thus, in particular, \( Z(\pi[m]) = Z(\pi \cdot \pi[m-1]) = \pi Z(\pi[m-1]) + \pi[m-1]Z(\pi) + Z(\pi[m-1])Z(\pi) \). Hence \( Z(\pi[2]) \in (\pi)[n+1] \) and by induction, \( Z(\pi[m]) \in (\pi)[m+n-1] \).

Thus \( a \in R \) implies
\[
Z(a) = \pi[n]a_i,
\]
\[
\pi[2](a) = Z(\pi[n]a_i) = a_iZ(\pi[n]) + \pi[n]Z(a_i) + Z(\pi[n])Z(a_i) \in \pi[2n-1].
\]

Hence by induction \( Z[q](a) \in (\pi)[qn-(q-1)] \) and the lemma follows.

**Corollary.** If \( n \geq 1 \) and \( T \in \overline{G}_n \), then \( Z[q]: R \rightarrow (\pi)[qn] \).

**Proof.** Since \( Z(\pi) \in (\pi)[n+1] \), it follows that \( Z(\pi[m]) \in (\pi)[m+n] \). Hence \( Z[2](a) \in \pi[2n] \) and by induction \( Z[q](a) \in (\pi)[qn] \).

Now suppose that \( n \geq (e+1)/(p-1) \), \( q \geq 2 \). Then if \( q = p[s]t \), it follows that \( p[s]t(n-1)+1-se \geq n+1 \). Since \( q \in (\pi)[se] \), it follows that \( Z[q]/q: R \rightarrow \pi[n+1] \) for all \( q \geq 2 \).

**Lemma 2.** Suppose \( 1 \leq i \leq p[\mu] \), then
\[
i\binom{p[\mu]}{i} \in p[\mu]R.
\]

**Proof.** Consider
\[
i\binom{p[\mu]}{i} = 1/(i-1)! \{ p[\mu](p[\mu] - 1) \cdots (p[\mu] - (i-1)) \}.
\]

In the expansion we obtain terms of the form
\[
p[\mu]\binom{p[\mu]}{i_1} \cdots \binom{p[\mu]}{i_l},
\]
\( l \geq 0 \), \( 1 \leq i_1 < i_2 < \cdots < i_l \leq i - 1 \). Since \( (p[\mu]/i_i) \in R \), the lemma follows.

**Theorem 4.** Let \( p \) be an odd prime and suppose \( n \geq (e+1)/(p-1) \). Suppose \( T \in G_n \), or, if \( n = 1 \), \( T \in \overline{G}_i \). Suppose \( T = I + \pi[n]T' \). Then there
is a derivation \( D(T)_n \) on \( K \) such that \( T' = D(T)_n: R \to (\tau) \). Hence \( T \subseteq \bar{G}_n \) if and only if \( D(T)_n \) is an inducing derivation.

**Proof.** Since \( T = I + Z \), it follows that

\[
T[p[\mu + 1]] - T[p[\mu]] = \sum_{i=1}^{p[\mu]} \left( \binom{p[\mu + 1]}{i} - \binom{p[\mu]}{i} \right) Z[i] + \sum_{i=p[\mu]+1}^{p[\mu+1]} \binom{p[\mu + 1]}{i} Z[i].
\]

If \( i \geq 2 \), then by Lemma 2 and the fact that \( Z[i]/i: R \to \tau[n + 1] \) we get

\[
T[p[\mu + 1]] - T[p[\mu]] = p[\mu](p - 1)Z + p[\mu]Z^*, \quad \text{where}
\]

\[
Z^*: R \to (\tau)[n + 1].
\]

Let

\[
T_\mu = p[-\mu](T[p[\mu + 1]] - T[p[\mu]])
\]

\[
= (p - 1)Z + Z_\mu^* = -Z + pZ + Z^*.
\]

Now consider \( T_{\mu+1} - T_{\mu} \). This map is given by

\[
\sum_{i=1}^{p[\mu]} g_i p[-\mu - 1]Z[i] + \sum_{i=p[\mu]+2}^{p[\mu+1]} g'_i p[-\mu - 1]Z[i],
\]

where

\[
g_i = \left( \binom{p[\mu + 2]}{i} - \binom{p[\mu + 1]}{i} \right) (1 + p) + \binom{p[\mu]}{i} p, \quad 1 \leq i \leq p[\mu],
\]

\[
g'_i = \left( \binom{p[\mu + 2]}{i} - \binom{p[\mu + 1]}{i} \right) (1 + p), \quad p[\mu] + 1 \leq i \leq p[\mu + 1].
\]

Note that \( g_1 = 0 \). For \( i \geq 2 \), we get

\[
ig_i = 1/(i - 1)! \left( p[\mu + 1][(p[\mu + 2] - 1) \cdots (p[\mu + 2] - (i - 1)) - (p[\mu + 1] - 1) \cdots (p[\mu + 1] - (i - 1))] - p[\mu + 1][(p[\mu + 1] - 1) \cdots (p[\mu + 1] - (i - 1))] - (p[\mu] - 1) \cdots (p[\mu] - (i - 1))) \right).
\]

Now suppose we pick \( \mu \) such that \( p[\mu/3] \geq 2(\mu + 1)e \). Then \( i \geq p[\mu/3] \) implies \( Z[i]: R \to p[2(\mu + 1)]R \). Also, if \( 1 \leq i < p[\mu/3] \), then by Lemma 2, and the fact that on the right side of (4) the terms involving \( (i - 1)! \) are cancelled, it follows that \( p[-\mu - 1]g_i Z[i]: R \to p(\mu - [(\mu/3) + 1])R \), where the inner
\[ T_{n+1} - T_n: R \rightarrow p(\lfloor \mu \rfloor - \lfloor \mu/3 \rfloor + 1)R \]

and

\[ \lim_{n \to \infty} T_n = T_{n_0} + \sum_{i=n_0}^{\infty} (T_{i+1} - T_i) \]

is a well-defined map.

Since

\[ T[p[\mu]] - I = \sum_{i=1}^{p[\mu]} \binom{p[\mu]}{i} Z[i], \]

then by Lemma 2, \( T[p[\mu]] - I: R \rightarrow p[\mu]R \). Thus since

\[ T_n(ab) - aT_n(b) - bT_n(a) = p[-\mu]((T[p[\mu + 1]](a) - a)(T[p[\mu + 1]](b) - b) \]

\[ - (T[p[\mu]](a) - a)(T[p[\mu]](b) - b) \] \( (p \neq 2) \)

it follows that for \( \mu \) large enough \( T_n \) is a derivation modulo \( p[\mu] \). Hence \( \lim_{n \to \infty} T_n \) is a derivation.

Now \( \lim_{n \to \infty} T_n = -Z + pZ + \lim_{n \to \infty} Z_n^* \). Thus if we let \( D(T)_n = -\pi[-n] \cdot \lim_{n \to \infty} T_n \), then \( D(T)_n = -T' \cdot pT' - \pi[-n] \lim_{n \to \infty} Z_n^* \) and \( T' - D(T)_n: R \to (\pi) \). Hence the theorem follows.

**Theorem 5.** Suppose \( n \geq (e + 1)/(p - 1) \) and suppose \( D: K \to K \) is an integral derivation on \( K \). Then \( D_n = I + \sum_{i=0}^{n_0} \pi[n]/i! D[i] \) is an automorphism on \( K \), \( D_n \in G_n \) and \( D_n \in \overline{G}_n \) if and only if \( D(\pi) \in (\pi) \).

**Proof.** Observe that \( V(i) < ie/(p - 1) \). Thus \( n \cdot V(i) \leq ni - ie/(p - 1) \)

\[ \geq (e + 1)i/(p - 1) - ie/(p - 1) \] \( \geq \lim_{n \to \infty} V(\pi[n]/i)! = \infty. \) Thus \( D_n \) is a well-defined map, \( D_n(R) \subseteq R \). \( D_n \) is additive since \( D[i] \) is additive. Also, by a straightforward computation \( D_n(ab) = D_n(a) \cdot D_n(b) \). Since \( D_n(1) = 1 \), it follows that \( D_n \in G \).

By a straightforward computation \( i \geq 2 \) implies \( V(\pi[n]/i)! \geq n + 1 \), if \( n \geq (e + 1)/(p - 1) \). Hence \( D_n = I + \pi[n](D + \pi D_n^*) \), where \( D_n^*(R) \subseteq R \). Hence \( D_n \in G_n \) and \( D_n \in \overline{G}_n \) if and only if \( D(\pi) \in (\pi) \).

**Theorem 6.** Let \( D \) be the additive group of integral derivations on \( K \) and let \( \overline{D} \) be the additive group of derivations on the residue field of \( K \) which are induced, then if \( p \) is odd and \( n \geq (e + p)/(p - 1) \), \( G_n/G_{n+1} \) is isomorphic to \( D/\pi D \) and if \( n \geq (e + p)/(p - 1) \), \( \overline{G}_n/G_{n+1} \) is isomorphic to \( \overline{D} \).

**Proof.** Define \( H_n(TG_{n+1}) = D(T)_n + \pi \overline{D} \) and \( H_n^*(TG_{n+1}) = d \), where \( d \) is induced by \( D(T)_n \) if \( T \subseteq G_n \). Theorem 4 implies \( H_n \) and \( H_n^* \) are well-defined since \( D(T)_n: R \to (\pi) \) if and only if \( T \subseteq G_{n+1} \). Since \( D(T_1 \cdot T_2)_n = T_1 + T_2 \)
+ \mathcal{T}_1(\pi n T_2), \text{ it follows that } D(T_1 \cdot T_2)_n \equiv D(T_1)_n + D(T_2)_n \pmod{\pi}, \text{ i.e., } H_n \text{ and } H_n^* \text{ are homomorphisms. } H_n \text{ and } H_n^* \text{ are monomorphisms by Theorem 4. By Theorem 5, since } D(D_n) - D: R \to (\pi), \text{ they are epimorphisms. Thus they are isomorphisms and the theorem follows.}

**Corollary.** K has property (H) if and only if \( \overline{G}_n = G_n \) for all \( n \geq (e + p)/(p - 1) \).

**Proof.** Suppose \( \overline{G}_n = G_n \). Then \( T \in G_n \) implies \( D(T)_n \) is inducing. Since \( D \in \mathcal{D} \) implies \( D = D(T)_n - \pi D' \) for some \( T \) and \( D' \), every \( D \) is inducing. Thus \( K \) has property (H). If \( K \) has property (H), then every \( D(T)_n \) is inducing, which implies \( T \in \overline{G}_n \). Hence \( G_n = \overline{G}_n \).

**Conjecture.** Theorem 6 holds for all \( n \geq 1 \).

Notice that if \( T \in G_n \) \( (n \geq (e + p)/(p - 1)) \), we can obtain \( D(T)_n \). Thus by Theorem 5, \( (D(T)_n)_n \) is an automorphism with derivation \( D((D(T)_n)_n) - D(T)_n \in \pi \mathcal{D} \). Hence \( T = T_1 \cdot ((D(T)_n)_n) \), where \( T_1 \in G_{n+1} \). Proceeding in this fashion we obtain

\[
T = T_j((D(T_{j-1})_{n+j-1})_{n+j-1}) \cdots ((D(T)_n)_n), \quad T_j \in G_{n+j}
\]

and since \( \bigcap G_n = \mathbb{I} \), we obtain

\[
T = \lim_{j \to \infty} ((D(T)_{n+j})_{n+j}) \cdots ((D(T)_n)_n),
\]

i.e., \( T \) has a Taylor-series-like expansion in terms of derivations.

Theorem 6 is an extension of a well-known result of MacLane [2], to \( \overline{p} \)-adic fields.

**References**


Florida State University, Tallahassee, Florida