DERIVATIONS ON $\bar{p}$-ADIC FIELDS[1]

BY
JOSEPH NEGGERS

1. **Introduction.** In a recent paper Heerema [1] has shown that if K^* is a p-adic field, i.e., a field which is complete and unramified with respect to a discrete rank 1 valuation, with ring of integers R^* and natural place H^*: $K^* \to \{k, \infty\}$, then for every derivation $d: k \to k$ there exists a derivation D^*: $K^* \to K^*$ such that $D(R^*) \subset R^*$ and $d(H^*(a)) = H^*(D^*(a))$ for every $a \in R^*$.

It follows from this, for example, that the inertial automorphisms of K^* are Taylor-series-like expressions in powers of p using integral derivations, i.e., derivations D^*: $K^* \to K^*$ such that $D^*(R^*) \subset R^*$. The fact that every derivation on k is induced by one on K^* also yields a simple way of constructing an example to show that if K is a ramified $\bar{p}$-adic field and K^* is unramified having the same residue field, then K^* is uniquely embedded in K if and only if k is perfect [4].

In this paper we study ramified $\bar{p}$-adic fields K with ring of integers R, residue field k and natural place H: $K \to \{k, \infty\}$ with the property that for every derivation $d: k \to k$ there exists a derivation $D: K \to K$ such that $D(R) \subset R$, $D(\pi) \in (\pi)$ for a prime element π of R and $a \in R$ implies $d(H(a)) = H(D(a))$. For convenience of discussion we will call this property of $\bar{p}$-adic fields property (H).

We derive several characterizations of $\bar{p}$-adic fields with property (H). These characterizations are essentially of two kinds. The first characterization gives a condition on the Eisenstein equation of the $\bar{p}$-adic field with respect to a given fixed p-adic subfield with the same residue field in the restricted valuation, while the other characterizations are intrinsic and yield properties this class of $\bar{p}$-adic fields must have.

The first characterization mentioned above makes use of the valuation topology of the fixed p-adic subfield and depends on the "distance" arbitrary elements can be moved by derivations in the metric generating the valuation topology. We have found a theoretically simple way of determining this distance which allows us to assign a numerical value to the Eisenstein polynomial with the property that the given $\bar{p}$-adic field has property (H) if and only if this numerical value is greater than zero.

The other characterizations are in terms of the derivations on the $\bar{p}$-adic field K and automorphisms on this field. We show that K has property (H)

Presented to the Society, January 23, 1964; received by the editors December 5, 1963.

(1) This research was supported by NSF GP-1084. It is based on the author's doctoral dissertation written under the supervision of Professor N. Heerema of the Florida State University.

if and only if every integral derivation on K is an inducing derivation, i.e., if and only if $D(R) \subset R$ implies $D((\pi)) \subset (\pi)$.

Finally, after establishing some connections between integral derivations on an arbitrary $\overline{p}$-adic field K and inertial automorphisms we are able to obtain some information on the structure of the pseudo-ramification groups of K. We are also able to give a third characterization of property (H), namely, if p is an odd prime and if $n \geqq (e+p)/(p-1)$, where e is the ramification of K, then K has property (H) if and only if for every automorphism $T: K \to K$ such that $T(a) - a \in (\pi)^n$ for all $a \in R$, it is true that $T(\pi) - \pi \in (\pi)^{n+1}$. This third result is a corollary to an extension of a result of MacLane [2] from the case of p-adic fields to the $\overline{p}$-adic field situation.

2. **The index of inertia.** As in the introduction, we assume that K^* is a p-adic field, R^* its valuation ring and $k = R^*/(p)$ its residue field. We assume that $H^*: K^* \to \{k, \infty\}$ is the natural place. For $a \in R^*$, let $\Delta(a) = \min\{V(D^*(a))\}$, where D^* ranges over all integral derivations on K^*. The symbol $\Delta(a)$ is called the index of inertia of a. If we assume $V(p) = 1$, then $\Delta(a)$ will have value a non-negative integer. If $D^*(a) = 0$ for all integral derivations D^*, then we will assign $\Delta(a)$ the value ∞. If k_0 is the maximum perfect subfield of k, then R^* contains a unique subring R_0^* such that $H^*(R_0^*) = k_0$ [2]. It is easily seen that if $a \in R_0^*$ then $\Delta(a) = \infty$. We will show below that if $\Delta(a) = \infty$, then $a \in R_0^*$.

For notational convenience we will introduce the symbol $a[b]$ meaning a^b.

If $a \in R^*$, then one can easily show that $a \in R_0^*$ or $a = \sum_{i=0}^{\infty} p[i]a_i[p[n_i]]$ and for some i, $H(a_i) \in k \backslash k^p$. Let $\mathscr{I}$ denote the set of all such subscripts.

THEOREM 1. *$\Delta(a) = \min_{i \in \mathscr{I}}(i + n_i)$ if $\mathscr{I} \neq \emptyset$, $\Delta(a) = \infty$ otherwise.*

Proof. Let $a \in R^*$ and decompose

$$a = \sum_{i \in \mathscr{I}} p[i]a_i[p[n_i]] + \sum_{j \notin \mathscr{I}} p[j]a_j.$$

Let $n = \min_{i \in \mathscr{I}}(i + n_i) = N(a)$. Let $i_0 < i_1 < \cdots < i_q$ be the collection of indices such that $i_0 + n_{i_0} = \cdots = i_q + n_{i_q} = n$. Let $a' = \sum_{v=0}^{q} p[i_v]a[p[n_{i_v}]]$. We have $a = a' + a''$, where $N(a') = N(a)$ and $N(a'') \geqq N(a) + 1$. Thus if $\Delta(a) = N(a)$, then $\Delta(a') = N(a')$ and conversely. Hence, it suffices to show that $\Delta(a') = N(a')$. Now $a' = p[i_0]a^*$, where a^* is a unit. Also, $H^*(a^*) = \gamma_{i_0}[p[n_{i_0}]]$, where $H^*(a_{i_0}) = \gamma_{i_0} \in k \backslash k^p$. Let $\{\gamma_{i_0}, m\}$ be a p-basis for k Then $\{\gamma_{i_0}[p[n_{i_0}]], m\}$ is a p-basis for $k^* = [p[n_{i_0}]](m)$ and $[k : k^*] = p[n_{i_0}] = p[n_{i_0}]$ with $k = k^*(\gamma_{i_0})$, where the minimum polynomial of γ_{i_0} is $X[p[n_{i_0}]] - \gamma_{i_0}[p[n_{i_0}]]$. Let m^* be a set of representatives in R^* of m. Let $K_{m^*}^*$ be the complete closure of $K_0^*(m^*)$, where K_0^* is the unique subfield of K^* having residue field k_0. Let $R_{m^*}^*$ be its ring of integers and construct $R' = R_{m^*}^*[a^*, R^*[p[n_{i_0}]]]$. Then R' is an integral domain and $H^*(R') = k^*$.

Let K' be the complete closure of the quotient field of R', then K' is the quotient field of R', since K^* is complete and unramified. Now let c be any representative for γ_{i_0}. Then $K^* = K'(c)$ and $[K^*:K'] = p[n_{i_0}]$, since K^* is unramified. Thus the minimum polynomial of c is of the form

$$\text{(A)} \qquad X[p[n_{i_0}]] + \alpha_1 X[p[n_{i_0}] - 1] + \cdots = f(X).$$

Now $f(c) = 0$ implies $H^*(f(c)) = f^{H^*}(\gamma_{i_0}) = 0$, where

$$\text{(B)} \qquad X[p[n_{i_0}]] + H^*(\alpha_1)X[p[n_{i_0}] - 1] + \cdots = f^{H^*}(X).$$

Thus $X[p[n_{i_0}]] - \gamma_{i_0}[p[n_{i_0}]] | f^{H^*}(X)$, but since they have the same degree, $f^{H^*}(X) = X[p[n_{i_0}]] - \gamma_{i_0}[p[n_{i_0}]]$. Hence

$$\text{(C)} \quad \begin{array}{ll} \text{(i)} & H^*(\alpha_i) \in (p) \quad \text{for } 1 \leqq i \leqq p[n_{i_0}] - 1, \\ \text{(ii)} & H^*(f(0)) = \gamma_{i_0}[p[n_{i_0}]]. \end{array}$$

Now $\{a^*, m^*\}$ is a collection of representatives of a p-basis of k^*. Thus, as Heerema has shown [1], we determine a unique integral derivation $D: K' \to K'$ by letting

$$\text{(D)} \quad \begin{array}{lll} \text{(i)} & D(a) = 0, \qquad a \in K_0^*, & a \in m^*, \\ \text{(ii)} & D(a^*) = p[n_{i_0}]u, & u \text{ some unit in } R'. \end{array}$$

Now consider the extension of D to K^*. By the construction of the fields and a straightforward argument on the valuation, $R^* = R'[c]$. Hence, if we can show that $V(D(c)) \geqq 0$, then $D: K^* \to K^*$ is an integral derivation. We know that $D(c)$ is uniquely determined and $D(c) = -f^D(c)/f'(c)$. We observe that $a \in R'$ implies $V(D(a)) \geqq n_{i_0}$. Thus $V(D(\alpha_i)) \geqq 1 + n_{i_0}$ for all i, $1 \leqq i \leqq p[n_{i_0}] - 1$ by condition (C), (i). Next observe that $f(0) = a^* + p \cdot v$ and $D(f(0)) = D(a^*) + pD(v) = p[n_{i_0}]u + p \cdot p[n_{i_0}]v^* = p[n_{i_0}](u + pv^*) = p[n_{i_0}]u^*$, where u^* is a unit. Thus $V(f^D(c)) = n_{i_0}$.

Observe that $f'(c)$ has degree $\leqq p[n_{i_0}] - 1$. Thus from the fact that $R^* = R'[c]$, $V(f'(c))$ is equal to the minimum of the values of the coefficients. Now the coefficient of $c[p[n_{i_0}] - 1]$ in $f'(c)$ is $p[n_{i_0}]$. Thus the minimum value of the coefficients is $\leqq n_{i_0}$. Thus $V(f'(c)) \leqq n_{i_0}$. Hence $V(D(c)) \geqq 0$. Thus $D: K^* \to K^*$ is an integral derivation. Now $D(a') = D(p[i_0]a^*) = p[i_0]D(a^*) = p[i_0 + n_{i_0}]u$ and $V(D(a')) = i_0 + n_{i_0} = n$. Since $V(D^*(a')) \geqq n$ for all D^*, it follows that $\Delta(a') = N(a') = n$ and thus $\Delta(a) = N(a) = n$ and the theorem follows.

3. **$\bar{p}$-adic fields and property (H).** In this section we shall be concerned with $\bar{p}$-adic fields K, with ring of integers R, maximal ideal (π), residue field $k = R/(\pi)$ and natural place $H: K \to \{k, \infty\}$. If V is the valuation on K and if $V(\pi) = 1$, $V(p) = e$, then $K = K^*(\pi)$, where K^* is a p-adic field in the restricted valuation $V^* = V/K^*$, $[K:K^*] = e$ and π is the root of an Eisen-

stein polynomial $f(X) = X^e + p\sum_{i=0}^{e-1} f_i X^i$, $f_i \in R^* = R \cap K^*$ and $f_0 \in R^* \setminus (p)$, $(p) = R^* \cap pR$ [3].

It is also true that if $H^* = H/K^*$, then $H^*(R^*) = k$. Throughout the discussion we will assume that K is given and that K^* and π have been fixed once chosen.

Let $\Delta_{K|K^*} = \min\{(\Delta(f_i) + 1)e + i\} - V(f'(\pi))$, where $f'(X)$ is the derivative with respect to X of $f(X)$. Notice that since $0 \leqq i \leqq e - 1$, then by the properties of V, $\min\{(\Delta(f_i) + 1)e + i\}$ is uniquely determined and equal to $(\Delta(f_{i_0}) + 1)e + i_0$ for some index i_0, $0 \leqq i_0 \leqq e - 1$. Hence $\Delta_{K|K^*} = (\Delta(f_{i_0}) + 1)e + i_0 - V(f'(\pi))$ for some index i_0. Since $V(f'(\pi))$ is fixed, $\Delta_{K|K^*}$ depends only on the coefficients of $f(X)$ once K^* and π have been chosen.

Let an integral derivation D on K be an integral$_n$ derivation if $D(\pi) \in (\pi)^n$.

THEOREM 2. *K has the property that every integral derivation is an integral$_n$ derivation, $n \geqq 1$, if and only if $\Delta_{K|K^*} \geqq n$. If $n \geqq 1$, then it is also true that for any such D, $D((\pi)) \subset (\pi)$.*

Proof. Suppose $\Delta_{K|K^*} = n \geqq 0$. Let D^* be an integral derivation on K^*. Then D^* has a unique extension D: $K \to K$ which is completely determined by $D(\pi)$. Now $D(\pi) = -f^{D^*}(\pi)/f'(\pi)$. Hence $V(D(\pi)) = V(f^{D^*}(\pi)) - V(f'(\pi)) \geqq \Delta_{K|K^*} = n \geqq 0$. Thus $D(\pi) \in (\pi)^n \subset R$.

Since $a \in R$ implies $a = g_a(\pi)$, where $g_a(X) \in R^*[X]$, it follows that $D(a)$ $g_a^{D^*}(\pi) + g_a'(\pi)D(\pi) \subset R$. Hence $D(R) \subset R$.

Notice that $n \geqq 1$ implies $D(\pi d) = \pi D(a) + aD(\pi) \in (\pi)$ if $a \in R$, hence $D((\pi)) \subset (\pi)$.

Now suppose that D is an integral derivation on K, then $a = g_a(\pi)$ implies $D(a) = g_a^D(\pi) + g_a'(\pi)D(\pi)$. In particular, if $a \in R^*$ then $D(a) = g_a^D(\pi) = g_{a,D}(\pi)$. Thus $D = D|K^*$ is given by $D'(a) = g_{a,D'}(\pi)$.

Consider $g_{a,D}(X)$. Since $[K : K^*] = e$, we may choose $g_{a,D}(X)$ of degree at most $e - 1$. Hence, if we do this, then

$$g_{a,D}(X) + g_{b,D'}(X) = g_{a+b,D'}(X)$$

and

$$ag_{b,D}(X) + bg_{a,D}(X) = g_{ab,D}(X)$$

for all $a, b \in R^*$.

Thus we may write $D' = \sum_{i=0}^{e-1} \pi^i D_i^*$, where $D_i^*(a)$ is the coefficient of X^i in $g_{a,D'}(X)$, and so D_i^*: $K^* \to K^*$ is an integral derivation on K^*. The fact that the polynomials $g_{a,D'}(X)$ are uniquely determined implies that the representation $D' = \sum_{i=0}^{e-1} \pi^i D_i^*$ is unique.

Since $\Delta_{K|K^*} = n \geqq 0$, each derivation D_i^* has a unique extension D_i to K such that $D_i(R) \subset R$, $D_i(\pi) \in (\pi)^n$ and $D_i((\pi)) \subset (\pi)$ if $n \geqq 1$. Hence since

D' has unique extension D to K, it follows that $D = \sum_{i=0}^{e-1} \pi^i D_i$. Thus $D(\pi) \in (\pi)^n$ and $D((\pi)) \subset (\pi)$ if $n \geqq 1$.

Conversely, suppose that every integral derivation D on K is an integral$_n$ derivation, $n \geqq 1$. Suppose also that $\Delta_{K|K^*} = m < n$. Suppose $\Delta_{K|K^*} = (\Delta(f_{i_0}) + 1)e + i_0 - V(f'(\pi))$ and suppose that D^* is an integral derivation on K^* such that $V^*(D^*(f_{i_0})) = \Delta(f_{i_0})e$. Then if we extend D^* to a derivation D on K, $V(D(\pi)) = m < n$.

If $m \geqq 0$, then $D(R) \subset R$ and hence $V(D(\pi) = n > m$, a contradiction. If $m < 0$, then $\pi^{-m}D(R) \subset R$ and hence $V(\pi^{-m}D(\pi)) = 0 \geqq n \geqq 1$, a contradiction. Thus $m \geqq n$ and the theorem follows.

COROLLARY. *Every integral derivation D^* on K^* has an integral extension D to K if and only if $\Delta_{K|K^*} \geqq 0$.*

THEOREM 3. *K has property* (H) *if and only if $\Delta_{K|K^*} \geqq 1$.*

Proof. Suppose $\Delta_{K|K^*} \geqq 1$. Let $d: k \to k$ be any derivation.

Then there is an integral derivation D^* on K^* such that $a \in R^*$ implies $H^*(D^*(a)) = d(H^*(a)) = d(H(a))$. Since $\Delta_{K|K^*} \geqq 1$, then D^* has a unique extension D to K which is at least an integral$_1$ derivation. Hence D induces a derivation on k. Since $a \in R^*$ implies $D(a) = D^*(a)$, it follows that D induces $d: k \to k$. Hence K has property (H).

Conversely, suppose that $\Delta_{K|K^*} = m \leqq 0$. Then there is an integral derivation D^* on K^* such that if D is the unique extension of D^* to K, then $V(D(\pi)) = m \leqq 0$. If D_0^* is any derivation such that D^* and D_0^* induce the same derivation on k, then $D_0^* = D^* + pD^*$ and thus if D_0 is the unique extension of D_0^* to K, then $V(D_0(\pi)) = V(D(\pi)) = m$. Now suppose $d: k \to k$ is induced by $D^*: K^* \to K^*$ and suppose that $D: K \to K$ also induces $d: k \to k$. Then if $D' = D|K^*$, $D' = \sum_{i=0}^{e-1} \pi^i D_i^*$, where $D_i^*(R^*) \subset R^*$. For $a \in R^*$, $H^*(D^*(a)) = H(D(a)) = H(D'(a)) = H^*(D_0^*(a))$ and thus D_0^* induces $d: k \to k$.

Now suppose D_i is the unique extension of D_i^* to K, then $V(\pi^i D_i(\pi)) \geqq m + i > V(D_0(\pi))$ and thus $V(\sum_{i=0}^{e-1} \pi^i D_i(\pi)) = V(D_0(\pi)) = m \leqq 0$. But

$$D = \sum_{i=0}^{e-1} \pi^i D_i$$

and thus $V(D(\pi)) = V(D_0(\pi)) = m \leqq 0$. However, D is an inducing derivation and so $V(D(\pi)) \geqq 1$. Hence $d: k \to k$ is not induced by any $D: K \to K$ and K does not have property (H).

COROLLARY 1. *K has property* (H) *if and only if every integral derivation D on K is an inducing derivation.*

COROLLARY 2. *If K is a $\overline{p}$-adic field of ramification e and if $(e, p) = 1$, then K has property* (H).

4. **Automorphisms on $\bar{p}$-adic fields.** In this section we shall be concerned with establishing a connection between derivations and inertial automorphisms on $\bar{p}$-adic fields K. Let $G = \{T \mid T$ is an automorphism on $K\}$; for $n \geqq 1$, let $G_n = \{T \in G \mid T(a) - a \in (\pi)^n$ for all $a \in R\}$;

$$\bar{G}_n = \{T \in G_n \mid T(\pi) - \pi \in (\pi)^{n+1}\}.$$

As in §2, $a[b]$ means a^b.

Lemma 1. *Let $n \geqq 1$ and let $T \in G_n$. Then if $Z = T - I$, $Z[q]: R \to (\pi)[q(n-1)+1]$.*

Proof. $Z(ab) = T(ab) - ab = aZ(b) + bZ(a) + Z(a)Z(b)$. Thus, in particular, $Z(\pi[m]) = Z(\pi \cdot \pi[m-1]) = \pi Z(\pi[m-1]) + \pi[m-1]Z(\pi) + Z(\pi[m-1])Z(\pi)$. Hence $Z(\pi[2]) \in (\pi)[n+1]$ and by induction, $Z(\pi[m]) \in (\pi)[m+n-1]$.

Thus $a \in R$ implies

$$Z(a) = \pi[n]a_1,$$

$$Z[2](a) = Z(\pi[n]a_1) = a_1 Z(\pi[n]) + \pi[n]Z(a_1) + Z(\pi[n])Z(a_1) \in \pi[2n-1].$$

Hence by induction $Z[q](a) \in (\pi)[qn - (q-1)]$ and the lemma follows.

Corollary. *If $n \geqq 1$ and $T \in \bar{G}_n$, then $Z[q]: R \to (\pi)[qn]$.*

Proof. Since $Z(\pi) \in (\pi)[n+1]$, it follows that $Z(\pi[m]) \in (\pi)[m+n]$. Hence $Z[2](a) \in \pi[2n]$ and by induction $Z[q](a) \in (\pi)[qn]$.

Now suppose that $n \geqq (e+p)/(p-1)$, $q \geqq 2$. Then if $q = p[s]t$, it follows that $p[s]t(n-1) + 1 - se \geqq n+1$. Since $q \in (\pi)[se]$, it follows that $Z[q]/q: R \to \pi[n+1]$ for all $q \geqq 2$.

Lemma 2. *Suppose $1 \leqq i \leqq p[\mu]$, then*

$$i\binom{p[\mu]}{i} \in p[\mu]R.$$

Proof. Consider

$$i\binom{p[\mu]}{i} = 1/(i-1)!\,\{p[\mu](p[\mu]-1)\cdots(p[\mu]-(i-1))\}.$$

In the expansion we obtain terms of the form

$$p[\mu]\binom{p[\mu]}{i_1}\cdots\binom{p[\mu]}{i_l},$$

$l \geqq 0$, $1 \leqq i_1 < i_2 < \cdots < i_l \leqq i-1$. Since $(p[\mu]/i_v) \in R$, the lemma follows.

Theorem 4. *Let p be an odd prime and suppose $n \geqq (e+1)/(p-1)$. Suppose $T \in G_n$, or, if $n = 1$, $T \in \bar{G}_1$. Suppose $T = I + \pi[n]T'$. Then there* *is a derivation $D(T)_n$ on K such that $T' - D(T)_n: R \to (\pi)$. Hence $T \in \bar{G}_n$ if and only if $D(T)_n$ is an inducing derivation.*

Proof. Since $T = I + Z$, it follows that

$$T[p[\mu+1]] - T[p[\mu]]$$
$$(1)\qquad = \sum_{i=1}^{p[\mu]}\left(\binom{p[\mu+1]}{i} - \binom{p[\mu]}{i}\right)Z[i] + \sum_{i=p[\mu]+1}^{p[\mu+1]}\binom{p[\mu+1]}{i}Z[i].$$

If $i \geqq 2$, then by Lemma 2 and the fact that $Z[i]/i: R \to \pi[n+1]$ we get

$$(2)\qquad \begin{gathered} T[p[\mu+1]] - T[p[\mu]] = p[\mu](p-1)Z + p[\mu]Z^*, \text{ where} \\ Z^*_\mu: R \to (\pi)[n+1]. \end{gathered}$$

Let

$$\begin{aligned} T_\mu &= p[-\mu](T[p[\mu+1]] - T[p[\mu]]) \\ &= (p-1)Z + Z^*_\mu = -Z + pZ + Z^*_\mu. \end{aligned}$$

Now consider $T_{\mu+1} - T_\mu$. This map is given by

$$(3)\qquad \begin{gathered} \sum_{i=1}^{p[\mu]} g_i p[-\mu-1]Z[i] + \sum_{i=p[\mu]+1}^{p[\mu+1]} g_i' p[-\mu-1]Z[i], \\ + \sum_{i=p[\mu+1]+1}^{p[\mu+2]}\binom{p[\mu+2]}{i}p[-\mu-1]Z[i], \end{gathered}$$

where

$$g_i = \left(\binom{p[\mu+2]}{i} - \binom{p[\mu+1]}{i}(1+p) + \binom{p[\mu]}{i}p\right), \qquad 1 \leqq i \leqq p[\mu],$$

$$g_i' = \left(\binom{p[\mu+2]}{i} - \binom{p[\mu+1]}{i}(1+p)\right), \qquad p[\mu]+1 \leqq i \leqq p[\mu+1].$$

Note that $g_1 = 0$. For $i \geqq 2$, we get

$$(4)\qquad \begin{aligned} ig_i = 1/(i-1)!\,\{&p[\mu+1]((p[\mu+2]-1)\cdots(p[\mu+2]-(i-1)) \\ &-(p[\mu+1]-1)\cdots(p[\mu+1]-(i-1))) \\ &-p[\mu+1]((p[\mu+1]-1)\cdots(p[\mu+1]-(i-1)) \\ &-(p[\mu]-1)\cdots(p[\mu]-(i-1)))\}. \end{aligned}$$

Now suppose we pick μ such that $p[\mu/3] \geqq 2(\mu+1)e$. Then $i \geqq p[\mu/3]$ implies $Z[i]: R \to p[2(\mu+1)]R$. Also, if $1 \leqq i < p[\mu/3]$, then by Lemma 2, and the fact that on the right side of (4) the terms involving $(i-1)!$ are cancelled, it follows that $p[-\mu-1]g_iZ[i]: R \to p([\mu] - [[\mu/3]+1])R$, where the inner

$[\mu/3]$ denotes the greatest integer function. Hence

$$T_{\mu+1} - T_\mu \colon R \to p([\mu] - [[\mu/3] + 1])R$$

and

$$\lim_{\mu\to\infty} T_\mu = T_{\mu_0} + \sum_{i=\mu_0}^{\infty} (T_{i+1} - T_i)$$

is a well-defined map.

Since

$$T[p[\mu]] - I = \sum_{i=1}^{p[\mu]} \binom{p[\mu]}{i} Z[i],$$

then by Lemma 2, $T[p[\mu]] - I \colon R \to p[\mu]R$. Thus since

$$\begin{aligned} T_\mu(ab) - aT_\mu(b) - bT_\mu(a) = p[-\mu]\{ &(T[p[\mu+1]](a) - a)(T[p[\mu+1]](b) - b) \\ &- (T[p[\mu]](a) - a)(T[p[\mu]](b) - b)\} \qquad (p \neq 2) \end{aligned}$$

it follows that for μ large enough T_μ is a derivation modulo $p[\mu]$. Hence $\lim_{\mu\to\infty} T_\mu$ is a derivation.

Now $\lim_{\mu\to\infty} T_\mu = -Z + pZ + \lim_{\mu\to\infty} Z_\mu^*$. Thus if we let $D(T)_n = -\pi[-n] \cdot \lim_{\mu\to\infty} T_\mu$, then $D(T)_n = +T' - pT' - \pi[-n]\lim_{\mu\to\infty} Z_\mu^*$ and $T' - D(T)_n \colon R \to (\pi)$, since $Z_\mu^* \colon R \to (\pi)[n+1]$. Hence the theorem follows.

Theorem 5. *Suppose $n \geqq (e+1)/(p-1)$ and suppose $D \colon K \to K$ is an integral derivation on K. Then $D_n = I + \sum_{i=0}^{\infty} \pi[ni]/i!\, D[i]$ is an automorphism on K, $D_n \in G_n$ and $D_n \in \overline{G_n}$ if and only if $D(\pi) \in (\pi)$.*

Proof. Observe that $V(i!) < ie/(p-1)$. Thus $ni - V(i!) \geqq ni - ie/(p-1) \geqq (e+1)i/(p-1) - ie/(p-1)$ and $\lim_{i\to\infty} V(\pi[ni]/i!) = \infty$. Thus D_n is a well-defined map, $D_n(R) \subset R$. D_n is additive since $D[i]$ is additive. Also, by a straightforward computation $D_n(ab) = D_n(a) \cdot D_n(b)$. Since $D_n(1) = 1$, it follows that $D_n \in G$.

By a straightforward computation $i \geqq 2$ implies $V(\pi[ni]/i!) \geqq n+1$, if $n \geqq (e+1)/(p-1)$. Hence $D_n = I + \pi[n](D + \pi D_n^*)$, where $D_n^*(R) \subset R$. Hence $D_n \in G_n$ and $D_n \in \overline{G_n}$ if and only if $D(\pi) \in (\pi)$.

Theorem 6. *Let $\mathscr{D}$ be the additive group of integral derivations on K and let $\overline{\mathscr{D}}$ be the additive group of derivations on the residue field of K which are induced, then if p is odd and $n \geqq (e+p)/(p-1)$, G_n/G_{n+1} is isomorphic to $\mathscr{D}/\pi\mathscr{D}$ and if $n \geqq (e+p)/(p-1)$, $\overline{G_n}/G_{n+1}$ is isomorphic to $\overline{\mathscr{D}}$.*

Proof. Define $H_n(TG_{n+1}) = D(T)_n + \pi\mathscr{D}$ and $H_n^*(TG_{n+1}) = d$, where d is induced by $D(T)_n$ if $T \in \overline{G_n}$. Theorem 4 implies H_n and H_n^* are well-defined since $D(T)_n \colon R \to (\pi)$ if and only if $T \in G_{n+1}$. Since $D(T_1 \cdot T_2)_n = T_1' + T_2'$

$+ T_1'(\pi[n]T_2')$, it follows that $D(T_1 \cdot T_2)_n \equiv D(T_1)_n + D(T_2)_n \pmod{\pi}$, i.e., H_n and H_n^* are homomorphisms. H_n and H_n^* are monomorphisms by Theorem 4. By Theorem 5, since $D(D_n) - D: R \to (\pi)$, they are epimorphisms. Thus they are isomorphisms and the theorem follows.

Corollary. *K has property* (H) *if and only if $\overline{G_n} = G_n$ for all $n \geqq (e+p)/(p-1)$.*

Proof. Suppose $\overline{G_n} = G_n$. Then $T \in G_n$ implies $D(T)_n$ is inducing. Since $D \in \mathscr{D}$ implies $D = D(T)_n - \pi D'$ for some T and D', every D is inducing. Thus K has property (H). If K has property (H), then every $D(T)_n$ is inducing, which implies $T \in \overline{G_n}$. Hence $G_n = \overline{G_n}$.

Conjecture. Theorem 6 holds for all $n \geqq 1$.

Notice that if $T \in G_n$ $(n \geqq (e+p)/(p-1))$, we can obtain $D(T)_n$. Thus by Theorem 5, $(D(T)_n)_n$ is an automorphism with derivation $D((D(T)_n)_n) - D(T)_n \in \pi\mathscr{D}$. Hence $T = T_1 \cdot ((D(T)_n)_n)$, where $T_1 \in G_{n+1}$. Proceeding in this fashion we obtain

$$T = T_j((D(T_{j-1})_{n+j-1})_{n+j-1}) \cdots ((D(T)_n)_n), \qquad T_j \in G_{n+j}$$

and since $\bigcap G_n = I$, we obtain

$$T = \lim_{j \to \infty} ((D(T_j)_{n+j})_{n+j}) \cdots ((D(T)_n)_n),$$

i.e., T has a Taylor-series-like expansion in terms of derivations.

Theorem 6 is an extension of a well-known result of MacLane [2], to $\bar{p}$-adic fields.

References

1. N. Heerema, *Derivations on p-adic fields,* Trans. Amer. Math. Soc. **102** (1962), 346-351.
2. Saunders MacLane, *Subfields and automorphism groups of p-adic fields,* Ann. of Math. **40** (1939), 423-442.
3. O. F. G. Schilling, *The theory of valuations,* Math. Surveys No. 4, Amer. Math. Soc., Providence, R. I., 1950.
4. O. Teichmüller, *Diskret bewertete perfekte Korper mit unvollkomenem Restklassen Korper,* J. Reine Angew. Math. **176** (1937), 141-152.
5. O. Zariski and P. Samuel, *Commutative algebra,* Vols. I, II, Van Nostrand, Princeton, N. J., 1958, 1960.

Florida State University,
Tallahassee, Florida