CONTINUOUS TRANSFORMATIONS AND STOCHASTIC DIFFERENTIAL EQUATIONS(1)

BY
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Introduction. It is the purpose of this paper to construct stochastic processes $\{y(t), y(0) = 0, 0 \leqq t \leqq 1\}$ satisfying the differential equation

$$dy(t) = f(y|t)dt + \sigma(t, y(t))dx(t), \tag{0.1}$$

where $\{x(t), x(0) = 0, 0 \leqq t \leqq 1\}$ is a Brownian motion process. Equation (0.1) has been studied by S. Bernstein [1], J. L. Doob [5] and others [2], [10]. In general, the solution given here is different from that given by these authors. Equation (0.1) is almost purely formal since the derivative dx/dt fails to exist with probability one. In [2], [5], [10], the stochastic integral of K. Ito [7], [8] is used to define an integrated form of (0.1), which is solved as in [8] to obtain a transformation of sample functions. The present work involves a transformation of sample functions but the integral used is the functional integral of R. Cameron and R. Fagen [4]. E. B. Dynkin [6] has given a different method of attacking similar problems. Note that $f(y|t)$ is a functional.

1. **Comparison with previous work.** Let $m, n,$ and v be constants and specialize (0.1) to

$$dy(t) = m(y(t) + v)dt + n(y(t) + v)dx(t). \tag{1.1}$$

If the first derivatives of y and x are continuous, (1.1) is equivalent to

$$y(t) = v(e^{mt+nx(t)} - 1). \tag{1.2}$$

Equation (1.2) represents a transformation which is the unique continuous (in the uniform topology, say) extension of (1.2) to all continuous $x(t)$ with $x(0) = 0$. The expected value of $y(t)$ as given by (1.2) is $v(m + n^2/2)t + o(t)$ as $t \to 0^+$, where the $\{x(t)\}$ process has variance parameter one. According to [5, p. 275], the expected value of $y(t)$ defined by (1.1) is $mvt + o(t)$ as $t \to 0^+$.

The backward and forward differencing schemes (8) and (8 bis) of [1, p. 12] suggest different stochastic differential equations but are equivalent. Following this method, the equation for $y(t)$ near $t = 0$,

$$y(t) = f(y|t/2)t + \sigma(t/2, y(t/2))x(t), \tag{1.3}$$

may be obtained by neglecting terms of higher order than t from equation (1.4) below.

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Notation. Subscripts on functions of more than one real variable denote partial derivatives. Thus $(\partial/\partial t)g(t,u) = g_1$ and $(\partial/\partial u)g(t,u) = g_2$.

$$y(t) = f(y|0)t + \sigma(0,0)\sigma_2(0,0)x(t/2)x(t) + \sigma(0,0)x(t). \tag{1.4}$$

For $f = m(y(t) + v)$ and $\sigma = n(y(t) + v)$ the expected value of $y(t)$ as given by (1.4) is $mvt + nvnt/2$, which agrees with that given by (1.2) as $t \to 0^+$.

The solution (1.2) of (1.1) was given by A. Rosenbloom [9].

2. **Preliminary results.** Let C be the space of functions which are continuous on $[0,1]$ and vanish at zero. The topology on C is that induced by the uniform norm denoted by $\|x(t)\|$. Let C' be the subspace of C of functions which have continuous derivatives on $[0,1]$.

DEFINITION. Let U be an open subset of C, $F(x|t)$ be defined on $U\otimes[0,1]$ and integrable in t for each x, and $F(x|1)$ be continuous on U. Define the functional integral of F at y on $[0,t]$ to be the following limit if it exists as a finite number:

$$\lim_{x\to y, x\in C'} \int_0^t F(x|s)(dx(s)/ds)\,ds = \int_0^t F(y|s)\,d^*y(s).$$

For $t = 1$, the functional integral defined here is, if it exists, the same as that defined in [4] as can be seen from the proof of Theorem 1 there. The following theorem is similar to Theorem 2 of [4].

THEOREM 1. *Let K be an open convex subset of $C, k \in K \cap C'$, and $g(t,u)$ and $g_1(t,u)$ be continuous on $D = \{(t,u)\,|\,0 \leqq t \leqq 1, u = y(t)$ for some $y \in K\}$, and if $(t,u),(t,v) \in D$, define*

$$G(t,u,v) = \int_v^u g(t,r)\,dr.$$

Then if $y \in K$ the following functional integral exists and

$$\int_0^t g(s,y(s))\,d^*y(s) = G(t,y(t),k(t)) - \int_0^t G_1(s,y(s),k(s))\,ds + \int_0^t g(s,k(s))k'(s)\,ds. \tag{2.1}$$

Proof. The following argument depends only on the continuity of $G(t,u,k(t))$ on D. Let $\eta > 0$ and $y_0 \in K$ be given. The graph of $y_0(t)$ is compact, so an open covering of sets

$$E(t) = \{(t',u)\,|\,|t-t'| < \delta(t), |u - y_0(t)| < \delta(t)\},$$

where $\delta(t) > 0$ is so small that $(t',u) \in E(t)$ implies

$$|G(t,y_0(t),k(t)) - G(t',u,k(t'))| < \eta,$$

may be reduced to a finite covering $\{E(t_k)\}_{k=1,n}$. The union of this finite covering contains a strip of half-width $\delta > 0$ about the graph of $y_0(t)$ such that $\|y(t) - y_0(t)\| < \delta$ implies

$$\|G(t, y_0(t), k(t)) - G(t, y(t), k(t))\| < \eta.$$

Thus $G(t, y(t), k(t))$ and $G_1(t, y(t), k(t))$ are continuous transformations in the uniform topologies. Equation (2.1) follows because its right-hand side with y replaced by x is $\int_0^t g(s, x(s))dx(s)$ for $x \in C' \cap K$. Since C' is dense in C we have the

COROLLARY. *Under the hypothesis of Theorem* 1, *the transformation* $\int_0^t g(s, y(s))d^*y(s)$ *defined on* K *into* C *is the unique continuous extension of* $\int_0^t g(s, y(s))dy(s)$ *defined on* $K \cap C'$.

Let $\sigma(t, u)$ be defined on an open set and

$$\Omega_0 = \{y \in C \mid \sigma(t, y(t)) > 0, 0 \leqq t \leqq 1\}.$$

THEOREM 2. *If* $\sigma(t, u)$ *is defined and continuous on an open set, the set* Ω_0 *is open and may be partitioned into at most a countable number of disjoint open convex components.*

Proof. If $y_0 \in \Omega_0, \sigma(t, y_0(t))$ takes on its minimum value $v > 0$. Use the argument in the proof of the theorem above, but choose $\delta(t)$ such that $(t', u) \in E(t)$ implies $\sigma(t', u) > v/2$. It follows that Ω_0 is open. Define the equivalence relation $y_1 \sim y_2$ if and only if $y_1, y_2 \in \Omega_0$ and $y \in C$ and

$$\min(y_1(t), y_2(t)) \leqq y(t) \leqq \max(y_1(t), y_2(t)), \qquad 0 \leqq t \leqq 1,$$

imply $y \in \Omega_0$. The partition elements, called components, are disjoint and convex. The components are open since Ω_0 is and members of a uniform sphere contained in Ω_0 are equivalent. There are at most a countable number of components since C has a countable dense subset.

3. **Transformations.** Let $\{K_i\}_{i=1,\infty}$ be the set of components of Ω_0 according to Theorem 2. Let $k_i \in K_i \cap C'$, $D_i = \{(t, u) \mid 0 \leqq t \leqq 1, u = y(t)$ for some $y \in K_i\}$, $i = 1, \infty$, and $G(t, u, v) = \int_v^u dr/\sigma(t, r)$ provided $(t, u), (t, v) \in D_i$ for some $i, i = 1, \infty$. The inverse function $H(t, w, v)$ such that $u = H(t, w, v)$ and $w = G(t, u, v)$ is defined and continuous on

$$\{(t, w, v) \mid (t, u), (t, v) \in D_i, w = G(t, u, v)\},$$

$i = 1, \infty$, by the implicit function theorem if $\sigma(t, u)$ is continuous. Define the transformations

$$S_i : z(t) = G(t, y(t), k_i(t))$$

on $K_i, i = 1, \infty$,

$$T_i : x(t) = z(t) + \Lambda_i(z \mid t)$$

on $S_i(K_i), i = 1, \infty$, and

$$R : x(t) = T_i S_i y \quad \text{on } K_i,\ i = 1, \infty,$$

on Ω_0, where

$$\Lambda_i(z|t) = M_i(H(\cdot, z(\cdot), k_i(\cdot))|t)$$

and

$$M_i(y|t) = \int_0^t \{k_i'(s)/\sigma(s, k_i(s)) - G_1(s, y(s), k_i(s)) - f(y|s)/\sigma(s, y(s))\} ds.$$

Suppose $y \in \Omega_0 \cap C'$ and $\sigma(t,u)$ and $\sigma_1(t,u)$ are continuous. Then $x = Ry$ implies $x \in C'$ and (0,1) holds. From the corollary to Theorem 1 we have the

THEOREM 3. *Let $\sigma(t,u)$ and $\sigma_1(t,u)$ be defined and continuous on the same open set and $f(y|t)$ be continuous on $\Omega_0 \otimes [0,1]$. Then* (0.1) *defines the unique continuous mapping R on Ω_0 into C.*

Suppose $x \in C'$ and $x = Ry$ for $y \in K_i$. Then $G(t, y(t), k_i(t)) \in C'$ and since H has continuous partial derivatives, $y \in C'$ and we have the

COROLLARY. *Under the hypothesis of Theorem* 3, *the pre-image of $C' \cap R(\Omega_0)$ is $C' \cap \Omega_0$.*

Let Ω_1 be the set of all extended real-valued functions on (0,1] which are not members of Ω_0. In this section Ω_1 is used only to normalize the measure on the $\{y(t)\}$ process. For example, the infinite-valued solutions of

$$dy(t) = -y^2(t)dt + dx(t) \tag{3.1}$$

are members of Ω_1 which then has positive measure as shown in [11]. Let $\mathbf{S}_y$ be the σ-ring generated by Ω_1 and the Borel subsets of Ω_0. Let $R(\Omega_1)$ be the set of all extended real-valued functions on (0,1] which are not members of $R(\Omega_0)$ and $\mathbf{S}_x$ be the σ-ring generated by $R(\Omega_1)$ and the Borel subsets of $R(\Omega_0)$.

Lemma 7 and Theorem 4 of [4] are stated here in the slightly weaker form of

THEOREM A. *Let the transformation*

$$T: \ (t) = z(t) + \Lambda(z|t)$$

defined on the open subset Γ of C satisfy the following conditions.

(A1) *On a uniform neighborhood U_0 of each $z_0 \in \Gamma$ let*

$$\frac{\partial}{\partial v}\Lambda(z + vr|t)\Big|_{v=0} = \int_0^1 N(z|t,s)r(s)ds$$

if $(z,t,r) \in U_0 \otimes [0,1] \otimes C$, where

$$N(z|t,s) = \begin{cases} N^1(z|t,s), & 0 \leqq t < s \leqq 1 \\ (1/2)N^1(z|t,s) + (1/2)N^2(z|t,s), & 0 \leqq t = s \leqq 1 \\ N^2(z|t,s), & 0 \leqq s < t \leqq 1 \end{cases}$$

and N^1 *and* N^2 *are continuous on* $U_0 \otimes \{0 \leqq t \leqq s \leqq 1\}$ *and* $U_0 \otimes \{0 \leqq s \leqq t \leqq 1\}$, *respectively.*

Let $N(z|t,s)$ *be bounded on* $U_0 \otimes [0,1] \otimes (0,1]$.

Let $D(z) \neq 0$ *on* Γ *where* $D(z)$ *is the Fredholm determinant,*

$$D(z) = 1 + \sum_{j=1}^{\infty} \frac{1}{j!} \int_0^1 \cdots \int_0^1 \begin{vmatrix} N(z|s_1,s_1) & \cdots & N(z|s_1,s_j) \\ \cdots & \cdots & \cdots \\ N(z|s_j,s_1) & \cdots & N(z|s_j,s_j) \end{vmatrix} ds_1 \cdots ds_j.$$

(A2) *Let* $N(z|0,s) = 0$ *for* $(z,s) \in \Gamma \otimes (0,1]$. *Let* $\Lambda(z|t) \in C$ *when* $z \in \Gamma$ *and* $\partial\Lambda(z|t)/\partial t$ *be continuous on* $\Gamma \otimes [0,1]$.

Then T *is open. Moreover, suppose the following conditions hold.*

(A3) *On a uniform neighborhood* U_0 *of each* $z_0 \in \Gamma$ *let, for each fixed* $t \in [0,1]$,

$$\int_0^1 (N(z_1|t,s) - N(z_2|t,s))^2 ds \to 0$$

uniformly in $z_1, z_2 \in U_0$ *as* $\|z_1(t) - z_2(t)\| \to 0$.

(A4) *On a uniform neighborhood* U_0 *of each* $z_0 \in \Gamma$ *let*

$$\int_0^1 \left[(\partial/\partial t)(1/\epsilon) \int_t^{t+\epsilon} \Lambda\left((1/\epsilon) \int_{(\cdot)-\epsilon}^{(\cdot)} z(u)\,du \,\middle|\, s \right) ds \right] dz(t),$$

where functions of a real variable are assumed to be defined continuously outside the interval $(0,1)$, *be defined and bounded below in* (x,ϵ) *on* $U_0 \otimes (0,\epsilon_0)$ *for some positive* $\epsilon_0 < (1/2)$.

(A5) $\int_0^1 (\partial/\partial t)\Lambda(z|t)\,d^*z(t)$ *exists as defined in* §2 *for all* $z \in \Gamma$.

(A6) T *is one-to-one on* Γ.

Then if F *is a Wiener measurable functional such that either of the following Wiener integrals (expectations on the Brownian motion process with variance parameter* $(1/2)$*) exists, they both exist and are equal.*

$$\int_{T_\Gamma}^{w} F(x)\,d_w x = \int_\Gamma^{w} F(Tz) J_T(z)\,d_w z \tag{3.2}$$

where

$$J_T(z) = |D(z)| \exp\left\{ -2\int_0^1 (\partial/\partial t)\Lambda(z|t)\,d^*z(t) - \int_0^1 [(\partial/\partial t)\Lambda(z|t)]^2 dt \right\}. \tag{3.3}$$

An application of Theorem A is our main result.

THEOREM 4. *Let* $\sigma(t,u)$ *and* $\sigma_1(t,u)$ *be defined and continuous on the same open set and* $f(y|t)$ *be continuous on* $\Omega_0 \otimes [0,1]$. *Let the solutions* $y \in C' \cap \Omega_0$ *of* (0.1) *be unique for each* $x \in C'$. *For each* $i, i = 1, \infty$, *let* T_i *defined on* $S_i(K_i)$ *satisfy* (A1). *Then* R *is one-to-one, open, and continuous. Thus a measure on one of the* σ*-rings,* $\mathbf{S}_x$ *and* $\mathbf{S}_y$, *and* R *define a measure on the* *other with the same total measure. Suppose the measure on* $\mathbf{S}_x$ *is that of the restriction of the Brownian motion process with variance parameter* (1/2) *and the measure defined by* R^{-1} *on* $\mathbf{S}_y$ *is denoted by* P_y. *Suppose also that* T_i *satisfies conditions* (A3) *and* (A5) *on* $S_i(K_i), i = 1, \infty$. *Then if* F *is a measurable functional on* Ω_0 *such that either side of* (3.4) *exists,* (3.4) *holds.*

$$\int_{\Omega_0} F(y)\,dP_y = \sum_{i=1}^{\infty} \int_{S_i(K_i)}^{w} F(H(\cdot\,,z(\cdot),k_i(\cdot)))\, J_{T_i}(z)\,d_w z, \tag{3.4}$$

where J_{T_i} *is given by* (3.3) *and the Wiener integral notation is used on the right.*

Proof. For $i = 1, \infty, T_i$ satisfies (A1) by hypothesis and (A2) because of the form of Λ_i. From the first part of Theorem A, T_i is open, $i = 1, \infty$. Since S_i is open, $i = 1, \infty, R$ is open. The uniqueness of solutions $y \in C' \cap \Omega_0$ of (0.1) for $x \in C'$, openness of R, and the corollary to Theorem 3 imply that R is one-to-one. The continuity of R follows from Theorem 3. Now assume all the hypothesis of Theorem 4. Since F is Borel measurable and R is open, $F(R^{-1}x)$ is Wiener measurable on $R(K_i), i = 1, \infty$. For $i = 1, \infty, T_i$ satisfies (A6) because $T_i = RS_i^{-1}$. From the definition of the functional integral used here, for each $i = 1, \infty$ and $z_0 \in S_i(K_i)$ there exists a uniform sphere of radius $r > 0$ about z_0 such that $\int_0^1 (\partial/\partial t)\Lambda_i(z|t)(dz(t)/dt)dt$ is bounded for $z \in C' \cap V$. Choose $\epsilon_0 > 0$ such that $\epsilon < \epsilon_0$ implies

$$\left\| \int_{\min(0,\, t-\epsilon)}^{t} (1/\epsilon)(z_0(s) - z_0(t))\,ds \right\| < r/2.$$

Then (A4) is satisfied on $V_1 \otimes (0, \epsilon_0)$ where V_1 is the uniform sphere of radius $r/2$ about z_0. From Theorem A the right-hand side of (3.4) is

$$\sum_{i=1}^{\infty} \int_{T_i S_i(K_i)}^{w} F(S_i^{-1} T_i^{-1} x)\,d_w x = \int_{R(\Omega_0)}^{w} F(R^{-1}x)\,d_w x.$$

Equation (3.4) follows by definition.

As an example, the probability of finite continuous solutions $y(t)$ of (3.1) is shown in [3] to be

$$\int_C^w \exp\left\{ \int_0^1 (x(t) - (x(t))^4)\,dt - 2(x(1))^3/3 \right\} d_w x.$$

4. **Generalizations.** A formal solution of (1.1) is

$$y(t) = e^{mt+nx(t)} \left\{ \int_0^t e^{-ms-nx(s)} nv\, d^*x(s) + \int_0^t e^{-ms-nx(s)} mv\, ds \right\}. \tag{4.1}$$

That (4.1) is the same as (1.2) may be seen either from Theorem 1 or its corollary. Stochastic differential equations of more general type than (0.1) may be solved by using standard integration techniques. For example, if the total differential equation $Pdy + Qdt + Rdx = 0$ is complete, then the transformation $x(t) \to y(t)$ is implicit. On the other hand (1.4) suggests a new type of "differential" equation.

The theory of §3 does not apply to the equation

$$dy(t) = h(t)dt + ny(t)dx(t) \tag{4.2}$$

since $\sigma(0, y(0)) = ny(0) = 0$. A solution of (4.2) is

$$y(t) = e^{nx(t)} \int_0^t h(s) e^{-nx(s)} ds.$$

In this case $y'(0) = h(0)$ with probability one if h is continuous.

In §3 the transformation R is factored into $T_i S_i$ on K_i. In place of S_i one might wish to use

$$S_i': v(t) = \int_0^t d^* y(s)/\sigma(s, y(s)) = V_i S_i y,$$

where V_i is the transformation

$$V_i: v(t) = z(t) + \int_0^t \{ k_i(s)/\sigma(s, k_i(s)) - G_1(s, H(s, z(s), k_i(s)), k_i(s)) \} ds,$$

which is of Volterra type and consequently one-to-one. Similarly, if σ is a functional $\sigma(y|t)$, such that the transformation

$$S_i': v(t) = \int_0^t d^* y(s)/\sigma(y|s)$$

is one-to-one, open, and continuous, then the more general problem could be treated. An interesting formula in this connection may be found in §8 of [3].

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