# $p$-VALENT CLOSE-TO-CONVEX FUNCTIONS

BY
A. E. LIVINGSTON[1]

1. **Introduction.** Let $S(p)$ denote the class of functions, which are regular and $p$-valently star-like in $|z| < 1$. A function

$$f(z) = a_1 z + a_2 z^2 + \cdots \qquad (|z| < 1)$$

is a member of $S(p)$, if there exists a positive number $\rho$ such that for $\rho < |z| < 1$

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0 \tag{1.1}$$

and

$$\int_0^{2\pi} \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] d\theta = 2p\pi. \tag{1.2}$$

The class $S(p)$ has been studied previously by Goodman [4], Robertson [9] and others. Goodman [4] has shown that a function in $S(p)$ is $p$-valent and has exactly $p$ roots in $|z| < 1$.

Goodman [4] also defined the class of $p$-valent convex functions, which we will refer to as $C(p)$. A function

$$f(z) = a_1 z + a_2 z^2 + \cdots \qquad (|z| < 1)$$

is said to be in $C(p)$, if there exists a $\rho$ such that for $\rho < |z| < 1$

$$1 + \operatorname{Re}\left[\frac{zf''(z)}{f'(z)}\right] > 0 \tag{1.3}$$

and

$$\int_0^{2\pi} \left[1 + \operatorname{Re}\frac{zf''(z)}{f'(z)}\right] d\theta = 2p\pi. \tag{1.4}$$

A function in $C(p)$ is at most $p$-valent and has $(p-1)$ critical points in $|z| < 1$. $S(p)$ and $C(p)$ are related to each other in the same way as $S(1)$ and $C(1)$. Namely, $f(z)$ is in $C(p)$ if and only if $zf'(z)$ is in $S(p)$.

Kaplan [5] defined the class of close-to-convex functions. A function $F(z)$,

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regular for $|z| < 1$, with $F(0) = 0$ and $F'(0) \neq 0$ is said to be close-to-convex if there exists $\phi(z)$ in $C(1)$ such that

$$\operatorname{Re}\left[\frac{F'(z)}{\phi'(z)}\right] > 0 \qquad (|z| < 1).$$

Notice that we may rewrite the last inequality to read

$$\operatorname{Re}\left[\frac{zF'(z)}{f(z)}\right] > 0 \qquad (|z| < 1)$$

for some function $f(z)$ in $S(1)$.

Umezawa [13] extended this definition to the case of $p$-valent functions. According to Umezawa, a function

$$F(z) = z^q + a_{q+1}z^{q+1} + \cdots \qquad (|z| < 1)$$

is $p$-valently close-to-convex, if there exists

$$\phi(z) = z^q + b_{q+1}z^{q+1} + \cdots \qquad (|z| < 1)$$

in $C(p)$ such that

$$\operatorname{Re}\left[\frac{F'(z)}{\phi'(z)}\right] > 0 \qquad (|z| < 1). \tag{1.5}$$

It is known that a function in this class is at most $p$-valent in $|z| < 1$ [13]. However, Umezawa's definition requires that the zeros of $F'(z)$ and $\phi'(z)$ have the same positions and multiplicities. We will redefine the concept of a close-to-convex function by requiring that (1.5) should hold only in some range $\rho < |z| < 1$. Furthermore, we will not require that our functions be normalized.

**Definition.** We shall say that a function

$$F(z) = a_1z + a_2z^2 + \cdots \qquad (|z| < 1),$$

regular for $|z| < 1$, is $p$-valently close-to-convex, or is in $\mathscr{K}(p)$, if it satisfies one of the following conditions.

(A) There exists a function $f(z)$ in $S(p)$ and a positive number $\rho$ such that

$$\operatorname{Re}\left[\frac{zF'(z)}{f(z)}\right] > 0 \qquad (\rho < |z| < 1). \tag{1.6}$$

(B) $F(z)$ is regular on $|z| = 1$ and there exists a function $f(z)$ in $S(p)$, also regular on $|z| = 1$, such that (1.6) holds on $|z| = 1$.

Notice that if $F(z)$ satisfies (A), then there exists a $\delta$ such that $G(z) = F(\beta z)$ satisfies (B) for $\delta < \beta < 1$.

If $F(z)$ is in $S(p)$, then taking $f(z) = F(z)$, we see that $F(z)$ is in $\mathscr{K}(p)$. Also, if $F(z)$ is in $C(p)$, then taking $f(z) = zF'(z)$, we see that $F(z)$ is in $\mathscr{K}(p)$.

In §2 we will show that a function in $\mathscr{K}(p)$ is at most $p$-valent in $|z| < 1$. We are also able to obtain sufficient conditions for a function $F(z)$ to be in $\mathscr{K}(p)$, provided $F(z)$ is regular on $|z| = 1$. If $F(z)$ has $p$ zeros at the origin, then we are able to remove the condition of regularity on $|z| = 1$.

Considerable interest has been shown in the coefficient problem for functions, which are at most $p$-valent in $|z| < 1$. Goodman [3] has conjectured that if

$$F(z) = a_1 z + a_2 z^2 + \cdots \qquad (|z| < 1)$$

is regular and at most $p$-valent in $|z| < 1$, then

$$|a_n| < \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)} |a_k|$$

for $n > p$.

The conjecture was proven by Goodman and Robertson [2] for a function in $S(p)$, in case all its coefficients are real and by Robertson [9] for $F(z)$ in $S(p)$, in case $a_1 = a_2 = \cdots = a_{p-2} = 0$, the remaining coefficients being complex. In §3 we will prove the conjecture for the $(p+1)$st coefficient of an arbitrary function in $\mathscr{K}(p)$. This is the largest class of $p$-valent functions for which the exact bound on the $(p+1)$st coefficient is known. We also obtain some sharp upper and lower bounds on $|F'(z)|$ for $F(z)$ in $\mathscr{K}(p)$.

§4 deals with the radii of close-to-convexity and convexity for a function in $\mathscr{K}(p)$. If

$$F(z) = a_q z^q + a_{q+1} z^{q+1} + \cdots \qquad (|z| < 1)$$

is in $\mathscr{K}(p)$, then we obtain a $r_q < 1$ such that $F(z)$ is $q$-valently close-to-convex in $|z| < r_q$ and $\beta_q < 1$ such that $F(z)$ is $q$-valently convex in $|z| < \beta_q$. The numbers $r_q$ and $\beta_q$ depend upon the nonzero critical points of $F(z)$. We are able to show that the number $\beta_q$ gives us the best possible result. However, we are not able to show this for the number $r_q$.

**2. The class $\mathscr{K}(p)$.** We will make use of the following lemma due to Umezawa [12].

LEMMA 1. *Let $f(z)$ be regular for $|z| \leqq r$ and $f'(z) \neq 0$ on $|z| = r$. Suppose that for $z = re^{i\theta}$*

$$\int_0^{2\pi} d \arg df(z) = \int_0^{2\pi} \frac{\partial}{\partial\theta}[\arg zf'(z)]\, d\theta = \int_0^{2\pi} \mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] d\theta = 2p\pi(^2).$$

*If, furthermore,*

(2) Geometrically this says that the angle that the tangent to the image of $|z| = r$ makes with the positive $x$-axis goes through a change of $2p\pi$ as $z$ traverses $|z| = r$. In other words, the image of $|z| = r$, under $w = f(z)$, makes $p$-loops.

$$\int_{\theta_1}^{\theta_2} d\arg df(z) = \int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta}[\arg zf'(z)]\,d\theta > -\pi \quad \textit{for } \theta_1 < \theta_2,$$

*then $f(z)$ is at most p-valent in $|z| < r$.*

THEOREM 1. *If $F(z)$ is in $\mathscr{K}(p)$, then $F(z)$ is at most p-valent in $|z| < 1$.*

**Proof.** There exists $f(z)$ in $S(p)$ and $\rho < 1$ such that

$$\mathrm{Re}\left[\frac{zF'(z)}{f(z)}\right] > 0 \qquad (\rho < |z| < 1). \tag{2.1}$$

Since $zF'(z)/f(z) \neq 0$ and $zF'(z) \neq 0$ for $|z| = r$ $(\rho < r < 1)$, we may define $\arg[zF'(z)/f(z)]$ and $\arg[zF'(z)]$ to be single-valued and continuous on $|z| = r$. Since $f(z) = [f(z)/zF'(z)][zF'(z)]$, then $\arg f(z) = \arg[zF'(z)] - \arg[zF'(z)/f(z)]$ will be uniquely determined and by (2.1) we have for $z = re^{i\theta}$ $(\rho < r < 1)$,

$$-\frac{\pi}{2} < \arg zF'(z) - \arg f(z) < \frac{\pi}{2}.$$

Let $\theta_1 < \theta_2$, then

$$-\frac{\pi}{2} < \arg re^{i\theta_2}F'(re^{i\theta_2}) - \arg f(re^{i\theta_2}) < \frac{\pi}{2} \tag{2.2}$$

and

$$-\frac{\pi}{2} < -\arg re^{i\theta_1}F'(re^{i\theta_1}) + \arg f(re^{i\theta_1}) < \frac{\pi}{2}. \tag{2.3}$$

Combining (2.2) and (2.3), we obtain

$$\begin{aligned} &-\pi + \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \\ &\quad < \arg[re^{i\theta_2}F'(re^{i\theta_2})] - \arg[re^{i\theta_1}F'(re^{i\theta_1})] \\ &\quad < \pi + \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \end{aligned} \tag{2.4}$$

or

$$\begin{aligned} -\pi + \int_{\theta_1}^{\theta_2} d\arg f(re^{i\theta}) &< \int_{\theta_1}^{\theta_2} d\arg dF(re^{i\theta}) \\ &< \pi + \int_{\theta_1}^{\theta_2} d\arg f(re^{i\theta}). \end{aligned} \tag{2.5}$$

Since $f(z)$ is in $S(p)$,

$$\int_{\theta_1}^{\theta_2} d\arg f(re^{i\theta}) > 0.$$

Thus the left side of (2.5) gives

(2.6) $$\int_{\theta_1}^{\theta_2} d\arg dF(re^{i\theta}) > -\pi.$$

Taking $\theta_1 = 0$ and $\theta_2 = 2\pi$ in (2.5) and using the fact that

$$\int_0^{2\pi} d\arg f(re^{i\theta}) = 2p\pi$$

we obtain

(2.7) $$(2p-1)\pi < \int_0^{2\pi} d\arg dF(re^{i\theta}) < (2p+1)\pi.$$

However, the integral in (2.7) is an integral multiple of $2\pi$. Therefore,

(2.8) $$\int_0^{2\pi} d\arg dF(re^{i\theta}) = 2p\pi.$$

Thus, by Lemma 1, $F(z)$ is at most $p$-valent for $|z| < r$. Since $r$ was arbitrary ($\rho < r < 1$), $F(z)$ is at most $p$-valent for $|z| < 1$.

Since (2.8) holds for any function in $\mathscr{K}(p)$ for some range $\rho < |z| < 1$, we easily obtain the following corollary.

COROLLARY. *If $F(z)$ is in $\mathscr{K}(p)$, then $F'(z)$ has exactly $(p-1)$ zeros in $|z| < 1$.*

Necessary and sufficient conditions for a function $F(z)$, regular in $|z| < 1$, with $F(0) = 0$ and $F'(z) \neq 0$ to be in $\mathscr{K}(1)$ have been given by Kaplan [5]. We see from the proof of Theorem 1 that necessary conditions for $F(z)$ to be in $\mathscr{K}(p)$ are that (2.6) and (2.8) hold in some range $\rho < |z| < 1$. We will now show these conditions to be sufficient in two particular cases. The method of proof used is that established by Kaplan [5].

LEMMA 2. *Let*

$$F(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots$$

*be regular for $|z| \leqq 1$. If*

(2.9) $$\int_0^{2\pi} d\arg dF(z) = 2p\pi$$

*and*

(2.10) $$\int_{\theta_1}^{\theta_2} d\arg dF(z) > -\pi \qquad (\theta_1 < \theta_2)$$

*for $|z| = 1$, then $F(z)$ is in $\mathscr{K}(p)$.*

REMARK. We will show that there exists a function $f(z)$ in $S(p)$ with all its zeros at the origin, which is regular for $|z| < 1 + \epsilon$ for some $\epsilon > 0$, and such that $\operatorname{Re}[zF'(z)/f(z)] > 0$ for $|z| < 1 + \epsilon$. This is actually more than we need to prove the lemma, but it is needed in the proof of Theorem 3.

**Proof.** Since $F(z)$ is regular on $|z| = 1$, it is regular in some circle containing $|z| \leqq 1$. By continuity we then have the existence of some $\epsilon > 0$ such that (2.9) and (2.10) hold for $1 \leqq |z| \leqq (1 + \epsilon)$. Now, the function $z^{(1-p)}F'(z)$ is free of zeros in $|z| \leqq (1 + \epsilon)$. Hence, we may define $\arg z^{(1-p)}F'(z)$ to be single-valued and continuous in $|z| \leqq 1 + \epsilon$.

Let

$$p(r,\theta) = \arg[(re^{i\theta})^{(1-p)}F'(re^{i\theta})] \qquad (r \leqq 1 + \epsilon)$$

and

$$P(r,\theta) = p(r,\theta) + p\theta.$$

Then, since (2.9) and (2.10) hold for $|z| = 1 + \epsilon$, we have

$$P(1+\epsilon,\theta+2\pi) - P(1+\epsilon,\theta) = 2p\pi,$$

$$P(1+\epsilon,\theta_2) - P(1+\epsilon,\theta_1) > -\pi \quad \text{for } \theta_1 < \theta_2.$$

Using an argument identical to Kaplan's [5], we may show the existence of a function $S(1+\epsilon,\theta)$, which is increasing in $\theta$ and such that

$$S(1+\epsilon,\theta+2\pi) - S(1+\epsilon,\theta) = 2p\pi \tag{2.11}$$

and

$$|S(1+\epsilon,\theta) - P(1+\epsilon,\theta)| \leqq \frac{\pi}{2}. \tag{2.12}$$

Let

$$q(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{[(1+\epsilon)^2 - r^2][S(1+\epsilon,\alpha) - p\alpha]d\alpha}{(1+\epsilon)^2 + r^2 - 2(1+\epsilon)r\cos(\alpha-\theta)}. \tag{2.13}$$

Then, $q(r,\theta)$ is harmonic for $r < 1 + \epsilon$.

Let $Q(r,\theta) = q(r,\theta) + p\theta$ for $r < 1 + \epsilon$. Using the fact that $S(1+\epsilon,\alpha) - p\alpha$ has period $2\pi$, we obtain for $r < 1 + \epsilon$ and $\theta_1 < \theta_2$,

$$Q(r,\theta_2) - Q(r,\theta_1)$$
$$= \int_0^{2\pi} \frac{[(1+\epsilon)^2 - r^2][S(1+\epsilon,\alpha+\theta_2) - S(1+\epsilon,\alpha+\theta_1)]d\alpha}{(1+\epsilon)^2 + r^2 - 2(1+\epsilon)r\cos\alpha}.$$

Since $S(1+\epsilon,\alpha)$ is increasing

$$Q(r,\theta_2) - Q(r,\theta_1) \geqq 0.$$

Thus $(\partial/\partial\theta)\,Q(r,\theta) \geqq 0$ for $r < 1 + \epsilon$.

Let $h(z)$ be a function, regular for $|z| < 1 + \epsilon$, such that $\operatorname{Im}[h(re^{i\theta})] = q(r,\theta)$ and let

$$f(z) = z^p e^{h(z)} = b_p z^p + \cdots \qquad (|z| < 1 + \epsilon).$$

For $|z| < 1 + \epsilon$,

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] = \frac{\partial}{\partial\theta} \arg f(z) = \frac{\partial}{\partial\theta}(p\theta + q(r,\theta)) = \frac{\partial}{\partial\theta} Q(r,\theta) \geqq 0.$$

But $zf'(z)/f(z)$ is regular for $|z| < 1 + \epsilon$. Thus,

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0 \quad \text{for } |z| < 1 + \epsilon.$$

Since $f(z)$ has $p$ zeros, all of them at the origin,

$$\int_0^{2\pi} \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] d\theta = 2p\pi \qquad (|z| < 1 + \epsilon).$$

Hence, $f(z)$ is $p$-valently star-like for $|z| < 1 + \epsilon$.

Now, for $z = re^{i\theta}$, $r < 1 + \epsilon$, we have

$$\begin{aligned} \left|\arg \frac{zF'(z)}{f(z)}\right| &= |\arg zF'(z) - \arg f(z)| \\ &= |P(r,\theta) - q(r,\theta) - p\theta| \\ &= |p(r,\theta) - q(r,\theta)|. \end{aligned}$$

Since $p(r,\theta)$ is harmonic for $|z| < 1 + \epsilon$, we may write

$$p(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{[(1+\epsilon)^2 - r^2] p(1+\epsilon,\alpha)}{(1+\epsilon)^2 + r^2 - 2(1+\epsilon) r\cos(\alpha - \theta)} d\alpha. \tag{2.14}$$

Then, using (2.12), (2.13) and (2.14), we obtain

$$\begin{aligned} \left|\arg \frac{zF'(z)}{f(z)}\right| &= |p(r,\theta) - q(r,\theta)| \\ &= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{[(1+\epsilon)^2 - r^2][P(1+\epsilon,\alpha) - S(1+\epsilon,\alpha)]d\alpha}{(1+\epsilon)^2 + r^2 - 2(1+\epsilon) r\cos(\alpha-\theta)}\right| \\ &\leqq \frac{\pi}{2}. \end{aligned}$$

Thus $\operatorname{Re}[zF'(z)/f(z)] \geqq 0$ for $|z| < 1 + \epsilon$. Hence, either $\operatorname{Re}[zF'(z)/f(z)] > 0$ for $|z| < 1 + \epsilon$, in which case $F(z)$ is in $\mathscr{K}(p)$, or $zF'(z)/f(z)$ reduces to a constant for $|z| < 1 + \epsilon$. In the second case $F(z)$ is in $C(p) \subset \mathscr{K}(p)$.

THEOREM 2. *Let*

$$F(z) = a_p z^p + a_{p+1} z^{p+1}, \cdots \qquad (|z| < 1)$$

*be regular for* $|z| < 1$. *If* (2.9) *and* (2.10) *hold for some range* $\rho < |z| < 1$, *then* $F(z)$ *is in* $\mathscr{K}(p)$.

**Proof.** Let $\rho < \delta < 1$. Then the function $G_\delta(z) = F(\delta z)$ is regular on $|z| = 1$

and satisfies (2.9) and (2.10) on $|z| = 1$. Hence, by Lemma 2, $G_\delta(z)$ is in $\mathscr{K}(p)$ and there exists

$$f_\delta(z) = b_p z^p + \cdots \qquad (|z| < 1)$$

in $S(p)$ such that

$$\text{(2.15)} \qquad \operatorname{Re}\left[\frac{zG'_\delta(z)}{f_\delta(z)}\right] > 0 \qquad (|z| < 1).$$

We may assume that $|b_p| = 1$. Cartwright [1] has shown that the family of $p$-valent functions with the moduli of the first $p$ coefficients fixed is a normal family. Thus we may choose a sequence $\delta_n$ tending to 1, such that the sequence of functions $f_{\delta_n}(z)$ tends to a function $f(z)$ in $S(p)$. Since $zG'_{\delta_n}(z)$ tends to $zF'(z)$, we obtain from (2.15) that

$$\operatorname{Re}\left[\frac{zF'(z)}{f(z)}\right] \geqq 0 \quad \text{for } |z| < 1.$$

This implies that $F(z)$ is in $\mathscr{K}(p)$.

THEOREM 3. *Let*

$$F(z) = a_q z^q + \cdots \qquad (1 \leqq q \leqq p)$$

*be regular for* $|z| \leqq 1$. *If* (2.9) *and* (2.10) *hold on* $|z| = 1$, *then* $F(z)$ *is in* $\mathscr{K}(p)$.

**Proof.** By condition (2.9) $F'(z)$ has $(p-1)$ zeros in $|z| < 1$, $(q-1)$ of them at the origin. Let $\alpha_1, \alpha_2, \cdots, \alpha_{p-q}$ be the nonzero roots of $F'(z)$ and let

$$G(z) = \int_0^z \frac{z^{p-q}F'(z)\,dz}{\prod_{i=1}^{p-q}(z-\alpha_i)(1-\bar{\alpha}_i z)} = d_p z^p + \cdots.$$

$G(z)$ is regular for $|z| \leqq 1$ and

$$zG'(z) = \frac{z^{p-q}zF'(z)}{\prod_{i=1}^{p-q}(z-\alpha_i)(1-\bar{\alpha}_i z)}.$$

Since

$$\arg\left[\frac{z^{p-q}}{\prod_{i=1}^{p-q}(z-\alpha_i)(1-\bar{\alpha}_i z)}\right] = 0 \quad \text{for } |z| = 1,$$

$\arg zG'(z) = \arg zF'(z)$ for $|z| = 1$.

Thus, $G(z)$ satisfies (2.9) and (2.10) on $|z| = 1$. Hence, by Lemma 2, $G(z)$ is in $\mathscr{K}(p)$ and there exists $f(z)$ in $S(p)$, regular for $|z| \leqq 1$, such that

$$\operatorname{Re}\left[\frac{zG'(z)}{f(z)}\right] > 0 \qquad (|z| \leqq 1).$$

But using the same reasoning as above, we have

$$\arg\left[\frac{zG'(z)}{f(z)}\right] = \arg\left[\frac{zF'(z)}{f(z)}\right] \quad \text{on } |z| = 1.$$

Hence,

$$\operatorname{Re}\left[\frac{zF'(z)}{f(z)}\right] > 0 \quad \text{for } |z| = 1.$$

Thus, $F(z)$ is in $\mathscr{K}(p)$.

Theorem 3 immediately gives us the following lemma, which will prove useful in obtaining a bound for the $(p+1)$st coefficient of a function in $\mathscr{K}(p)$.

**Lemma** 3. *If $F(z)$ is regular in $|z| \leqq 1$ and in $\mathscr{K}(p)$, then there exists*

$$f(z) = b_p z^p + \cdots \qquad (|b_p| = 1)$$

*regular and in $S(p)$ for $|z| \leqq 1$, such that*

$$\operatorname{Re}\left[\frac{zF'(z)}{f(z)}\right] > 0 \quad \text{on } |z| = 1.$$

3. **Some extremal problems for the class** $\mathscr{K}(p)$. The following lemma has been proven by Royster [11]. However, the proof we give, which was communicated to me by Professor M. S. Robertson, seems to be different.

**Lemma** 4. *Let $f(z) = [h(z)]^{-p}$, where $h(z)$ is in $S(1)$, $h(0) = 0$, $h'(0) = 1$ and let*

$$f(z) = \sum_{n=-p}^{\infty} C_n z^n \qquad (0 < |z| < 1,\ C_{-p} = 1),$$

*then*

$$|C_n| \leqq \binom{2p}{n+p} \qquad (n = -p, \cdots, 1),$$

*and these inequalities are sharp.*

**Proof.** We write

$$z^p f(z) = z^p[h(z)]^{-p} = \sum_{n=0}^{\infty} d_n z^n \qquad (|z| < 1,\ d_0 = 1). \tag{3.1}$$

The lemma will then be proven, if we can show

$$|d_n| \leqq \binom{2p}{n} \qquad (n \leqq p+1).$$

Taking the logarithm of both sides of (3.1), differentiating and multiplying through by $z$, we obtain

$$-\frac{zf'(z)}{pf(z)} = \frac{zh'(z)}{h(z)}.$$

Thus, we have for $|z| < 1$

$$\text{(3.2)} \qquad \operatorname{Re}\left[-\frac{zf'(z)}{pf(z)}\right] = \operatorname{Re}\left[\frac{zh'(z)}{h(z)}\right] > 0 \qquad (|z| < 1).$$

Let

$$P(z) = -\frac{zf'(z)}{pf(z)},$$

then

$$\operatorname{Re}\left[\frac{1}{P(z)}\right] > 0 \qquad \text{for } |z| < 1.$$

Let

$$\frac{1}{P(z)} = 1 + \sum_{n=1}^{\infty} \mu_n z^n,$$

$$\frac{1}{P(z)} = -\frac{pf(z)}{zf'(z)} = -\frac{pz^p f(z)}{z^{p+1}f'(z)},$$

$$-\frac{1}{P(z)} z^{p+1}f'(z) = pz^p f(z),$$

or

$$\left[-\sum_{m=0}^{\infty} \mu_m z^m\right]\left[\sum_{s=0}^{\infty} (s-p) d_s z^s\right] = p\sum_{n=0}^{\infty} d_n z^n.$$

Equating coefficients, we obtain

$$pd_n = \sum_{r=0}^{n} (p-r) d_r \mu_{n-r},$$

$$nd_n = \sum_{r=0}^{n-1} (p-r) d_r \mu_{n-r}.$$

Since $|\mu_{n-r}| \leqq 2$, we obtain

$$\text{(3.3)} \qquad n|d_n| \leqq 2\sum_{r=0}^{n-1} (p-r)|d_r|$$

provided $p - r \geqq 0$. That is, provided $n \leqq p + 1$. Using (3.3) and a simple induction argument, we have

$$|d_n| \leqq \binom{2p}{n} \quad \text{for } n \leqq p + 1.$$

That the inequalities are sharp is shown by the function

$$f(z) = \left[\frac{z}{(1+z)^2}\right]^{-p}.$$

THEOREM 4. *Let*

$$F(z) = \sum_{n=1}^{\infty} a_n z^n \qquad (|z| < 1)$$

*be regular and in* $\mathscr{K}(p)$ *for* $|z| < 1$, *then*

$$|a_{p+1}| \leqq \sum_{k=1}^{p} \frac{2k(2p+1)!}{(p+k)!(p-k)![(p+1)^2 - k^2]} |a_k| \tag{3.4}$$

*and this inequality is sharp in all the variables* $|a_1|, \cdots, |a_p|$.

REMARK. This theorem was first proven for $p = 1$ be Reade [8].

**Proof.** We may assume without loss of generality that $F(z)$ is regular for $|z| \leqq 1$. Then, by Lemma 3 there exists a function

$$f(z) = b_p z^p + \cdots \qquad (|b_p| = 1),$$

regular for $|z| \leqq 1$ and in $S(p)$, such that

$$\text{Re}\left[\frac{zF'(z)}{f(z)}\right] > 0 \qquad (|z| = 1). \tag{3.5}$$

We may assume that $b_p = 1$ since $\arg[b_p]$ is not involved in the inequality to be obtained. Thus we may write $f(z)$ in the form $[\phi(z)]^p$, where

$$\phi(z) = z + \sum_{n=2}^{\infty} h_n z^n$$

is regular for $|z| < 1$ and in $S(1)$.

We may then write (3.5) in the form

$$\text{Re}[zF'(z)[\phi(z)]^{-p}] > 0 \quad \text{on } |z| = 1.$$

Let

$$[\phi(z)]^{-p} = \sum_{n=-p}^{\infty} C_n z^n \qquad (0 < |z| < 1, \ C_{-p} = 1).$$

Then

$$zF'(z)[\phi(z)]^{-p} = \left[\sum_{n=1}^{\infty} na_n z^n\right]\left[\sum_{n=-p}^{\infty} C_n z^n\right]$$

$$= \sum_{k=-(p-1)}^{\infty} d_k z^k,$$

where

$$d_k = \sum_{n=1}^{p+k} C_{-(n-k)} n a_n \qquad (k = -(p-1), \cdots).$$

Consider the function $G(z)$ given by

$$G(z) = zF'(z)[\phi(z)]^{-p} - \sum_{k=-(p-1)}^{-1} d_k z^k + \sum_{k=-(p-1)}^{-1} \bar{d}_k z^{-k}. \tag{3.6}$$

Since $\bar{z} = z^{-1}$ for $|z| = 1$, the last two terms in (3.6) add up to a purely imaginary number for $|z| = 1$. Thus,

$$\operatorname{Re}[G(z)] = \operatorname{Re}[zF'(z)[\phi(z)]^{-p}] > 0 \quad \text{for } |z| = 1.$$

But $G(z)$ is regular for $|z| \leqq 1$. Therefore,

$$\operatorname{Re}[G(z)] > 0 \quad \text{for } |z| \leqq 1.$$

Now

$$G(z) = d_0 + (d_1 + \bar{d}_{-1})z + \cdots \qquad (|z| \leqq 1).$$

Hence

$$|d_1 + \bar{d}_{-1}| \leqq 2\operatorname{Re}[d_0] \leqq 2|d_0|,$$

$$\left|\sum_{n=1}^{p+1} C_{-(n-1)} n a_n + \sum_{n=1}^{p-1} \bar{C}_{-(n+1)} n \bar{a}_n\right| \leqq 2\left|\sum_{n=1}^{p} C_{-n} n a_n\right|,$$

$$(p+1)|a_{p+1}| \leqq \sum_{n=1}^{p-1} [2n|C_{-n}| + n|C_{-(n-1)}| + n|C_{-(n+1)}|]|a_n| + [2p|C_{-p}| + p|C_{-(p-1)}|]|a_p|.$$

By Lemma 4

$$|C_{-k}| \leqq \binom{2p}{p-k} \qquad (k = 1, 2, \cdots, p).$$

Therefore,

$$(p+1)|a_{p+1}| \leqq \sum_{n=1}^{p-1}\left[2n\binom{2p}{p-n}+n\binom{2p}{p-n+1}+n\binom{2p}{p-n-1}\right]|a_n|$$

$$+\left[2p+p\binom{2p}{1}\right]|a_p|$$

$$=(p+1)\sum_{n=1}^{p}\frac{2n(2p+1)!}{(p+n)!(p-n)![(p+1)^2-n^2]}|a_n|$$

which is (3.4).

We remark that the inequality is sharp, since it is known to be sharp for $f(z)$ in $S(p)$ with real coefficients [2], [4].

In order to obtain bounds on $|F'(z)|$ for $F(z)$ in $\mathscr{K}(p)$, we will make use of the following lemma.

Lemma 5. *Let*

$$F(z) = a_q z^q + \cdots \qquad (|z| \leqq 1)$$

*be regular and in* $\mathscr{K}(p)$ *for* $|z| \leqq 1$. *Let* $\alpha_1, \alpha_2, \cdots, \alpha_{p-q}$ *be the nonzero critical points of* $F'(z)$ *in* $|z| < 1$. *Then the function*

$$H(z) = \int_0^z z^{p-q}F'(z)\left[\prod_{i=1}^{p-q}\left(\frac{\alpha_i}{|\alpha_i|}-\frac{z}{|\alpha_i|}\right)(\bar{\alpha}_i z-1)\right]^{-1}dz$$

*is regular for* $|z| \leqq 1$ *and in* $\mathscr{K}(p)$.

**Proof.** By Lemma 3, there exists

$$h(z) = b_p z^p + \cdots \qquad (|b_p| = 1),$$

regular and in $S(p)$ for $|z| \leqq 1$, such that

$$\operatorname{Re}\frac{zF'(z)}{h(z)} > 0 \quad \text{for } |z| = 1.$$

$$\frac{zH'(z)}{h(z)} = \frac{z^{p-q}zF'(z)\left[\prod_{i=1}^{p-q}\left(\frac{\alpha_i}{|\alpha_i|}-\frac{z}{|\alpha_i|}\right)(\bar{\alpha}_i z-1)\right]^{-1}}{h(z)}.$$

But,

$$\arg\left(z^{p-q}\left[\prod_{i=1}^{p-q}\left(\frac{\alpha_i}{|\alpha_i|}-\frac{z}{|\alpha_i|}\right)(\bar{\alpha}_i z-1)\right]^{-1}\right) = 0 \quad \text{on } |z| = 1.$$

Thus,

$$\frac{zH'(z)}{h(z)} = M\frac{zF'(z)}{h(z)}, \qquad M > 0 \quad \text{on } |z| = 1.$$

Hence,

$$\operatorname{Re}\left[\frac{zH'(z)}{h(z)}\right] > 0 \quad \text{for } |z| = 1.$$

Therefore, $H(z)$ is in $\mathscr{K}(p)$.

THEOREM 5. *Let*

$$F(z) = a_q z^q + \cdots \qquad (|z| < 1),$$

*be regular and in* $\mathscr{K}(p)$ *for* $|z| < 1$. *Let* $\alpha_1, \alpha_2, \cdots, \alpha_{p-q}$ *be the nonzero critical points of* $F(z)$ *and let* $\rho = \max|\alpha_i|$ *and* $\rho^* = \min|\alpha_i|$. *Then*

$$(3.7)\quad |F'(re^{i\theta})| \leqq \frac{(1+r)r^{q-1}}{(1-r)^{2p+1}} q|a_q| \left[\prod_{i=1}^{p-q}\left(1+\frac{r}{|\alpha_i|}\right)(1+r|\alpha_i|)\right] \qquad (r<1),$$

$$(3.8)\quad |F'(re^{i\theta})| \geqq \frac{(1-r)r^{q-1}}{(1+r)^{2p+1}} q|a_q| \left[\prod_{i=1}^{p-q}\left(\frac{r}{|\alpha_i|}-1\right)(1-r|\alpha_i|)\right] \qquad (\rho<r<1),$$

$$(3.9)\quad |F'(re^{i\theta})| \geqq \frac{(1-r)r^{q-1}}{(1+r)^{2p+1}} q|a_q| \left[\prod_{i=1}^{p-q}\left(1-\frac{r}{|\alpha_i|}\right)(1-r|\alpha_i|)\right] \qquad (r<\rho^*).$$

*All these inequalities are sharp, equality being attained by the function*

$$F_0(z) = \int_0^z \frac{(1+z)z^{q-1}}{(1-z)^{2p+1}} q|a_q| \prod_{i=1}^{p-q}\left(1+\frac{z}{|\alpha_i|}\right)(1+z|\alpha_i|)\,dz.$$

Note that inequality (3.7) was obtained by Umezawa [13] for his class of $p$-valent close-to-convex functions.

**Proof.** We may assume without loss of generality that $F(z)$ is regular for $|z| \leqq 1$. Consider the functions $H(z)$ and $h(z)$, given in Lemma 5 and in its proof.

$$\frac{zH'(z)}{h(z)} = d_0 + d_1 z + \cdots \qquad (|z| \leqq 1),$$

where

$$d_0 = \frac{qa_q}{b_p}\left[\prod_{i=1}^{p-q}(-e^{i\arg\alpha_i})\right]^{-1}.$$

Then

$$\frac{1}{\operatorname{Re}[d_0]}\left[\frac{zH'(z)}{h(z)} - i\operatorname{Im}[d_0]\right] = P(z),$$

where $\operatorname{Re} P(z) > 0$ for $|z| < 1$ and $P(0) = 1$. Thus,

$$\left| \frac{P(z)-1}{P(z)+1} \right| \leqq |z|.$$

Hence

$$\left| \frac{\dfrac{zH'(z)}{h(z)} - d_0}{\dfrac{zH'(z)}{h(z)} + \bar{d}_0} \right| \leqq |z| = r,$$

$$(1-r) \left| \frac{zH'(z)}{h(z)} \right| \leqq (1+r)|d_0| = (1+r)q|a_q|.$$

Using the known bound

$$|h(z)| \leqq \frac{r^p}{(1-r)^{2p}} \quad \text{for } |z| = r$$

and using the definition of $H(z)$, we obtain

$$|F'(re^{i\theta})| \leqq \frac{(1+r)}{(1-r)r^{p-q+1}} q|a_q|\,|h(z)| \left| \prod_{i=1}^{p-q} \left( \frac{\alpha_i}{|\alpha_i|} - \frac{z}{|\alpha_i|} \right) (\bar{\alpha}_i z - 1) \right|$$

$$\leqq \frac{(1+r)r^{q-1}}{(1-r)^{2p+1}} q|a_q| \prod_{i=1}^{p-q} \left( 1 + \frac{r}{|\alpha_i|} \right) (1 + r|\alpha_i|),$$

which is (3.7).

To obtain (3.8) and (3.9), we notice that for $z = re^{i\theta}$

$$\left| \frac{P(z)+1}{P(z)-1} \right| \geqq \frac{1}{r},$$

$$|h(z)| \geqq \frac{r^p}{(1+r)^{2p}},$$

$$\left| \frac{\alpha_i}{|\alpha_i|} - \frac{z}{|\alpha_i|} \right| \, |\bar{\alpha}_i z - 1| \geqq \left( \frac{r}{|\alpha_i|} - 1 \right) (1 - r|\alpha_i|) \qquad (|\alpha_i| < r),$$

and

$$\left| \frac{\alpha_i}{|\alpha_i|} - \frac{z}{|\alpha_i|} \right| |\bar{\alpha}_i z - 1| \geqq \left( 1 - \frac{r}{|\alpha_i|} \right) (1 - r|\alpha_i|) \qquad (r < |\alpha_i|).$$

Going through the same type of argument as before, we obtain (3.8) and (3.9).

The function $F_0(z)$ is in $\mathscr{K}(p)$ relative to

$$f(z) = \frac{z^q}{(1-z)^{2p}} \prod_{i=1}^{p-q} \left( 1 + \frac{z}{|\alpha_i|} \right) (1 + z|\alpha_i|).$$

Equality in (3.7) is attained by $F_0'(r)$, in (3.8) by $F_0'(-r)$, $r>\rho$, and in (3.9) by $F_0'(-r)$, $r<\rho^*$.

**4. Radii of close-to-convexity and convexity for functions in $\mathscr{K}(p)$.** Goodman [4] has proven that if

$$f(z) = a_q z^q + \cdots \qquad (|z|<1)$$

is in $S(p)$, then

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geqq J_q(r) \quad \text{for } r<\rho, \tag{4.1}$$

where

$$J_q(r) = q - r\left[\frac{2p}{1+r} + \sum_{i=1}^{p-q} \frac{1}{|\alpha_i| - r} + \frac{|\alpha_i|}{1-|\alpha_i|r}\right],$$

$\alpha_1, \cdots, \alpha_{p-q}$ being the nonzero roots of $f(z)$ and $\rho = \min|\alpha_i|$. $J_q(r)$ is a decreasing function of $r$ for $r<\rho$, is positive for $r=0$ and tends to $-\infty$ as $r$ tends to $\rho$. Thus, $J_q(r)$ has a least positive root $r_q$ and $J_q(r)>0$ for $r<r_q$.

We thus have that $f(z)$ is $q$-valently star-like for $|z|<r_q$. This estimate is sharp, since (4.1) was shown to be sharp [4], equality being attained at $z=-r$ by the function

$$f(z) = z^q(1-z)^{-2p}\prod_{i=1}^{p-q}\left(1+\frac{z}{|\alpha_i|}\right)(1+z|\alpha_i|). \tag{4.2}$$

THEOREM 6. *Let*

$$F(z) = a_q z^q + \cdots \qquad (|z|<1)$$

*be in* $\mathscr{K}(p)$. *Let* $\alpha_1, \cdots, \alpha_{p-q}$ *be the nonzero roots of* $F'(z)$ *and let* $r_q$ *be the least positive root of* $J_q(r)$, *defined in* (4.1). *Then* $F(z)$ *is q-valently close-to-convex for* $|z|<r_q$.

**Proof.** We first prove the theorem for $F(z)$, regular on $|z|=1$. Then there exists

$$f(z) = b_p z^p + \cdots \qquad (|z| \leqq 1),$$

regular and in $S(p)$ for $|z|\leqq 1$, such that

$$\operatorname{Re}\left[\frac{zF'(z)}{f(z)}\right] > 0 \quad \text{on } |z|=1.$$

Since

$$\arg\left(z^{p-q}\left[\prod_{i=1}^{p-q}(z-\alpha_i)(1-\bar{\alpha}_i z)\right]^{-1}\right) = 0 \quad \text{on } |z|=1,$$

we have

$$\text{Re}\left[\frac{z^{p-q}zF'(z)}{\prod_{i=1}^{p-q}(z-\alpha_i)(1-\bar{\alpha}_i z)\cdot f(z)}\right]>0 \quad \text{for } |z|\leqq 1.$$

Let

$$g(z)=z^{q-p}\left[\prod_{i=1}^{p-q}(z-\alpha_i)(1-\bar{\alpha}_i z)\right]f(z).$$

Then, $g(z)$ is in $S(p)$ since $\text{Re}[zg'(z)/g(z)]>0$ on $|z|=1$. But $g(z)$ has nonzero roots at $\alpha_1,\alpha_2,\cdots,\alpha_{p-q}$. Therefore, $g(z)$ is $q$-valently star-like for $|z|<r_q$. Since

$$\text{Re}\left[\frac{zF'(z)}{g(z)}\right]>0 \quad \text{for } |z|\leqq r_q,$$

$F(z)$ is $q$-valently close-to-convex for $|z|<r_q$.

If $F(z)$ is not regular on $|z|=1$, there exists a $\rho^*<1$ such that for $\rho^*<\delta<1$ the function $G_\delta(z)=F(\delta z)$ is in $\mathscr{V}(p)$ and regular on $|z|=1$. $G'_\delta(z)=0$ for $z=\alpha_i/\delta$. Thus, $G_\delta(z)$ is $q$-valently close-to-convex for $|z|<r_{q,\delta}$, where $r_{q,\delta}$ is the least positive root of

$$J_{q,\delta}(r)=q-r\left[\frac{2p}{1+r}+\sum_{i=1}^{p-q}\frac{\delta}{|\alpha_i|-r\delta}+\frac{|\alpha_i|}{\delta-|\alpha_i|r}\right].$$

Thus, there exists

$$f_\delta(z)=C_q z^q+\cdots \qquad (|z|<r_{q,\delta},\ |C_q|=1)$$

$q$-valently star-like for $|z|<r_{q,\delta}$, such that

$$\text{Re}\left[\frac{zG'_\delta(z)}{f_\delta(z)}\right]>0 \quad \text{for } |z|<r_{q,\delta}.$$

But $r_{q,\delta}\geqq r_q$, since $J_{q,\delta}(r)\geqq J_q(r)$ for $r<\min|\alpha_i|$. Thus $f_\delta(z)$ is $q$-valently star-like for $|z|<r_q$.

By a result of M. Cartwright [1] the family of $q$-valent functions $f(z)=a_q z^q+\cdots$ ($|a_q|=1$) is a normal family. Thus we may choose an increasing sequence $\delta_i$ tending to 1, such that the functions $f_{\delta_i}(z)$ tend to a function $f(z)$, which is $q$-valently star-like for $|z|<r_q$. Since for each $i$

$$\text{Re}\left[\frac{zG'_{\delta_i}(z)}{f_{\delta_i}(z)}\right]>0 \quad \text{for } |z|<r_q$$

and since $zG'_{\delta_i}(z)$ tends to $zF'(z)$, we have

$$\text{Re}\left[\frac{zF'(z)}{f(z)}\right]\geqq 0 \quad \text{for } |z|<r_q.$$

Thus either $\operatorname{Re}[zF'(z)/f(z)] > 0$ for $|z| < r_q$, in which case $F(z)$ is $q$-valently close-to-convex for $|z| < r_q$, or $[zF'(z)/f(z)]$ reduces to a constant for $|z| < r_q$. In the second case $F(z)$ is $q$-valently convex and hence $q$-valently close-to-convex for $|z| < r_q$.

THEOREM 7. *Let*

$$F(z) = a_q z^q + \cdots \qquad (|z| < 1),$$

*be in* $\mathscr{K}(p)$, *then* $F(z)$ *is q-valently convex for* $|z| < \beta_q$, *where* $\beta_q$ *is the least positive root of*

$$K_q(r) = J_q(r) - \frac{2r}{1 - r^2}$$

*and this estimate is the best possible.*

**Proof.** Let us first assume that $F(z)$ is regular on $|z| = 1$. Then, as we have seen before, there exists

$$g(z) = b_q z^q + \cdots \qquad (|z| < 1),$$

which is in $S(p)$ for $|z| < 1$, such that

$$\operatorname{Re}\left[\frac{zF'(z)}{g(z)}\right] > 0 \quad \text{for } |z| \leqq 1.$$

Let

$$\frac{zF'(z)}{g(z)} = P(z), \qquad \operatorname{Re}[P(z)] > 0 \qquad \text{for } |z| \leqq 1,$$

$$1 + \frac{zF''(z)}{F'(z)} = \frac{zP'(z)}{P(z)} + \frac{zg'(z)}{g(z)}.$$

Now $g(z)$ has the same zeros as $F'(z)$. Therefore,

$$\operatorname{Re}\left[\frac{zg'(z)}{g(z)}\right] \geqq J_q(r) \quad \text{for } r < \min|\alpha_i|.$$

By a result, obtained independently by Libera [6], MacGregor [7] and Robertson [10], we have

$$\operatorname{Re}\left[\frac{zP'(z)}{P(z)}\right] \geqq -\frac{2r}{1 - r^2}.$$

Thus

$$\operatorname{Re}\left[1 + \frac{zF''(z)}{F'(z)}\right] \geqq -\frac{2r}{1 - r^2} + J_q(r) = K_q(r)$$

for $r < \min|\alpha_i|$.

Thus, if $|z| < \beta_q$

$$\operatorname{Re}\left[1+\frac{zF''(z)}{F'(z)}\right]>0.$$

Since $F'(z)$ has $(q-1)$ zeros in $|z|<\beta_q$, all of them at the origin,

$$\int_0^{2\pi}\operatorname{Re}\left[1+\frac{zF''(z)}{F'(z)}\right]d\theta=2q\pi\qquad(|z|<\beta_q).$$

Thus $F(z)$ is $q$-valently convex for $|z|<\beta_q$.

Arguing as in Theorem 6, we may remove the assumption of regularity on $|z|=1$.

The function

$$F(z)=\int_0^z\frac{(1+z)z^{q-1}}{(1-z)^{2p+1}}\prod_{i=1}^{p-q}\left(1+\frac{z}{|\alpha_i|}\right)(1+z|\alpha_i|)\,dz$$

shows that the radius found is sharp, since

$$1+\frac{zF''(z)}{F'(z)}=K_q(r)$$

for $z=-r$, $r<\min|\alpha_i|$.

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LAFAYETTE COLLEGE,
EASTON, PENNSYLVANIA