1. Introduction. Let $S(p)$ denote the class of functions, which are regular and $p$-valently star-like in $|z| < 1$. A function

$$f(z) = a_1 z + a_2 z^2 + \cdots \quad (|z| < 1)$$

is a member of $S(p)$, if there exists a positive number $\rho$ such that for $\rho < |z| < 1$

\begin{equation}
\Re \left[ \frac{zf'(z)}{f(z)} \right] > 0
\end{equation}

and

\begin{equation}
\int_0^{2\pi} \Re \left[ \frac{zf'(z)}{f(z)} \right] d\theta = 2p\pi.
\end{equation}

The class $S(p)$ has been studied previously by Goodman [4], Robertson [9] and others. Goodman [4] has shown that a function in $S(p)$ is $p$-valent and has exactly $p$ roots in $|z| < 1$.

Goodman [4] also defined the class of $p$-valent convex functions, which we will refer to as $C(p)$. A function

$$f(z) = a_1 z + a_2 z^2 + \cdots \quad (|z| < 1)$$

is said to be in $C(p)$, if there exists a $\rho$ such that for $\rho < |z| < 1$

\begin{equation}
1 + \Re \left[ \frac{zf''(z)}{f'(z)} \right] > 0
\end{equation}

and

\begin{equation}
\int_0^{2\pi} \left[ 1 + \Re \frac{zf''(z)}{f'(z)} \right] d\theta = 2p\pi.
\end{equation}

A function in $C(p)$ is at most $p$-valent and has $(p - 1)$ critical points in $|z| < 1$. $S(p)$ and $C(p)$ are related to each other in the same way as $S(1)$ and $C(1)$. Namely, $f(z)$ is in $C(p)$ if and only if $zf'(z)$ is in $S(p)$.

Kaplan [5] defined the class of close-to-convex functions. A function $F(z)$,
regular for $|z| < 1$, with $F(0) = 0$ and $F'(0) \neq 0$ is said to be close-to-convex if there exists $\phi(z)$ in $C(1)$ such that

$$\text{Re} \left[ \frac{F'(z)}{\phi'(z)} \right] > 0 \quad (|z| < 1).$$

Notice that we may rewrite the last inequality to read

$$\text{Re} \left[ \frac{zF'(z)}{f(z)} \right] > 0 \quad (|z| < 1)$$

for some function $f(z)$ in $S(1)$.

Umezawa [13] extended this definition to the case of $p$-valent functions. According to Umezawa, a function

$$F(z) = z^q + a_{q+1}z^{q+1} + \cdots \quad (|z| < 1)$$

is $p$-valently close-to-convex, if there exists

$$\phi(z) = z^q + b_{q+1}z^{q+1} + \cdots \quad (|z| < 1)$$

in $C(p)$ such that

(1.5) \quad $$\text{Re} \left[ \frac{F'(z)}{\phi'(z)} \right] > 0 \quad (|z| < 1).$$

It is known that a function in this class is at most $p$-valent in $|z| < 1$ [13].

However, Umezawa’s definition requires that the zeros of $F'(z)$ and $\phi'(z)$ have the same positions and multiplicities. We will redefine the concept of a close-to-convex function by requiring that (1.5) should hold only in some range $\rho < |z| < 1$. Furthermore, we will not require that our functions be normalized.

**Definition.** We shall say that a function

$$F(z) = a_1z + a_2z^2 + \cdots \quad (|z| < 1),$$

regular for $|z| < 1$, is $p$-valently close-to-convex, or is in $\mathcal{K}(p)$, if it satisfies one of the following conditions.

(A) There exists a function $f(z)$ in $S(p)$ and a positive number $\rho$ such that

(1.6) \quad $$\text{Re} \left[ \frac{zF'(z)}{f(z)} \right] > 0 \quad (\rho < |z| < 1).$$

(B) $F(z)$ is regular on $|z| = 1$ and there exists a function $f(z)$ in $S(p)$, also regular on $|z| = 1$, such that (1.6) holds on $|z| = 1$.

Notice that if $F(z)$ satisfies (A), then there exists a $\delta$ such that $G(z) = F(\beta z)$ satisfies (B) for $\delta < \beta < 1$.

If $F(z)$ is in $S(p)$, then taking $f(z) = F(z)$, we see that $F(z)$ is in $\mathcal{K}(p)$. Also, if $F(z)$ is in $C(p)$, then taking $f(z) = zF'(z)$, we see that $F(z)$ is in $\mathcal{K}(p)$. 

In §2 we will show that a function in $S(p)$ is at most $p$-valent in $|z| < 1$. We are also able to obtain sufficient conditions for a function $F(z)$ to be in $S(p)$, provided $F(z)$ is regular on $|z| = 1$: If $F(z)$ has $p$ zeros at the origin, then we are able to remove the condition of regularity on $|z| = 1$.

Considerable interest has been shown in the coefficient problem for functions, which are at most $p$-valent in $|z| < 1$. Goodman [3] has conjectured that if

$$F(z) = a_1 z + a_2 z^2 + \cdots \quad (|z| < 1)$$

is regular and at most $p$-valent in $|z| < 1$, then

$$|a_n| < \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p+1)!(n^2-k^2)} |a_k|$$

for $n > p$.

The conjecture was proven by Goodman and Robertson [2] for a function in $S(p)$, in case all its coefficients are real and by Robertson [9] for $F(z)$ in $S(p)$, in case $a_1 = a_2 = \cdots = a_{p-2} = 0$, the remaining coefficients being complex. In §3 we will prove the conjecture for the $(p+1)$st coefficient of an arbitrary function in $S(p)$. This is the largest class of $p$-valent functions for which the exact bound on the $(p+1)$st coefficient is known. We also obtain some sharp upper and lower bounds on $|F'(z)|$ for $F(z)$ in $S(p)$.

§4 deals with the radii of close-to-convexity and convexity for a function in $S(p)$. If

$$F(z) = a_2 z^2 + a_{p+1} z^{p+1} + \cdots \quad (|z| < 1)$$

is in $S(p)$, then we obtain a $r_q < 1$ such that $F(z)$ is $q$-valently close-to-convex in $|z| < r_q$ and $\beta_q < 1$ such that $F(z)$ is $q$-valently convex in $|z| < \beta_q$. The numbers $r_q$ and $\beta_q$ depend upon the nonzero critical points of $F(z)$. We are able to show that the number $\beta_q$ gives us the best possible result. However, we are not able to show this for the number $r_q$.

2. The class $S(p)$. We will make use of the following lemma due to Umezawa [12].

**Lemma 1.** Let $f(z)$ be regular for $|z| \leq r$ and $f'(z) \neq 0$ on $|z| = r$. Suppose that for $z = re^{i\theta}$

$$\int_0^{2\pi} d\arg df(z) = \int_0^{2\pi} \frac{\partial}{\partial \theta} \left[ \arg zf'(z) \right] d\theta = \int_0^{2\pi} \Re \left[ 1 + \frac{zf''(z)}{f'(z)} \right] d\theta = 2p\pi(\dagger).$$

If, furthermore,

(\dagger) Geometrically this says that the angle that the tangent to the image of $|z| = r$ makes with the positive $x$-axis goes through a change of $2p\pi$ as $z$ traverses $|z| = r$. In other words, the image of $|z| = r$, under $w = f(z)$, makes $p$-loops.
\[ \int_{\theta_1}^{\theta_2} d \arg f(z) = \int_{\theta_1}^{\theta_2} \frac{\partial}{\partial \theta} [\arg zf'(z)] d\theta > -\pi \quad \text{for } \theta_1 < \theta_2, \]

then \( f(z) \) is at most \( p \)-valent in \( |z| < r \).

**Theorem 1.** If \( F(z) \) is in \( \mathcal{H}(p) \), then \( F(z) \) is at most \( p \)-valent in \( |z| < 1 \).

**Proof.** There exists \( f(z) \) in \( S(p) \) and \( \rho < 1 \) such that

\[ (2.1) \quad \Re \left[ \frac{zF'(z)}{f(z)} \right] > 0 \quad (\rho < |z| < 1). \]

Since \( zF'(z)/f(z) \neq 0 \) and \( zF'(z) \neq 0 \) for \( |z| = r \) (\( \rho < r < 1 \)), we may define \( \arg [zF'(z)/f(z)] \) and \( \arg [zF'(z)] \) to be single-valued and continuous on \( |z| = r \). Since \( f(z) = [f(z)/zF'(z)] [zF'(z)] \), then \( \arg f(z) = \arg [zF'(z)] - \arg [zF'(z)/f(z)] \) will be uniquely determined and by (2.1) we have for \( z = re^{i\theta} \) (\( \rho < r < 1 \)),

\[ -\frac{\pi}{2} < \arg zF'(z) - \arg f(z) < \frac{\pi}{2}. \]

Let \( \theta_1 < \theta_2 \), then

\[ (2.2) \quad -\frac{\pi}{2} < \arg re^{i\theta_2} F'(re^{i\theta_2}) - \arg f(re^{i\theta_2}) < \frac{\pi}{2} \]

and

\[ (2.3) \quad -\frac{\pi}{2} < - \arg re^{i\theta_1} F'(re^{i\theta_1}) + \arg f(re^{i\theta_1}) < \frac{\pi}{2}. \]

Combining (2.2) and (2.3), we obtain

\[ -\pi + \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \]

\[ < \arg [re^{i\theta_2} F'(re^{i\theta_2})] - \arg [re^{i\theta_1} F'(re^{i\theta_1})] \]

\[ < \pi + \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \]

or

\[ -\pi + \int_{\theta_1}^{\theta_2} d \arg f(re^{i\theta}) < \int_{\theta_1}^{\theta_2} d \arg F(re^{i\theta}) \]

\[ < \pi + \int_{\theta_1}^{\theta_2} d \arg f(re^{i\theta}). \]

Since \( f(z) \) is in \( S(p) \),

\[ \int_{\theta_1}^{\theta_2} d \arg f(re^{i\theta}) > 0. \]

Thus the left side of (2.5) gives
\[ \int_{\theta_1}^{\theta_2} d\arg dF(re^{i\theta}) > -\pi. \]

Taking \( \theta_1 = 0 \) and \( \theta_2 = 2\pi \) in (2.5) and using the fact that
\[ \int_{0}^{2\pi} d\arg f(re^{i\theta}) = 2p\pi \]
we obtain
\[ (2.7) \quad (2p - 1)\pi < \int_{0}^{2\pi} d\arg dF(re^{i\theta}) < (2p + 1)\pi. \]

However, the integral in (2.7) is an integral multiple of \( 2\pi \). Therefore,
\[ (2.8) \quad \int_{0}^{2\pi} d\arg dF(re^{i\theta}) = 2p\pi. \]

Thus, by Lemma 1, \( F(z) \) is at most \( p \)-valent for \( |z| < r \). Since \( r \) was arbitrary \((\rho < r < 1)\), \( F(z) \) is at most \( p \)-valent for \( |z| < 1 \).

Since (2.8) holds for any function in \( \mathcal{H}(p) \) for some range \( \rho < |z| < 1 \), we easily obtain the following corollary.

**Corollary.** If \( F(z) \) is in \( \mathcal{H}(p) \), then \( F'(z) \) has exactly \( (p - 1) \) zeros in \( |z| < 1 \).

Necessary and sufficient conditions for a function \( F(z) \), regular in \( |z| < 1 \), with \( F(0) = 0 \) and \( F'(z) \neq 0 \) to be in \( \mathcal{H}(1) \) have been given by Kaplan [5]. We see from the proof of Theorem 1 that necessary conditions for \( F(z) \) to be in \( \mathcal{H}(p) \) are that (2.6) and (2.8) hold in some range \( \rho < |z| < 1 \). We will now show these conditions to be sufficient in two particular cases. The method of proof used is that established by Kaplan [5].

**Lemma 2.** Let
\[ F(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots \]
be regular for \( |z| \leq 1 \). If
\[ (2.9) \quad \int_{0}^{2\pi} d\arg dF(z) = 2p\pi \]
and
\[ (2.10) \quad \int_{\theta_1}^{\theta_2} d\arg dF(z) > -\pi \quad (\theta_1 < \theta_2) \]
for \( |z| = 1 \), then \( F(z) \) is in \( \mathcal{H}(p) \).

**Remark.** We will show that there exists a function \( f(z) \) in \( S(p) \) with all its zeros at the origin, which is regular for \( |z| < 1 + \epsilon \) for some \( \epsilon > 0 \), and
such that \( \text{Re}[zF'(z)/f(z)] > 0 \) for \( |z| < 1 + \epsilon \). This is actually more than we need to prove the lemma, but it is needed in the proof of Theorem 3.

**Proof.** Since \( F(z) \) is regular on \( |z| = 1 \), it is regular in some circle containing \( |z| \leq 1 \). By continuity we then have the existence of some \( \epsilon > 0 \) such that (2.9) and (2.10) hold for \( 1 \leq |z| \leq (1 + \epsilon) \). Now, the function \( z^{(1-p)}F'(z) \) is free of zeros in \( |z| \leq (1 + \epsilon) \). Hence, we may define \( \arg z^{(1-p)}F'(z) \) to be single-valued and continuous in \( |z| \leq 1 + \epsilon \).

Let

\[
p(r, \theta) = \arg [(re^{i\theta})^{1-p}F'(re^{i\theta})] \quad (r \leq 1 + \epsilon)
\]

and

\[
P(r, \theta) = p(r, \theta) + p\theta.
\]

Then, since (2.9) and (2.10) hold for \( |z| = 1 + \epsilon \), we have

\[
P(1 + \epsilon, \theta + 2\pi) - P(1 + \epsilon, \theta) = 2p\pi,
\]

\[
P(1 + \epsilon, \theta_2) - P(1 + \epsilon, \theta_1) > -\pi \quad \text{for } \theta_1 < \theta_2.
\]

Using an argument identical to Kaplan's [5], we may show the existence of a function \( S(1 + \epsilon, \theta) \), which is increasing in \( \theta \) and such that

\[
S(1 + \epsilon, \theta + 2\pi) - S(1 + \epsilon, \theta) = 2p\pi
\]

and

\[
|S(1 + \epsilon, \theta) - P(1 + \epsilon, \theta)| \leq \frac{\pi}{2}.
\]

Let

\[
q(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{[(1 + \epsilon)^2 - r^2][S(1 + \epsilon, \alpha) - p\alpha]}{(1 + \epsilon)^2 + r^2 - 2(1 + \epsilon)r \cos(\alpha - \theta)} d\alpha.
\]

Then, \( q(r, \theta) \) is harmonic for \( r < 1 + \epsilon \).

Let \( Q(r, \theta) = q(r, \theta) + p\theta \) for \( r < 1 + \epsilon \). Using the fact that \( S(1 + \epsilon, \alpha) - p\alpha \) has period \( 2\pi \), we obtain for \( r < 1 + \epsilon \) and \( \theta_1 < \theta_2 \),

\[
Q(r, \theta_2) - Q(r, \theta_1) = \int_0^{2\pi} \frac{[(1 + \epsilon)^2 - r^2][S(1 + \epsilon, \alpha + \theta_2) - S(1 + \epsilon, \alpha + \theta_1)]}{(1 + \epsilon)^2 + r^2 - 2(1 + \epsilon)r \cos \alpha} d\alpha.
\]

Since \( S(1 + \epsilon, \alpha) \) is increasing

\[
Q(r, \theta_2) - Q(r, \theta_1) \geq 0.
\]

Thus \( (\partial/\partial\theta) Q(r, \theta) \geq 0 \) for \( r < 1 + \epsilon \).

Let \( h(z) \) be a function, regular for \( |z| < 1 + \epsilon \), such that \( \text{Im}[h(re^{i\theta})] = q(r, \theta) \) and let
$$f(z) = z^p e^{\theta(z)} = b_p z^p + \cdots \quad (|z| < 1 + \epsilon).$$

For $|z| < 1 + \epsilon$,

$$\text{Re} \left[ \frac{zf'(z)}{f(z)} \right] = \frac{\partial}{\partial \theta} \arg f(z) = \frac{\partial}{\partial \theta} (p \theta + q(r, \theta)) = \frac{\partial}{\partial \theta} Q(r, \theta) \geq 0.$$ 

But $zf'(z)/f(z)$ is regular for $|z| < 1 + \epsilon$. Thus,

$$\text{Re} \left[ \frac{zf'(z)}{f(z)} \right] > 0 \quad \text{for } |z| < 1 + \epsilon.$$ 

Since $f(z)$ has $p$ zeros, all of them at the origin,

$$\int_0^{2\pi} \text{Re} \left[ \frac{zf'(z)}{f(z)} \right] d\theta = 2p\pi \quad (|z| < 1 + \epsilon).$$

Hence, $f(z)$ is $p$-valently star-like for $|z| < 1 + \epsilon$.

Now, for $z = re^{\theta}$, $r < 1 + \epsilon$, we have

$$\arg \frac{zF'(z)}{f(z)} = \arg zF'(z) - \arg f(z) = \arg (p(r, \theta) - q(r, \theta) - p\theta) = \arg (p(r, \theta) - q(r, \theta)).$$

Since $p(r, \theta)$ is harmonic for $|z| < 1 + \epsilon$, we may write

$$\text{(2.14)} \quad p(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{[(1 + \epsilon)^2 - r^2]p(1 + \epsilon, \alpha)}{(1 + \epsilon)^2 + r^2 - 2(1 + \epsilon)r \cos(\alpha - \theta)} d\alpha.$$ 

Then, using (2.12), (2.13) and (2.14), we obtain

$$\left| \frac{zF'(z)}{f(z)} \right| = \left| \frac{p(r, \theta) - q(r, \theta)}{x} \right|$$

$$= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{[(1 + \epsilon)^2 - r^2][p(1 + \epsilon, \alpha) - S(1 + \epsilon, \alpha)]d\alpha}{(1 + \epsilon)^2 + r^2 - 2(1 + \epsilon)r \cos(\alpha - \theta)} \right|$$

$$\leq \frac{x}{2}.$$

Thus $\text{Re} \left[ zF'(z)/f(z) \right] \geq 0$ for $|z| < 1 + \epsilon$. Hence, either $\text{Re} \left[ zF'(z)/f(z) \right] > 0$ for $|z| < 1 + \epsilon$, in which case $F(z)$ is in $\mathcal{H}(p)$, or $zF'(z)/f(z)$ reduces to a constant for $|z| < 1 + \epsilon$. In the second case $F(z)$ is in $C(p) \subset \mathcal{H}(p)$.

**Theorem 2.** Let

$$F(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots \quad (|z| < 1)$$

be regular for $|z| < 1$. If (2.9) and (2.10) hold for some range $\rho < |z| < 1$, then $F(z)$ is in $\mathcal{H}(p)$.

**Proof.** Let $\rho < \delta < 1$. Then the function $G_{\delta}(z) = F(\delta z)$ is regular on $|z| = 1$
and satisfies (2.9) and (2.10) on \(|z| = 1\). Hence, by Lemma 2, \(G(z)\) is in \(\mathcal{A}(p)\) and there exists

\[
f_n(z) = b_p z^p + \cdots \quad (|z| < 1)
\]

in \(S(p)\) such that

\[
\text{Re} \left[ \frac{zG'(z)}{f_n(z)} \right] > 0 \quad (|z| < 1).
\]  

We may assume that \(|b_p| = 1\). Cartwright [1] has shown that the family of \(p\)-valent functions with the moduli of the first \(p\) coefficients fixed is a normal family. Thus we may choose a sequence \(\delta_n\) tending to 1, such that the sequence of functions \(f_n(z)\) tends to a function \(f(z)\) in \(S(p)\). Since \(zG_n(z)\) tends to \(zF'(z)\), we obtain from (2.15) that

\[
\text{Re} \left[ \frac{zF'(z)}{f(z)} \right] \geq 0 \quad |z| < 1.
\]

This implies that \(F(z)\) is in \(\mathcal{A}(p)\).

**Theorem 3.** Let

\[
F(z) = a_q z^q + \cdots \quad (1 \leq q \leq p)
\]

be regular for \(|z| \leq 1\). If (2.9) and (2.10) hold on \(|z| = 1\), then \(F(z)\) is in \(\mathcal{A}(p)\).

**Proof.** By condition (2.9) \(F'(z)\) has \((p - 1)\) zeros in \(|z| < 1\), \((q - 1)\) of them at the origin. Let \(\alpha_1, \alpha_2, \ldots, \alpha_{p-q}\) be the nonzero roots of \(F'(z)\) and let

\[
G(z) = \int_0^z \frac{z^{p-q}F'(z)\,dz}{\prod_{i=1}^{p-q} (z - \alpha_i)(1 - \overline{\alpha}_iz)}
\]

\(G(z)\) is regular for \(|z| \leq 1\) and

\[
zG'(z) = \frac{z^{p-q}zF'(z)}{\prod_{i=1}^{p-q} (z - \alpha_i)(1 - \overline{\alpha}_iz)}.
\]

Since

\[
\arg \left[ \frac{z^{p-q}}{\prod_{i=1}^{p-q} (z - \alpha_i)(1 - \overline{\alpha}_iz)} \right] = 0 \quad |z| = 1,
\]

\(\arg zG'(z) = \arg zF'(z)\) for \(|z| = 1\).

Thus, \(G(z)\) satisfies (2.9) and (2.10) on \(|z| = 1\). Hence, by Lemma 2, \(G(z)\) is in \(\mathcal{A}(p)\) and there exists \(f(z)\) in \(S(p)\), regular for \(|z| \leq 1\), such that
But using the same reasoning as above, we have
\[ \arg \left( \frac{zG'(z)}{f(z)} \right) = \arg \left( \frac{zF'(z)}{f(z)} \right) \text{ on } |z| = 1. \]

Hence,
\[ \text{Re} \left( \frac{zF'(z)}{f(z)} \right) > 0 \text{ for } |z| = 1. \]

Thus, \( F(z) \) is in \( \mathcal{K}(p) \).

Theorem 3 immediately gives us the following lemma, which will prove useful in obtaining a bound for the \((p + 1)\)st coefficient of a function in \( \mathcal{K}(p) \).

**Lemma 3.** If \( F(z) \) is regular in \(|z| \leq 1\) and in \( \mathcal{K}(p) \), then there exists
\[ f(z) = b_p z^p + \cdots \quad (|b_p| = 1) \]
regular and in \( S(p) \) for \(|z| \leq 1\), such that
\[ \text{Re} \left( \frac{zF'(z)}{f(z)} \right) > 0 \text{ on } |z| = 1. \]

3. Some extremal problems for the class \( \mathcal{K}(p) \). The following lemma has been proven by Royster [11]. However, the proof we give, which was communicated to me by Professor M. S. Robertson, seems to be different.

**Lemma 4.** Let \( f(z) = [h(z)]^{-p} \), where \( h(z) \) is in \( S(1) \), \( h(0) = 0 \), \( h'(0) = 1 \) and let
\[ f(z) = \sum_{n=1}^{\infty} C_n z^n \quad (0 < |z| < 1, \ C_{-1} = 1), \]
then
\[ |C_n| \leq \binom{2p}{n+p} (n = -p, \ldots, 1), \]
and these inequalities are sharp.

**Proof.** We write
\[ z^p f(z) = z^p [h(z)]^{-p} = \sum_{n=0}^{\infty} d_n z^n \quad (|z| < 1, \ d_0 = 1). \]

The lemma will then be proven, if we can show
\[ |d_n| \leq \binom{2p}{n} \quad (n \leq p + 1). \]

Taking the logarithm of both sides of (3.1), differentiating and multiplying through by \( z \), we obtain
\[ -zf'(z) - \frac{zh'(z)}{p} = \frac{zh'(z)}{h(z)}. \]

Thus, we have for \( |z| < 1 \)
\[ \text{Re} \left[ -zf'(z) \right] = \text{Re} \left[ \frac{zh'(z)}{h(z)} \right] > 0 \quad (|z| < 1). \]

Let
\[ P(z) = -zf'(z), \]
then
\[ \text{Re} \left[ \frac{1}{P(z)} \right] > 0 \quad \text{for } |z| < 1. \]

Let
\[ \frac{1}{P(z)} = 1 + \sum_{n=1}^{\infty} \mu_n z^n, \]
\[ \frac{1}{P(z)} = -\frac{pf(z)}{zf'(z)} = -\frac{pz^p f(z)}{z^{p+1} f'(z)}, \]
\[ -\frac{1}{P(z)} z^{p+1} f'(z) = pz^p f(z), \]
or
\[ \left[ -\sum_{m=0}^{\infty} \mu_m z^m \right] \left[ \sum_{s=0}^{\infty} (s - p) d_s z^s \right] = p \sum_{n=0}^{\infty} d_n z^n. \]

Equating coefficients, we obtain
\[ pd_n = \sum_{r=0}^{n} (p - r) d_r \mu_{n-r}, \]
\[ nd_n = \sum_{r=0}^{n-1} (p - r) d_r \mu_{n-r}. \]

Since \( |\mu_{n-r}| \leq 2 \), we obtain
\[ (3.3) \quad n |d_n| \leq 2 \sum_{r=0}^{n-1} (p - r) |d_r|. \]
provided $p - r \geq 0$. That is, provided $n \leq p + 1$. Using (3.3) and a simple induction argument, we have

$$|d_n| \leq \binom{2p}{n} \text{ for } n \leq p + 1.$$  

That the inequalities are sharp is shown by the function

$$f(z) = \left[ \frac{z}{(1 + z)^2} \right]^{-p}.$$  

**Theorem 4.** Let

$$F(z) = \sum_{n=1}^{\infty} a_n z^n \quad (|z| < 1)$$  

be regular and in $S(p)$ for $|z| < 1$, then

$$|a_{p+1}| \leq \sum_{k=1}^{p} \frac{2k(2p + 1)!}{(p + k)! (p - k)! [(p + 1)^2 - k^2]} |a_k|$$  

and this inequality is sharp in all the variables $|a_1|, \ldots, |a_p|$.  

**Remark.** This theorem was first proven for $p = 1$ by Reade [8].

**Proof.** We may assume without loss of generality that $F(z)$ is regular for $|z| \leq 1$. Then, by Lemma 3 there exists a function

$$f(z) = b_p z^p + \cdots \quad (|b_p| = 1),$$  

regular for $|z| \leq 1$ and in $S(p)$, such that

$$\text{Re} \left[ \frac{zf'(z)}{f(z)} \right] > 0 \quad (|z| = 1).$$  

We may assume that $b_p = 1$ since arg $|b_p|$ is not involved in the inequality to be obtained. Thus we may write $f(z)$ in the form $|\phi(z)|^p$, where

$$\phi(z) = z + \sum_{n=2}^{\infty} h_n z^n$$  

is regular for $|z| < 1$ and in $S(1)$.  

We may then write (3.5) in the form

$$\text{Re} \left[ zF'(z) |\phi(z)|^{-p} \right] > 0 \quad \text{on } |z| = 1.$$  

Let

$$|\phi(z)|^{-p} = \sum_{n=-p}^{\infty} C_n z^n \quad (0 < |z| < 1, \ C_{-p} = 1).$$  

Then
\[ zF'(z) [\phi(z)]^{-p} = \left[ \sum_{n=1}^{\infty} na_n z^n \right] \left[ \sum_{n=1}^{\infty} C_n z^n \right] \]

\[ = \sum_{k=-(p-1)}^{-} d_k z^k, \]

where

\[ d_k = \sum_{n=1}^{p+k} C_{-(n-k)} n a_n \quad (k = -(p-1), \ldots). \]

Consider the function \( G(z) \) given by

\[ G(z) = zF'(z) [\phi(z)]^{-p} - \sum_{k=-(p-1)}^{-1} d_k z^k + \sum_{k=-(p-1)}^{-1} \overline{d}_k z^{-k}. \] (3.6)

Since \( \overline{z} = z^{-1} \) for \( |z| = 1 \), the last two terms in (3.6) add up to a purely imaginary number for \( |z| = 1 \). Thus,

\[ \text{Re}[G(z)] = \text{Re}[zF'(z) [\phi(z)]^{-p}] > 0 \quad \text{for} \quad |z| = 1. \]

But \( G(z) \) is regular for \( |z| \leq 1 \). Therefore,

\[ \text{Re}[G(z)] > 0 \quad \text{for} \quad |z| \leq 1. \]

Now

\[ G(z) = d_0 + (d_1 + \overline{d}_{-1}) z + \cdots \quad (|z| \leq 1). \]

Hence

\[ |d_1 + \overline{d}_{-1}| \leq 2 \text{Re}[d_0] \leq 2 |d_0|, \]

\[ \left| \sum_{n=1}^{p+1} C_{-(n-1)} n a_n + \sum_{n=1}^{p-1} \overline{C}_{-(n+1)} n a_n \right| \leq 2 \left| \sum_{n=1}^{p} C_{-n} n a_n \right|, \]

\[ (p + 1) |a_{p+1}| \leq \sum_{n=1}^{p-1} \left[ 2n |C_{-n}| + n |C_{-(n-1)}| + n |C_{-(n+1)}| \right] |a_n| \]

\[ + \left[ 2p |C_{-p}| + p |C_{-(p-1)}| \right] |a_p|. \]

By Lemma 4

\[ |C_{-k}| \leq \binom{2p}{p-k} \quad (k = 1, 2, \ldots, p). \]

Therefore,
\[(p + 1)|a_{p+1}| \leq \sum_{n=1}^{p-1} \left[ 2n \left( \frac{2p}{p - n} \right) + n \left( \frac{2p}{p - n + 1} \right) + n \left( \frac{2p}{p - n - 1} \right) \right] |a_n| + \left[ 2p + p \left( \frac{2p}{1} \right) \right] |a_p| \]

\[
= (p + 1) \sum_{n=1}^{p} \frac{2n(2p + 1)!}{(p + n)!(p - n)![(p + 1)^2 - n^2]} |a_n|
\]

which is (3.4).

We remark that the inequality is sharp, since it is known to be sharp for \( f(z) \) in \( S(p) \) with real coefficients \([2], [4]\).

In order to obtain bounds on \(|F'(z)|\) for \( F(z) \) in \( \mathcal{A}(p) \), we will make use of the following lemma.

**Lemma 5.** Let

\[ F(z) = a_qz^q + \cdots \quad (|z| \leq 1) \]

be regular and in \( \mathcal{A}(p) \) for \(|z| \leq 1\). Let \( \alpha_1, \alpha_2, \ldots, \alpha_{p-q} \) be the nonzero critical points of \( F'(z) \) in \(|z| < 1\). Then the function

\[ H(z) = \frac{\prod_{i=1}^{p-q} \left( \frac{\alpha_i}{|\alpha_i|} - \frac{z}{|\alpha_i|} \right) (\alpha_i z - 1)}{\prod_{j=1}^{p} \left( \frac{\alpha_j}{|\alpha_j|} - \frac{z}{|\alpha_j|} \right) (\alpha_j z - 1)} \]

is regular for \(|z| \leq 1\) and in \( \mathcal{A}(p) \).

**Proof.** By Lemma 3, there exists

\[ h(z) = b_pz^p + \cdots \quad (|b_p| = 1), \]

regular and in \( S(p) \) for \(|z| \leq 1\), such that

\[ \text{Re} \frac{zF'(z)}{h(z)} > 0 \quad \text{for} \quad |z| = 1. \]

\[ \frac{zF'(z)}{h(z)} = \frac{z^{p-q}zF'(z) \left[ \prod_{i=1}^{p-q} \left( \frac{\alpha_i}{|\alpha_i|} - \frac{z}{|\alpha_i|} \right) (\alpha_i z - 1) \right]^{-1}}{h(z)}. \]

But,

\[ \arg \left( z^{p-q} \left[ \prod_{i=1}^{p-q} \left( \frac{\alpha_i}{|\alpha_i|} - \frac{z}{|\alpha_i|} \right) (\alpha_i z - 1) \right]^{-1} \right) = 0 \quad \text{on} \quad |z| = 1. \]

Thus,

\[ \frac{zF'(z)}{h(z)} = M \frac{zF'(z)}{h(z)}, \quad M > 0 \quad \text{on} \quad |z| = 1. \]

Hence,
Therefore, $H(z)$ is in $\mathcal{K}(p)$.

**Theorem 5.** Let

$$F(z) = a_q z^q + \cdots \quad (|z| < 1),$$

be regular and in $\mathcal{K}(p)$ for $|z| < 1$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{p-q}$ be the nonzero critical points of $F(z)$ and let $p = \max |\alpha_i|$ and $p^* = \min |\alpha_i|$. Then

\begin{equation}
|F'(re^{i\theta})| \leq \left( \frac{1 + r}{1 - r} \right)^{q-1} q |a_q| \left[ \prod_{i=1}^{p-q} \left( 1 + \frac{r}{|\alpha_i|} \right) \right] (1 + r|\alpha_i|) \quad (r < 1),
\end{equation}

\begin{equation}
|F'(re^{i\theta})| \geq \left( \frac{1 - r}{1 + r} \right)^{q-1} q |a_q| \left[ \prod_{i=1}^{p-q} \left( \frac{r}{|\alpha_i|} - 1 \right) \right] (1 - r|\alpha_i|) \quad (\rho < r < 1),
\end{equation}

\begin{equation}
|F'(re^{i\theta})| \geq \left( \frac{1 - r}{1 + r} \right)^{q-1} q |a_q| \left[ \prod_{i=1}^{p-q} \left( 1 - \frac{r}{|\alpha_i|} \right) \right] (1 - r|\alpha_i|) \quad (r < p^*).}
\end{equation}

All these inequalities are sharp, equality being attained by the function

$$F_0(z) = \int_0^z \frac{(1 + z) z^{q-1}}{(1 - z)^{p+1}} q |a_q| \prod_{i=1}^{p-q} \left( 1 + \frac{z}{|\alpha_i|} \right) (1 + z|\alpha_i|) \, dz.$$

Note that inequality (3.7) was obtained by Umezawa [13] for his class of $p$-valent close-to-convex functions.

**Proof.** We may assume without loss of generality that $F(z)$ is regular for $|z| \leq 1$. Consider the functions $H(z)$ and $h(z)$, given in Lemma 5 and in its proof.

\[ \frac{zH'(z)}{h(z)} = d_0 + d_1 z + \cdots \quad (|z| \leq 1), \]

where

\[ d_0 = \frac{qa_q}{b_p} \left[ \prod_{i=1}^{p-q} (-e^{i\arg \alpha_i}) \right]^{-1}. \]

Then

\[ \frac{1}{\text{Re}[d_0]} \left[ \frac{zH'(z)}{h(z)} - i \text{Im}[d_0] \right] = P(z), \]

where $\text{Re} P(z) > 0$ for $|z| < 1$ and $P(0) = 1$. Thus,
Hence
\[
\left| \frac{zH'(z)}{h(z)} - d_0 \right| \leq |z| = r,
\]
\[
(1 - r) \left| \frac{zH'(z)}{h(z)} \right| \leq (1 + r) |d_0| = (1 + r) q|a_q|.
\]

Using the known bound
\[
|h(z)| \leq \frac{r^p}{(1 - r)^{q^p}} \quad \text{for } |z| = r
\]
and using the definition of \( H(z) \), we obtain
\[
|F'(re^{i\theta})| \leq \frac{(1 + r)}{(1 - r)} r^{p-q+1} q|a_q| |h(z)| \left| \prod_{i=1}^{p-q} \left( \frac{\alpha_i}{|\alpha_i|} - \frac{z}{|\alpha_i|} \right) (\bar{\alpha}_i z - 1) \right|
\]
\[
\leq \frac{(1 + r)r^{p-1}}{(1 - r)^{q+p-1} q|a_q| \prod_{i=1}^{p-q} \left( 1 + \frac{r}{|\alpha_i|} \right) (1 + r|\alpha_i|)},
\]
which is (3.7).

To obtain (3.8) and (3.9), we notice that for \( z = re^{i\theta} \)
\[
\left| \frac{P(z) + 1}{P(z) - 1} \right| \geq \frac{1}{r},
\]
\[
|h(z)| \geq \frac{r^p}{(1 + r)^{q^p}},
\]
\[
\left| \frac{\alpha_i}{|\alpha_i|} - \frac{z}{|\alpha_i|} \right| \left| \bar{\alpha}_i z - 1 \right| \geq \left( \frac{r}{|\alpha_i|} - 1 \right) (1 - r|\alpha_i|) \quad (|\alpha_i| < r),
\]
and
\[
\left| \frac{\alpha_i}{|\alpha_i|} - \frac{z}{|\alpha_i|} \right| \left| \bar{\alpha}_i z - 1 \right| \geq \left( 1 - \frac{r}{|\alpha_i|} \right) (1 - r|\alpha_i|) \quad (r < |\alpha_i|).
\]

Going through the same type of argument as before, we obtain (3.8) and (3.9).

The function \( F_0(z) \) is in \( S(p) \) relative to
\[
f(z) = \frac{z^q}{(1 - z)^{q^p}} \prod_{i=1}^{p-q} \left( 1 + \frac{z}{|\alpha_i|} \right) (1 + z|\alpha_i|).
\]
Equality in (3.7) is attained by $F_0'(r)$, in (3.8) by $F_0'(-r)$, $r > \rho$, and in (3.9) by $F_0'(-r)$, $r < \rho^*$. 

4. Radii of close-to-convexity and convexity for functions in $s(p)$. Goodman [4] has proven that if

$$f(z) = a_0z^q + \cdots \quad (|z| < 1)$$

is in $S(p)$, then

$$\text{Re} \frac{zf'(z)}{f(z)} \geq J_q(r) \quad \text{for } r < \rho,$$

where

$$J_q(r) = q - r \left[ \frac{2p}{1 + r} + \sum_{i=1}^{p-q} \frac{1}{|\alpha_i| - r} + \frac{|\alpha_i|}{1 - |\alpha_i|r} \right],$$

$\alpha_1, \ldots, \alpha_{p-q}$ being the nonzero roots of $f(z)$ and $\rho = \min|\alpha_i|$. $J_q(r)$ is a decreasing function of $r$ for $r < \rho$, is positive for $r = 0$ and tends to $-\infty$ as $r$ tends to $\rho$. Thus, $J_q(r)$ has a least positive root $r_q$ and $J_q(r) > 0$ for $r < r_q$.

We thus have that $f(z)$ is $q$-valently star-like for $|z| < r_q$. This estimate is sharp, since (4.1) was shown to be sharp [4], equality being attained at $z = -r$ by the function

$$f(z) = z^q(1 - z)^{-2p} \prod_{i=1}^{p-q} \left( 1 + \frac{z}{|\alpha_i|} \right) (1 + z|\alpha_i|).$$

THEOREM 6. Let

$$F(z) = a_0z^q + \cdots \quad (|z| < 1)$$

be in $s(p)$. Let $\alpha_1, \ldots, \alpha_{p-q}$ be the nonzero roots of $F'(z)$ and let $r_q$ be the least positive root of $J_q(r)$, defined in (4.1). Then $F(z)$ is $q$-valently close-to-convex for $|z| < r_q$.

Proof. We first prove the theorem for $F(z)$, regular on $|z| = 1$. Then there exists

$$f(z) = b_0z^p + \cdots \quad (|z| \leq 1),$$

regular and in $S(p)$ for $|z| \leq 1$, such that

$$\text{Re} \left[ \frac{zf'(z)}{f(z)} \right] > 0 \quad \text{on } |z| = 1.$$

Since

$$\arg \left( z^{p-q} \prod_{i=1}^{p-q} \frac{1}{z - \alpha_i} (1 - \overline{\alpha_i}z) \right)^{-1} = 0 \quad \text{on } |z| = 1,$$

we have
Re \left[ \frac{z^{p-q} z F'(z)}{ \prod_{i=1}^{p-q} (z - \alpha_i) (1 - \overline{\alpha_i} z) \cdot f(z)} \right] > 0 \quad \text{for } |z| \leq 1.

Let

\[ g(z) = z^{q-p} \left[ \prod_{i=1}^{p-q} (z - \alpha_i) (1 - \overline{\alpha_i} z) \right] f(z). \]

Then, \( g(z) \) is in \( S(p) \) since \( \Re \left[ \frac{z g'(z)}{g(z)} \right] > 0 \) on \( |z| = 1 \). But \( g(z) \) has nonzero roots at \( \alpha_1, \alpha_2, \ldots, \alpha_{p-q} \). Therefore, \( g(z) \) is \( q \)-valently star-like for \( |z| < r_q \). Since

\[ \Re \left[ \frac{z F'(z)}{g(z)} \right] > 0 \quad \text{for } |z| \leq r_q, \]

\( F(z) \) is \( q \)-valently close-to-convex for \( |z| < r_q \).

If \( F(z) \) is not regular on \( |z| = 1 \), there exists a \( p^* < 1 \) such that for \( p^* < \delta < 1 \) the function \( G_i(z) = F(\delta z) \) is in \( \mathcal{V}(p) \) and regular on \( |z| = 1 \). \( G_i'(z) = 0 \) for \( z = \alpha_i/\delta \). Thus, \( G_i(z) \) is \( q \)-valently close-to-convex for \( |z| < r_{q,\delta} \), where \( r_{q,\delta} \) is the least positive root of

\[ J_{q,\delta}(r) = q - r \left[ \frac{2p}{1 + r} + \sum_{i=1}^{p-q} \frac{\delta}{|\alpha_i| - r\delta} + \frac{|\alpha_i|}{\delta - |\alpha_i|r} \right]. \]

Thus, there exists

\[ f_i(z) = C_q z^q + \ldots \quad (|z| < r_{q,\delta}, \quad |C_q| = 1) \]

\( q \)-valently star-like for \( |z| < r_{q,\delta} \) such that

\[ \Re \left[ \frac{z G_i'(z)}{f_i(z)} \right] > 0 \quad \text{for } |z| < r_{q,\delta}. \]

But \( r_{q,\delta} \geq r_q \), since \( J_{q,\delta}(r) \geq J_q(r) \) for \( r < \min|\alpha_i| \). Thus \( f_i(z) \) is \( q \)-valently star-like for \( |z| < r_q \).

By a result of M. Cartwright [1] the family of \( q \)-valent functions \( f(z) = a_q z^q + \ldots \) (\( |a_q| = 1 \)) is a normal family. Thus we may choose an increasing sequence \( \delta_i \) tending to 1, such that the functions \( f_i(z) \) tend to a function \( f(z) \), which is \( q \)-valently star-like for \( |z| < r_q \). Since for each \( i \),

\[ \Re \left[ \frac{z G_i'(z)}{f_i(z)} \right] > 0 \quad \text{for } |z| < r_q \]

and since \( z G_i'(z) \) tends to \( z F'(z) \), we have

\[ \Re \left[ \frac{z F'(z)}{f(z)} \right] \geq 0 \quad \text{for } |z| < r_q. \]
Thus either \( \text{Re}\left[\frac{zF'(z)}{f(z)}\right] > 0 \) for \(|z| < r_q\), in which case \( F(z) \) is \( q \)-valently close-to-convex for \(|z| < r_q\), or \( \left|\frac{zF'(z)}{f(z)}\right| \) reduces to a constant for \(|z| < r_q\). In the second case \( F(z) \) is \( q \)-valently convex and hence \( q \)-valently close-to-convex for \(|z| < r_q\).

**Theorem 7.** Let

\[
F(z) = a_qz^q + \cdots \quad (|z| < 1),
\]

be in \( \mathcal{S}(p) \), then \( F(z) \) is \( q \)-valently convex for \(|z| < \beta_q\), where \( \beta_q \) is the least positive root of

\[
K_q(r) = J_q(r) - \frac{2r}{1 - r^2}
\]

and this estimate is the best possible.

**Proof.** Let us first assume that \( F(z) \) is regular on \(|z| = 1\). Then, as we have seen before, there exists

\[
g(z) = b_qz^q + \cdots \quad (|z| < 1),
\]

which is in \( S(p) \) for \(|z| < 1\), such that

\[
\text{Re}\left[ \frac{zF'(z)}{g(z)} \right] > 0 \quad \text{for} \quad |z| \leq 1.
\]

Let

\[
\frac{zF'(z)}{g(z)} = P(z), \quad \text{Re}[P(z)] > 0 \quad \text{for} \quad |z| \leq 1,
\]

\[
1 + \frac{zF''(z)}{F'(z)} = \frac{zP''(z)}{P(z)} + \frac{zg'(z)}{g(z)}.
\]

Now \( g(z) \) has the same zeros as \( F'(z) \). Therefore,

\[
\text{Re}\left[ \frac{zg'(z)}{g(z)} \right] \geq J_q(r) \quad \text{for} \quad r < \min|\alpha_i|.
\]

By a result, obtained independently by Libera [6], MacGregor [7] and Robertson [10], we have

\[
\text{Re}\left[ \frac{zP'(z)}{P(z)} \right] \geq -\frac{2r}{1 - r^2}.
\]

Thus

\[
\text{Re}\left[ 1 + \frac{zF''(z)}{F'(z)} \right] \geq -\frac{2r}{1 - r^2} + J_q(r) = K_q(r)
\]

for \( r < \min|\alpha_i| \).

Thus, if \(|z| < \beta_q\)
Since $F'(z)$ has $(q - 1)$ zeros in $|z| < \beta_q$, all of them at the origin,

$$
\int_{0}^{2\pi} \Re \left[ 1 + \frac{zF''(z)}{F'(z)} \right] d\theta = 2q\pi \quad (|z| < \beta_q).
$$

Thus $F(z)$ is $q$-valently convex for $|z| < \beta_q$.

Arguing as in Theorem 6, we may remove the assumption of regularity on $|z| = 1$.

The function

$$
F(z) = \int_{0}^{z} \frac{(1 + z)z^{q-1} \prod_{i=1}^{p-q} \left( 1 + \frac{z}{|\alpha_i|} \right)}{(1 - z)^{2q+1}} dz
$$

shows that the radius found is sharp, since

$$
1 + \frac{zF''(z)}{F'(z)} = K_q(r)
$$

for $z = -r$, $r < \min |\alpha_i|$.

REFERENCES


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