ON CO-GROUPS IN THE CATEGORY OF GRADED ALGEBRAS[1]

BY

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1. **Introduction.** Let $\mathscr{C}$ be any category. An object $B \in \mathscr{C}$ is called a *group* in $\mathscr{C}$ (or, more strictly, is given a structure of a group in $\mathscr{C}$) if for any $A \in \mathscr{C}$ the set $\mathrm{Hom}(A, B)$ is given a group structure, such that for any $f: A' \to A''$ the induced map $f^*: \mathrm{Hom}(A'', B) \to \mathrm{Hom}(A', B)$ is a homomorphism. Of course not every object of $\mathscr{C}$ can be so structured. Dually, an object $A \in \mathscr{C}$ is called a *co-group* if it is a group in the dual category, i.e., for each $B \in \mathscr{C}$, $\mathrm{Hom}(A, B)$ is given a group structure, such that for any $g: B' \to B''$ the induced map $g_*: \mathrm{Hom}(A, B') \to \mathrm{Hom}(A, B'')$ is a homomorphism. The importance of groups and co-groups in different categories has been emphasized in [2].

In the category $\mathscr{T}_H$ of based topological spaces and homotopy classes of maps, the co-groups are the so-called H'-spaces[2] (e.g. any suspension is such an H'-space). The following facts are well known.

(1) The fundamental group functor π_1 (for spaces with "nice" base point) preserves free products (in $\mathscr{T}_H$ the free product is the wedge of spaces $X \vee Y$ and $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$).

(2) The fundamental group of an H'-space is free.

Now, in a category with zero maps and free products a more direct and workable definition of co-groups may be given (see Definition 2.1). It follows from it that if a functor between two categories preserves free products, it also preserves co-groups. Therefore, the "categorical" explanation of (2) in view of (1) follows from a result of D. M. Kan [4], who has proved that a co-group in the category $\mathscr{G}$ of groups is always a free group. Kan's theorem shows that the co-group structure of $G \in \mathscr{G}$ completely determines a set of free generators of G (the so called "primitive" elements).

On the other hand it is a fact that

(1′) The Pontrjagin algebra of the loop space $H_*(\Omega; K)$, where K is a field, is another functor which preserves free products (since no published proof of this probably well-known result is known to the author, we shall give one in §3). Moreover,

(2′) The Bott-Samelson theorem [1] states that $H_*(\Omega\Sigma X; K)$ (Σ being the suspension functor) is a free algebra (generated by $H_*(X; K)$).

The aim of this paper is to prove some purely algebraic results which

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(2) Or G'-spaces in more recent terminology.

will enable us to derive (2′) from (1′) in the same way as the result of Kan gives the possibility of deriving (2) from (1).

Kan's theorem quoted above has been later extended in a slightly weakened form to the more general case when no associativity is required [3]. Certain of our results will also hold for "co-multiplicative objects"(3) rather than for co-groups. The definition of a co-multiplicative object (c.-m. object) may be obtained from the above definition of a co-group by replacing the words "group structure" by "structure of a multiplicative system with two-sided unit." Again a more convenient equivalent definition will be given in §2.

Let $\mathscr{A}$ be the category of connected graded associative algebras over a commutative ring Λ and let $\mathscr{A}_H$ be the "primitive" [2, Part II] category of c.-m. objects in $\mathscr{A}$. Our first main result is

Theorem 1.1. *There exists a functor* $N:\mathscr{A}_H\rightarrow\mathscr{M}$ *(where* $\mathscr{M}$ *is the category of positively graded modules over* Λ*) such that*

(i) $N(A)$ *is a submodule of* A $(A\in\mathscr{A}_H)$ *called the module of normed elements and*

(ii) *as an algebra,* A *is freely generated by* $N(A)$, *i.e.,* $A=T(N(A))$, *where* $T(N)$ *is the tensor algebra of* N; *in particular, any object of* $\mathscr{A}_H$ *is a free algebra.*

In the case of co-groups we have a more precise statement. Let $\mathscr{A}_G$ be the category of co-groups in $\mathscr{A}$ and let $\mathscr{C}$ be the category of connected graded co-associative co-algebras (with co-unit) over Λ.

Theorem 1.2. *There exist two covariant functors* S: $\mathscr{A}_G\rightarrow\mathscr{C}$ *and* T_G: $\mathscr{C}\rightarrow\mathscr{A}_G$ *such that*

(i) *both compositions* T_GS *and* ST_G *are equal to the identity,*

(ii) $S(A)$ *for* $A\in\mathscr{A}_G$ *is precisely the submodule of normed elements of* A, *and*

(iii) *as an algebra,* $T_G(S(A))$ *is the free (tensor) algebra generated by* $S(A)$. *Thus, in particular,* $\mathscr{A}_G$ *and* $\mathscr{C}$ *are category isomorphic.*

The strength of Theorem 1.2 resides in the fact that the co-algebra structure of $S(A)$ completely determines the co-group structure of A and conversely. The precise way in which this is done will be described in §2. Since the elements of $S(A)$ are "semi-primitive" in a certain sense, Theorem 1.2 is the analogue for algebras of Kan's theorem [4].

Let now $\Lambda=K$ be a field and suppose that $\mathscr{A}$ is the category of connected graded algebras over K, of finite type. Let C^* be the dual algebra of the co-algebra C.

(3) H'-objects in the terminology of [2].

COROLLARY 1.3. *The contravariant functor* $S^*: \mathscr{A}_G \to \mathscr{A}$ *defined by* $S^*(A) = (S(A))^*$ *establishes an anti-isomorphism between the two categories.*

Let Y be a topological space with finitely generated homology, and let ΣY be the (reduced) suspension of Y. It is well known that ΣY is a co-group in $\mathscr{T}_H$; as shown in the proof of Corollary 3.3 this co-group structure induces a canonical co-group structure in the Pontrjagin algebra of the loop space $\Omega\Sigma Y$ of ΣY (provided we use a field K of coefficients or no torsion is present).

COROLLARY 1.4. $S^*(H_*(\Omega\Sigma Y, K)) = H^*(Y, K)$, *the latter being the cohomology algebra of* Y.

§2 contains the main definitions used in the paper and also the proof of the results stated above (modulo two key lemmas, whose proof is relegated to the completely technical §4). §3 is devoted to the topological implication of our results.

2. **Definitions and proof of the theorems.** A category $\mathscr{C}$ is said to possess zero maps (morphisms) if for each $A, B, \in \mathscr{C}$, the set $\mathrm{Hom}(A,B)$ contains a distinguished element 0 satisfying $0f = 0$, $g0 = 0$ for all $f \in \mathrm{Hom}(A', A)$, $g \in \mathrm{Hom}(B, B')$.

We shall denote by 1 both the identity functor (in any category) and the identity map (of any object).

Let $A_1, A_2 \in \mathscr{C}$. A *free* (or inverse) product [2] of the objects A_1, A_2 is an object Q and a pair of maps $i_j: A_j \to Q$, $j = 1, 2$, such that for any $X \in \mathscr{C}$ and any two maps $f_j: A_j \to X$, $j = 1, 2$, there exists a unique map f: $Q \to X$ with $fi_j = f_j$. We write $f = \langle f_1, f_2 \rangle$; thus $\langle f_1, f_2 \rangle i_j = f_j$. The free product, if it exists, is unique up to an equivalence. We will denote by $A_1 * A_2$ an arbitrary respresentative of the class of equivalent free products of A_1, A_2. The free product is associative in the obvious sense. If g_j: $A_j' \to A_j$, $j = 1, 2$, we write $g_1 * g_2$ for the map $\langle i_1 g_1, i_2 g_2 \rangle : A_1' * A_2' \to A_1 * A_2$.

From now on $\mathscr{C}$ will be a category with zero maps and free products. The following definitions are equivalent to the ones given in Introduction [2, Part I, Theorem 4.6].

DEFINITION 2.1. A co-multiplication on $A \in \mathscr{C}$ is a map

$$\Phi: A \to A * A$$

such that $\langle 1, 0 \rangle \Phi = \langle 0, 1 \rangle \Phi = 1: A \to A$. The pair (A, Φ) will be then called a co-multiplicative object (c.-m. object). A co-multiplication Φ is associative if

$$(1 * \Phi)\Phi = (\Phi * 1)\Phi: A \to A * A * A.$$

Φ has an inverse $\nu: A \to A$ if $\langle \nu, 1 \rangle \Phi = \langle 1, \nu \rangle \Phi = 0$. If Φ is associative and has an inverse, then (A, Φ) is called a co-group.

If (A', Φ') and (A'', Φ'') are c.-m. objects in $\mathscr{C}$, a map $f: A' \to A''$ is a homomorphism if $\Phi'' f = (f * f)\Phi'$. The category of all c.-m. objects in $\mathscr{C}$ and their homomorphisms will be denoted by $\mathscr{C}_H$; the corresponding category of co-groups will be denoted by $\mathscr{C}_G$; obviously $\mathscr{C}_G \subset \mathscr{C}_H$.

EXAMPLE 2.2. In the category $\mathscr{T}_H$ of based topological spaces, the free product is the wedge $X \vee Y$ (union with a common base-point). A comultiplication Φ in $\mathscr{T}_H$ is a map $\Phi: X \to X \vee X$, such that $r_1\Phi \sim r_2\Phi \sim 1$: $X \to X$, where r_1, r_2: $X \vee X \to X$ are the retractions onto the two factors.

Let now Λ be a fixed commutative ring and $\mathscr{A}$ the category of connected associative graded algebras over Λ. Any $A \in \mathscr{A}$ is provided with an augmentation $\epsilon: A \to \Lambda$ which maps A^0 (the module of elements of degree 0) isomorphically onto Λ and which vanishes on elements of positive degree; we denote $\operatorname{Ker} \epsilon$ by $\bar{A}$. Any map f in $\mathscr{A}$ satisfies $\epsilon f = \epsilon$; the zero map 0: $A \to B, A, B \in \mathscr{A}$ is characterized by $0(\bar{A}) = 0$.

If N is a positively graded Λ-module we denote by $T(N)$ the tensor algebra of N.

In order to be able to describe in a convenient way the free product in $\mathscr{A}$ we shall introduce first some notations which will be used throughout the paper.

Let α, β be any two symbols. A sequence of the form $(\alpha, \beta, \alpha, \beta, \cdots)$ or $(\beta, \alpha, \beta, \alpha, \cdots)$ will be called *alternating*. There are exactly two alternating sequences of length n: the first starting with α and the second starting with β. The set of all alternating sequences will be denoted by $\mathscr{R} = \mathscr{R}(\alpha, \beta)$. If $I = (i_1, \cdots, i_n)$, $J = (j_1, \cdots, j_p)$ are two alternating sequences, define

$$\begin{aligned} I \vee J &= (i_1, \cdots, i_n, j_1, \cdots, j_p) \quad \text{if } i_n \neq j_1, \\ I \vee J &= (i_1, \cdots, i_n, j_2, \cdots, j_p) \quad \text{if } i_n = j_1. \end{aligned} \tag{2.1}$$

If $I \in \mathscr{R}(1,2)$ set

$$\bar{A}_I = \bar{A}_{i_1} \otimes \cdots \otimes \bar{A}_{i_n}, \qquad I = (i_1, \cdots, i_n). \tag{2.2}$$

Define, for $A_1, A_2 \in \mathscr{A}$

$$A_1 * A_2 = \Lambda + \Sigma \bar{A}_I, \qquad I \in \mathscr{R}(1,2). \tag{2.3}$$

$A_1 * A_2$ has the obvious grading and augmentation; it can be converted into an algebra by means of the multiplication

$$\mu_{IJ}: \bar{A}_I \otimes \bar{A}_J \to \bar{A}_{I \vee J},$$

where μ_{IJ} is the canonical isomorphism if $i_n \neq j_1$ and is induced by the multiplication $\bar{A}_{i_n} \otimes \bar{A}_{j_1} \to \bar{A}_{i_n}$ and the identity on the other factors if $i_n = j_1$. If we define $i_k: A_k \to A_1 * A_2$, $k = 1, 2$, as the obvious imbeddings, all the axioms of a free product are satisfied. For instance, if $f_k: A_k \to B$, $k = 1, 2$, then $\langle f_1, f_2 \rangle | \bar{A}_I, I \in \mathscr{R}$ is defined as being the composition

$$\overline{A}_I \xrightarrow{f_{i_1}\otimes \cdots \otimes f_{i_n}} B\otimes \cdots \otimes B \xrightarrow{\mu'} B,$$

where μ' is the multiplication in B. In particular,

$$(2.4)\qquad \begin{aligned} \langle 1,0\rangle | \Lambda + \overline{A}_1 &: \Lambda + \overline{A}_1 \approx A_1, \\ \langle 0,1\rangle | \Lambda + \overline{A}_2 &: \Lambda + \overline{A}_2 \approx A_2, \end{aligned}$$

all the other summands being mapped onto zero.

We now adopt the following convention: if $a \in A$ we use a prime sign (a') in order to indicate that a belongs to the first factor of $A * A$ and a double prime sign (a'') in order to indicate that a belongs to the second factor. The following is a consequence of Definition 2.1 and of (2.4).

Proposition 2.3. *A co-multiplication on $A \in \mathscr{A}$ is a map $\Phi: A \to A * A$ satisfying*

$$\Phi(a) = a' + a'' + b, \qquad b \in \Sigma A_I, \quad I \in R(',''), \qquad |I| \geqq 2$$

(*where $|I|$ is the length of the sequence I*) *for all $a \in A$* (*b depends on a*).

For $A_j \in \mathscr{A}$, $j = 1,2$, define $k_j: A_j \to A_1 \otimes A_2$ by $k_1(a_1) = a_1 \otimes 1, k_2(a_2) = 1 \otimes a_2$. Then

$$\omega = \langle k_1, k_2\rangle : A_1 * A_2 \to A_1 \otimes A_2$$

maps the free product of algebras onto their tensor product. If Φ: $A \to A * A$ is a co-multiplication, then the "diagonal map" $\Delta = \omega\Phi$ converts A to a Hopf algebra called *the underlying Hopf algebra* of (A, Φ). On the other hand if μ: $(A * A) \otimes (A * A) \to A * A$ is the multiplication in $A * A$ and i_j: $A \to A * A$, $j = 1,2$, are the two imbeddings, then

$$(2.5)\qquad \rho = \mu(i_1 \otimes i_2)\Delta = \mu(i_1 \otimes i_2)\omega\Phi : A \to A * A$$

is a map of graded modules.

Theorem 1.1 is an immediate consequence of the following

Lemma 2.4. *For any c.-m. object (A, Φ), $A \in \mathscr{A}$, we can define in a canonical way a submodule $N \subset A$ such that A is freely generated by N, i.e., $A = T(N)$; if $f: (A_1, \Phi_1) \to (A_2, \Phi_2)$ is a homomorphism then $f(N_1) \subset N_2$.*

The elements $a \in N$ are called *normed*. The construction of normed elements and the proof of Lemma 2.4 will be obtained in §4 by induction on the degree.

Let $S = \operatorname{Ker}(\Phi - \rho)$, where (A, Φ) is a c.-m. object in $\mathscr{A}$ and ρ is given by (2.5). Since

$$(2.6)\qquad \Delta(a) = a \otimes 1 + 1 \otimes a + \Sigma a_i \otimes a_j, \quad a \in A$$

for any $a \in S$, we have

$$(2.6')\qquad \Phi(a) = \rho(a) = \mu(i_1 \otimes i_2)\Delta(a) = a' + a'' + \Sigma a_i' a_j''.$$

Formula (2.6′) suggests calling the elements of S *semi-primitive*.

LEMMA 2.5. *If Φ is associative then*

(i) *all normed elements are semi-primitive, i.e., $N = S$, and*

(ii) *S is a sub-co-algebra of the underlying Hopf algebra of (A, Φ), i.e., $a \in S$ implies that in* (2.6) $a_i, a_j \in S$.

This lemma will also be proved in §4.

We are now able to prove Theorem 1.2.

Proof of Theorem 1.2. The functorial character of $S = S(A)$ follows from the fact that if $f: (A, \Phi) \to (A', \Phi')$ is a homomorphism of co-groups, $\Phi' f = (f * f)\Phi$ and $\rho' f = (f * f)\rho$ and thus $f(\mathrm{Ker}(\Phi - \rho)) \subset \mathrm{Ker}(\Phi' - \rho')$. Lemma 2.5 together with Theorem 1.1 show us that as an algebra, $A = T(S(A))$. Given an arbitrary co-algebra $C \in \mathscr{C}$ with diagonal map $\bar{\Delta}: C \to C \otimes C$, let $\alpha: C \to T(C)$ be the canonical inclusion; define

$$\Delta = (\alpha \otimes \alpha)\bar{\Delta}: C \to T(C) \otimes T(C)$$

and

$$\rho = \mu(i_1 \otimes i_2)\Delta: C \to T(C) * T(C).$$

By universality of the tensor algebra $T(C)$, ρ can be extended to a unique map of algebras

$$\Phi: T(C) \to T(C) * T(C).$$

One easily checks that (a) Φ is co-multiplication and (b) $T_G(C) = (T(C), \Phi)$ is a co-group in $\mathscr{A}$ if and only if C is co-associative. Notice that the canonical anti-automorphism ν of the underlying Hopf algebra of $T_G(C)$ serves as an inverse for $T_G(C)$; also, that the underlying Hopf algebra of $T_G(C)$ is co-commutative if and only if C was co-commutative.

A co-multiplication

$$\Phi: A \to A * A$$

is commutative if we have $P\Phi = \Phi$, where $P: A * A \to A * A$ is the permutation of the two factors. Obviously, for a commutative Φ any semi-primitive a is primitive, i.e., $\Phi(a) = a' + a''$. Thus we have

COROLLARY 2.6. *A commutative co-group in $\mathscr{A}$ is a free algebra generated by the submodule of primitive elements. In particular, the underlying Hopf algebra of A is also primitively generated.*

3. **Topological implications.** We shall first prove the fact quoted in the Introduction under (1′).

Let $\mathscr{G}$ be the category of c.s.s. groups. $\mathscr{G}$ admits a free product $G_1 * G_2$, $G_i \in \mathscr{G}$, where $(G_1 * G_2)_q = G_{1q} * G_{2q}$ and the boundary and degeneracy operators are defined in the obvious way. We shall consider homology with coefficients in a field K.

Theorem 3.1. *The homology functor H_*: $\mathscr{G} \to \mathscr{A}$ preserves free products, i.e., the map*

$$\nu = \langle i_{1*}, i_{2*} \rangle : H_*(G_1) * H_*(G_2) \to H_*(G_1 * G_2) \tag{3.1}$$

is an isomorphism (*here* $i_j: G_j \to G_1 * G_2$, $j = 1, 2$, *are the inclusions*).

Proof. Let $A_j = H_*(G_j)$, $\bar{A}_j = \tilde{H}_*(G_j)$ (reduced homology), $j = 1, 2$. With the notations of (2.3) define

$$D_0 \subset D_1 \subset \cdots \subset D_n \subset \cdots \subset A_1 * A_2$$

by setting

$$\begin{aligned} &D_0 = K, \qquad D_1 = K + \bar{A}_1 + \bar{A}_2, \\ &D_n = D_1 + \Sigma \tilde{A}_J, \qquad |J| \leqq n,\ n \geqq 2; \end{aligned} \tag{3.2}$$

moreover,

$$\text{Lim. dir.}\, D_n = A_1 * A_2 \tag{3.3}$$

and

$$D_n / D_{n-1} = \bar{A}_{I_n} + \bar{A}_{J_n}, \tag{3.4}$$

where I_n and J_n are the only alternating sequences of length n.

If $J = (j_1, \cdots, j_k)$, we shall use the notations $G_J = G_{j_1} \times \cdots \times G_{j_k}$ and $\bar{G}_J = G_{j_1} \wedge \cdots \wedge G_{j_k}$ (smashed product). By the Künneth formula we have

$$\bar{A}_J = \tilde{H}_*(\bar{G}_J). \tag{3.5}$$

On the other hand any $g \in G_1 * G_2$ admits a unique representation as a reduced word $g = x_1 \cdots x_k$, $e \neq x_s \in G_{j_s}$. If F_n is the subcomplex of all words of length $\leqq n$, we have

$$H_*(G_1 * G_2) = \text{Lim. dir.}\, H_*(F_n); \tag{3.6}$$

for homology commutes with direct limits. For any sequence $J = (j_1, \cdots, j_k)$, $k \leqq n$, define a map

$$\omega_J : G_J \to F_n$$

by setting $\omega_J(x_1, x_2, \cdots, x_k) = x_1 \cdots x_k$. The maps ω_J composed with the natural splitting $\bar{A}_J \to H_*(\bar{G}_J) \to H_*(G_J)$ induce maps

$$\nu_n : D_n \to H_*(F_n)$$

such that the diagram

$$\begin{CD} D_n @>{\nu_n}>> H_*(F_n) \\ @VVV @VVV \\ A_1 * A_2 @>{\nu}>> H_*(G_1 * G_2) \end{CD} \tag{3.7}$$

is commutative (the vertical arrows are induced by inclusion). By (3.3) and (3.6) it is enough to prove that for all n, ν_n is an isomorphism. This follows by repeated applications of the 5-lemma to the commutative diagram

$$\begin{array}{ccccccc} 0 \to D_n & \longrightarrow & D_{n+1} & \longrightarrow & \bar{A}_{I_n} + \bar{A}_{J_n} & \longrightarrow & 0 \\ \downarrow \nu_n & & \downarrow \nu_{n+1} & & \downarrow & & \\ H_*(F_n) & \to & H_*(F_{n+1}) & \to & \tilde{H}_*(F_{n+1}/F_n) = \tilde{H}_*(\bar{G}_{I_n} \vee \bar{G}_{J_n}). & & \end{array}$$

The upper exact sequence (see (3.4)) obviously splits; the third vertical arrow is an isomorphism by (3.5).

Corollary 3.2. *If $X_i, i = 1,2$, are spaces with "nice" base-point $*$ (i.e., if $*$ has in X_i a neighborhood U_i which can be deformed in X_i into $*$* (rel $*$)), *then*

$$H_*(\Omega(X_1 \vee X_2)) \approx H_*(\Omega X_1) * H_*(\Omega X_2).$$

Proof. If $S(X)$ denotes the singular complex of X and the base-points are "nice", then

$$S(X_1 \vee X_2) \approx S(X_1) \vee S(X_2).$$

Let $\mathscr{S}$ be the category of connected c.s.s. complexes, and let $G: \mathscr{S} \to \mathscr{G}$ be the Kan functor; according to [5] we have a natural equivalence of the functors $H*GS$ and $H_*\Omega$ and therefore

$$\begin{aligned} H_*(\Omega(X_1 \vee X_2)) &= H_*(GS(X_1 \vee X_2)) \\ &= H_*(G(S(X_1) \vee S(X_2))) \\ &= H_*(GS(X_1) * GS(X_2)); \end{aligned}$$

it suffices now to apply Theorem 3.1.

Corollary 3.3 (Bott-Samelson). *If (X, Φ) is a co-multiplicative object in $\mathscr{T}_H$ (i.e., X is a space of normalized Lusternik-Schnirelman category ≤ 1) then $H_*(\Omega X)$ is a free algebra. If Φ is (homotopy) associative, then $H_*(\Omega X)$ is freely generated by* $\mathrm{Ker}[(\Omega\Phi)_* - \rho_*]$, *where*

$$\rho = \mu(\Omega i_1 \times \Omega i_2)\Delta : \Omega X \to \Omega(X \vee X)$$

and Δ: $\Omega X \to \Omega X \times \Omega X$ is the diagonal map, i_j: $X_j \to X \times X$, $j = 1,2$, are the inclusions and μ is the multiplication of loops in $\Omega(X \vee X)$.

Proof. Φ being as in Example 2.2 it follows from Corollary 3.2 that $(\Omega\Phi)_* : H_*(\Omega X) \to H_*(\Omega(X \vee X))$ is a co-multiplication and thus by applying Lemmas 2.4 and 2.5 we get the result.

Proof of Corollary 1.4. In the particular case when X is a suspension, i.e., $X = \Sigma Y$, we know from the original proof of the Bott-Samelson theorem, that $H_*(\Omega\Sigma Y)$ is freely generated by the image of $H_*(Y)$ under the imbedding $e: Y \to \Omega\Sigma Y$. It is easy to check directly that $\operatorname{Im} e_* \subset \operatorname{Ker}[(\Omega\Phi)_* - \rho_*]$ and therefore by the second part of Corollary 3.3 $\operatorname{Im} e_* = \operatorname{Ker}[(\Omega\Phi)_* - \rho_*]$. (In fact, the last relation admits also a purely topological proof.) Since e_* is a monomorphism, $S(H_*(\Omega\Sigma Y)) = H_*(Y)$ as a co-algebra and its dual $S^*(H_*(\Omega\Sigma Y)) = (H_*(Y))^* = H^*(Y)$ as an algebra.

REMARK 3.4. In the previous analysis we could have replaced the use of a coefficient field by integer coefficients, provided no torsion is present.

REMARK 3.5. In the general case of a co-multiplicative object (X, Φ) in $\mathscr{T}_H$, the original proof of the Bott-Samelson theorem shows that $H_*(\Omega X) = T(M)$, where $M \subset H_*(\Omega X)$ is a submodule mapped isomorphically onto $H_*(X)$ by the homology suspension σ. We know on the other hand by Theorem 1.1, that also $H_*(\Omega X) = T(N)$, where N is the module of normed elements. This means that for any $m \in M, m = n + d$, where $n \in N$ and d is decomposable. Thus $\sigma(m) = \sigma(n)$ and we can always normalize M by choosing $M = N$. This normalization has the advantage that N is natural with respect to maps of co-multiplicative objects, i.e., if f: $(X_1, \Phi_1) \to (X_2, \Phi_2)$ then $(\Omega f)_*(N_1) \subset N_2$.

4. **Proof of the key lemmas.** Recall that we have denoted by $\mathscr{R}_n(\alpha, \beta)$ the set of all alternating sequences on length n formed by using the symbols α and β. For a fixed $n, \mathscr{R}_n(\alpha, \beta)$ consists of exactly two elements: one sequence $(\alpha, \beta, \alpha, \cdots)$ starting with α and another sequence $S_n = (\beta, \alpha, \beta, \cdots)$ starting with β. The sequences S_n, $n \geqq 2$, will be called special; $\mathscr{S}(\alpha, \beta)$ will denote the set of all special sequences of any length $(\geqq 2)$.

Let now $\alpha, \beta, \gamma, \cdots$ be any finite set of symobols and let $\mathscr{I}_n(\alpha, \beta, \gamma, \cdots)$ denote the set of all sequences of length n formed by using these symbols. A sequence $I \in \mathscr{I}_n(\alpha, \beta, \gamma, \cdots)$ is *mixed* if *all* the possible symbols occur in I. The set of all mixed sequences of length n will be denoted by $\mathscr{M}_n(\alpha, \beta, \gamma, \cdots)$. Finally

$$\mathscr{I}(\alpha, \beta, \gamma, \cdots) = \bigcup_n \mathscr{I}_n(\alpha, \beta, \gamma, \cdots),$$

$$\mathscr{M}(\alpha, \beta, \gamma, \cdots) = \bigcup_n \mathscr{M}_n(\alpha, \beta, \gamma, \cdots).$$

Obviously $\mathscr{R}_n(\alpha, \beta) \subset \mathscr{M}_n(\alpha, \beta)$, $n \geqq 2$.

Let N' and N'' be two positively graded modules over a fixed commutative ring Λ and let $T(N'), T(N'')$ and $T(N' + N'')$ denote the tensor algebras of the corresponding modules. We have then the natural isomorphism

$$\langle T(i_1), T(i_2) \rangle : T(N') * T(N'') \approx T(N' + N''),$$

$$i_1(a) = (a, 0), \qquad i_2(b) = (0, b), \qquad a \in N', \quad b \in N'',$$

whereby the two sides may be identified. Thus

$$T(N') * T(N'') = \Lambda + \Sigma N_J, \qquad J \in \mathscr{J}(',''), \tag{4.01}$$

where

$$N_J = N^{j_1} \otimes \cdots \otimes N^{j_2}, \qquad J = (j_1, \cdots, j_n). \tag{4.02}$$

Let (A, Φ) be a co-multiplicative object in $\mathscr{A}$ and let A^p denote the homogeneous component of A of degree p. We now proceed to give an inductive definition of the module N of normed elements. Let us assume that $N \cap A^1 = A^1$ and that $N^p = N \cap A^p$ has been defined for all $p \leqq k-1$. Let $N_{(k-1)} = N^1 + \cdots + N^{k-1}$. The imbedding $N_{(k-1)} \subset A$ can be extended to a unique algebra homomorphism

$$n : T(N) \to A$$

(we shall omit the subscript in $N_{(k-1)}$, since this cannot lead to confusion). Our inductive hypothesis is

$$n \,|\, T(N)^p : T(N)^p \approx A^p, \qquad p \leqq k-1. \tag{2.4$_{k-1}$}$$

As a consequence of (2.4_{k-1}) we can identify $(A_I)^k$, $I \in \mathscr{R}(',")$, with $|I|$ = length $I \geqq 2$, with a sum of form $(\sum N_J)^k$, $J \in \mathscr{M}(',")$. This yields

$$(A * A)^k = A'^k + A''^k + (\textstyle\sum N_J), \qquad J \in \mathscr{M}(',") \tag{4.03}$$

and correspondingly, for $a \in A^k$

$$\Phi(a) = a' + a'' + \textstyle\sum \Phi_J(a), \text{ where } \Phi_J(a) \in N_J, \quad J \in \mathscr{M}(','). \tag{4.04}$$

We shall always write Φ_r instead of Φ_{S_r}, where S_r is the unique special sequence of length r formed from the symbols $'$ and $''$.

DEFINITION N_k. An element $a \in A'$ is *normed* if in (4.04), $\Phi_s(a) = 0$ for all $s \geqq 2$.

The tensor algebra $T(N)$ is bigraded $T(N) = \sum F^{pk}$, where the first degree of $n_1 \otimes \cdots \otimes n_p$ is p and its second degree is $\deg n_1 + \cdots + \deg n_p$. $T(N) * T(N) = T(N' + N'') = \sum N_J$ is also bigraded by $\sum G^{pk}$. Let $\nabla = \langle 1, 1 \rangle : T(N') * T(N'') \to T(N)$ be the "folding" map. The following properties are easily checked

$$\Phi_s(A^k) \subset G^{sk}; \tag{4.1}$$

$$\Phi_s(n(F^{pk})) \subset \sum_{q \geqq p} G^{qk}; \tag{4.2}$$

$$\nabla(G^{pk}) \subset F^{pk}; \tag{4.3}$$

(4.4) ∇ is a monomorphism when restricted to any N_J.

By (4.1) and (4.2) we have

$$\Phi_s(n(F^{pk})) = 0 \quad \text{if } s < p; \tag{4.5}$$

moreover, it follows easily from the definition of Φ that

$$\nabla \Phi_s(n(a)) = a \quad \text{if } a \in F^{sk}. \tag{4.6}$$

Proof of Proposition 2.3$_k$. (i) $n: T(N)^k \to A^k$ is onto. Let $a \in A^k$; set $a_1 = a$, $a_s = a_{s-1} - n\nabla\Phi_s(a_{s-1})$, $s \geqq 2$. By (4.1) and (4.3) we have

$$\nabla \Phi_s(a_j) \subset F^{sk}; \tag{4.7}$$

therefore, by (4.5),

$$\Phi_m(n\nabla\Phi_s(a_{s-1})) = 0 \quad \text{and thus } \Phi_m(a_s) = \Phi_m(a_{s-1}) \quad \text{for } m < s. \tag{4.8}$$

Let us assume as an induction hypothesis that $\Phi_m(a_{s-1}) = 0$ for all $m \leqq s-1$; it follows that from (4.8) then $\Phi_m(a_s) = 0$ for all $m < s$. But according to (4.7) and to (4.6)

$$\nabla\Phi_s(a_s) = \nabla\Phi_s(a_{s-1}) - \nabla\Phi_s n\nabla\Phi_s(a_{s-1}) = 0.$$

By 4.4 this proves in particular that

$$\Phi_m(a_k) = 0 \quad \text{for all } m \leqq k; \tag{4.9}$$

since on the other hand $G^{mk} = 0$ for $m > k$ it follows from (4.1) that a_k is normed. This completes the proof of (i) since $a - a_k$ lies in the image of n by construction.

(ii) $n: T(N)^k \to A^k$ is a monomorphism. Let $a \in \ker n$, $a = a^1 + \cdots + a^k$, $a^j \in F^{jk}$. Obviously $\Phi_s(n(a^1)) = \Phi_s(a^1) = 0$ for $s \geqq 2$ (since $F^{1k} = N^k$). By (4.5) and (4.6), $a^2 = \nabla\Phi_2 n(a^2) = \nabla\Phi_2 n(a) = 0$. In the same way, by applying successively $\nabla\Phi_3$, $\nabla\Phi_4$, we prove that $a^s = 0$ for all $s \geqq 2$ and therefore $a = a^1 \in N^k$. Since $n|N^k$ is the identity, $n(a) = 0$ implies $a = 0$.

In order to prove Lemma 2.5 we shall use the notation described at the beginning of this section. By definition, $\mathscr{N}_n(1,2,3) = \mathscr{M}_n(1,2,3)$ for $n \geqq 4$ and $\mathscr{N}_3(1,2,3) = \mathscr{M}_3(1,2,3) - \{(1,2,3)\}$ (we exclude the increasing sequence $(1,2,3)$). Let us define the maps

$$\begin{aligned} \phi &: \mathscr{N}_n(1,2,3) \to \mathscr{M}_n(1,2), \\ \psi &: \mathscr{N}_n(1,2,3) \to \mathscr{M}_n(1,2) \end{aligned} \tag{4.10}$$

by

$$\phi((i_1, \cdots, i_n)) = (j_1, \cdots, j_n), \qquad j_k = \max(i_{k-1}, 1),$$
$$\psi((i_1, \cdots, i_n)) = (j_1, \cdots, j_n), \qquad j_k = \min(i_k, 2).$$

Consider the weakest equivalence relation on $\mathscr{M}_n(1,2)$ compatible with the relation $K_1 \sim K_2$ if $K_1 = \phi(I), K_2 = \psi(I)$ for the same $I \in \mathscr{N}_n(1,2,3)$.

LEMMA 4.1. *All the elements of $\mathscr{M}_n(1,2)$ are equivalent under the above relation.*

Proof. For $n = 3$ the proof is obtained by direct verification. If $n > 3$, let $K = (k_1, \cdots, k_n)$, $M = (m_1, \cdots, m_n)$, $k_i, m_i = 1$ or 2. If $k_i \leqq m_i$ for all i, then $K \sim M$; indeed $L = (l_1, \cdots, l_k)$, where $l_i = k_i + m_i - 1$ satisfies $\phi(L) = K$, $\psi(L) = M$.

(i) If for some i, $k_i = m_i = 1$, set $l_s = \max(k_s, m_s)$. Then $L = (l_1, \ldots, l_n) \in \mathscr{M}_n(1,2)$ and by the previous remark $K \sim L \sim M$. If for some $i, k_i = m_i = 2$, set $l_s = \min(k_s, m_s)$ and again $K \sim L \sim M$.

(ii) Assume now that for all i, $k_i \neq m_i$. Since $n \geqq 3$ we may assume that one of the sequences K or M, say K, contains the symbol 1 at least twice. By replacing one of these 1's by 2 we get another sequence K' such that $K \sim K'$ and we are able to apply (i) to K' and M.

Lemma 2.5 will be easily deduced from the following

LEMMA 4.2. *If (A, Φ) is an associative c.-m. object in A and N is the module of all normed elements then any $a \in N$ is semi-primitive, i.e, $\Phi(a) = a' + a'' + \sum a_1' a_j''$, where $a_i' \in N'$, $a_j'' \in N''$.*

Proof of Lemma 4.2. According to Lemma 2.4, $A = T(N)$ and therefore by (4.1)

$$A * A = T(N') + T(N'') + \sum N_J, \quad J \in \mathscr{M}(1,2),$$

$$A * A * A = T(N') + T(N'') + T(N''') + \sum N_K + \sum N_L,$$

where $K \in \mathscr{M}(1,2) \cup \mathscr{M}(1,3) \cup \mathscr{M}(2,3)$ and $L \in \mathscr{M}(1,2,3)$ (for technical reasons we write 1 for $'$, 2 for $''$ and 3 for $'''$).

Let $a \in N$. Then

$$(\Phi * 1)\Phi(a) = a' + a'' + a''' + \sum f_K(a) + \sum f_L(a),$$

$$(1 * \Phi)\Phi(a) = a' + a'' + a''' + \sum g_K(a) + \sum g_L(a),$$

where $f_K(a)$, $g_K(a) \in N_K$, $f_L(a)$, $g_L(a) \in N_L$. By associativity of Φ we must have

$$f_L(a) = g_L(a), \qquad L \in \mathscr{M}(1,2,3). \tag{4.11}$$

Lemma 4.2 will be proved by proving for all k and s, $s \leqq k$, the following statement

$$\Phi_J(a) = 0, \qquad J \in \mathscr{M}_n(1,2), \quad J \neq (1,2) \tag{4.2.k.s.}$$

for $a \in N^p$, $p \leqq k - 1$ and all $n \geqq 2$ and for $a \in N^k$, $2 \leqq n \leqq s$.

Since for any k, $(4.2.k - 1.k - 1)$ and $(4.2.k.s)$, $s = 1, 2$, are clearly equivalent, it will be enough to use induction on s. (4.2.1.1) being trivial we shall assume $(4.2.k.s - 1)$, $s \geqq 3$ and prove $(4.2.k.s.)$.

First we shall notice that any $b \in (N_J)^k$, $J \in \mathscr{M}(1,2)$ is a sum of products of elements lying in N'^{p_i} or N''^{p_i} with all $p_i \leqq k - 1$. This enables us by $(4.2.k.s - 1)$ to assume in computing $(\Phi * 1)(b)$ and $(1 * \Phi)(b)$, that

$$\Phi(a) = a' + a'' + \sum a_i' a_j''. \tag{4.12}$$

On the other hand it follows from the definition of Φ and that of the maps ϕ and ψ in (4.10) that if the sequences J and L have the same length, i.e., $|J| = |L|$ and $J \in \mathscr{M}(1,2)$, $L \in \mathscr{M}(1,2,3)$,

$$\begin{aligned} &(\Phi * 1)_L(N_J) = 0 \qquad \text{if } J \neq \phi(L), \quad |J| = |L|, \\ &(\Phi * 1)_L | N_{\phi(L)} : N_{\phi(L)} \approx N_L, \end{aligned} \tag{4.13}$$

and analogously

$$\begin{aligned} &(1 * \Phi)_L(N_J) = 0 \quad \text{if } J \neq \psi(L), \quad |J| = |L|, \\ &(1 * \Phi)_L | N_{\psi(L)} : N_{\psi(L)} \approx N_L. \end{aligned} \tag{4.13'}$$

By (4.12) we have for any $a \in N^k$

$$\begin{aligned} (\Phi * 1)\Phi_{(1,2)}(a) &= \Phi_{(1,3)}(a) + \Phi_{(2,3)}(a) + \Phi_{(1,2,3)}(a), \\ (1 * \Phi)\Phi_{(1,2)}(a) &= \Phi_{(1,2)}(a) + \Phi_{(1,3)}(a) + \Phi_{(1,2,3)}(a). \end{aligned} \tag{4.14}$$

Now the remark that by $(4.2.k.s - 1)$ the only $\Phi_J(a)$, with $|J| < s$ which can be $\neq 0$ for $a \in N^k$ is $\Phi_{(1,2)}(a)$, yields together with (4.14) and the first lines of (4.13) and (4.13′)

$$\begin{aligned} &f_L(a) = (\Phi * 1)_L \Phi_{\phi(L)}(a), \\ &g_L(a) = (1 * \Phi)_L \Phi_{\psi(L)}(a), \qquad L \in \mathscr{N}(1,2,3) \end{aligned} \tag{4.15}$$

(where $\mathscr{N}(1,2,3)$ is the set of all mixed sequences with the exception of $(1,2,3)$). By using now (4.11) and the second lines of (4.13) and (4.13′) and by referring to the definition of equivalence of sequences used in Lemma 4.1, we get that $\Phi_J(a) = 0$ implies that $\Phi_K(a) = 0$ if $J \sim K$. But, by Lemma 4.1 all sequences in $\mathscr{M}_s(1,2)$ are equivalent and by definition of normed elements, at least for one sequence $J \in \mathscr{M}_s(1,2)$, we have $\Phi_J(a) = 0$; thus $\Phi_J(a) = 0$ for all J. This completes the proof of $(4.2.k.s.)$ and of Lemma 4.2.

Proof of Lemma 2.5. According to Lemma 4.2, if $a \in N, \Phi(a)$ has the form (4.12) with $a_i' \in N'$, $a_j'' \in N''$. An easy computation based on (2.5) shows that $\Phi(a) = \rho(a)$ (see (2.6) and (2.6′)) and thus $a \in S$. The second part of Lemma 2.5 is an immediate consequence of the fact that N is a coalgebra.

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