ON SEMI-CYLINDERS, SPLINTERS, AND BOUNDED-TRUTH-TABLE REDUCIBILITY

BY
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In Part I of this paper we construct a bounded-truth-table-complete set which is not creative, and in Part II we construct an infinite and coinfinite recursively enumerable (r.e.) splinter which is not a cylinder. Although these results appear unrelated, the proofs given here are similar and so have been included in a single paper. The two parts may be read independently of one another. Each of these results yields as corollary the existence of pseudo-creative sets which are not cylinders. In fact, in Part II we construct a pseudo-creative set S for which there is no total recursive function f such that $x \in S$ implies $f(x) \in S - \{x\}$ and $x \in S'$ implies $f(x) \in S' - \{x\}$.

Preliminaries. We will use N to denote the set of all non-negative integers. Unless specifically mentioned otherwise, all sets are considered subsets of N. If A is a set, $A' = N - A$. Since we consider only sets of integers, we will not use Cartesian products of sets but will instead work with images of Cartesian products under some effective mapping. More specifically, if A and B are subsets of N, let $A \otimes B = \{(a, b) \mid a \in A \text{ and } b \in B\}$. Let τ be any effective one-to-one mapping of $N \otimes N$ onto N. Then we define $A \times B$ to be $\tau(A \otimes B)$, and we abbreviate $\tau((a, b))$ to $\langle a, b\rangle$. (This is the notation introduced by Rogers in [5].) Given integers a and b we can always effectively find the integer $\langle a, b\rangle$, and given the integer $\langle a, b\rangle$ we can always effectively find a and b.

ϕ_e is the partial recursive function with index e. If ϕ_e is defined for every member of N, we say that ϕ_e is total.

In [3], Myhill has called a set a cylinder if it is recursively isomorphic to $B \times N$ for some r.e. set B, however we will follow Rogers in calling a set a cylinder if it is recursively isomorphic to $B \times N$ for any set B. Let D_x be the nonempty finite set with canonical index x. In [5], Rogers shows that a set A is a cylinder if and only if there is a total recursive function f such that $D_x \subset A$ implies $f(x) \in A - D_x$ and $D_x \subset A'$ implies $f(x) \in A' - D_x$.

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DEFINITION. A set A is a *semi-cylinder* if there is a total recursive function h such that $x \in A$ implies $h(x) \in A - \{x\}$ and $x \in A'$ implies $h(x) \in A' - \{x\}$. If the function h can be taken one-to-one, we say that A is a 1-1 *semi-cylinder.*

It is readily shown that if A is a recursive set, A is a semi-cylinder if and only if neither A nor A' is a unit set.

LEMMA. *Every cylinder is a* 1-1 *semi-cylinder. Every* 1-1 *semi-cylinder is a cylinder.*

Proof. The second statement is obvious. To prove the first, let A be a cylinder. We may assume $A = B \times N$ for some set B. Define $h(\langle a, n\rangle) = \langle a, n+1\rangle$ for all a and n. h is then the required function.

We can also show that 1-1 semi-cylinders and cylinders differ on the r.e. sets: If S is an infinite r.e. set, $S \times S$ is recursively isomorphic to $S \times S \cup \{\langle n, n\rangle \mid n \in N\}$. $(S \times S) \cup \{\langle n, n\rangle \mid n \in N\}$ is easily shown to be a 1-1 semi-cylinder, and in [7] we have shown the existence of a simple set, S_0, for which $S_0 \times S_0$ is not a cylinder.

LEMMA (MYHILL). *Every creative set is a cylinder.*

Proof. Suppose K is creative. Since every r.e. set is 1-1 reducible to every creative set, $K \times N$ is 1-1 reducible to K. For any set A, A is 1-1 reducible to $A \times N$; thus K is 1-1 reducible to $K \times N$, and so K is recursively isomorphic to $K \times N$.

I. A BOUNDED-TRUTH-TABLE-COMPLETE SET WHICH IS NOT CREATIVE

Introduction. Let R be any of the reducibility relations introduced by Post in [4]. Post called a r.e. set R-complete if every r.e. set is R-reducible to it, and he proved that the class of Turing-complete sets, the class of truth-table-complete sets, and the class of bounded-truth-table-complete sets are all distinct. In [2], Myhill shows that the class of many-one-complete sets and the class of one-one-complete sets are each identical with the class of all creative sets.

Recently Fischer has shown that bounded-truth-table reducibility differs from many-one reducibility on the r.e. sets [1], but the question of whether every bounded-truth-table-complete set is creative (i.e., of whether the class of bounded-truth-table-complete sets is identical with the class of many-one-complete sets) has remained open.

In Part I we construct a bounded-truth-table-complete set which is not creative and obtain as corollaries a new proof of Fischer's result and the existence of a pseudo-creative set which is not a cylinder.

LEMMA. *If A is a cylinder, then there exists a recursive permutation t such that*

(i) *for all x, $t(x) \neq x$,*

(ii) *for all x, $t^2(x) = x$, and*

(iii) *for all* x, $x \in A$ *if and only if* $t(x) \in A$.

Proof. We may assume $A = B \times N$ for some set B. Define $t(\langle y, 2n\rangle) = \langle y, 2n+1\rangle$ and $t(\langle y, 2n+1\rangle) = \langle y, 2n\rangle$.

THEOREM. *Any creative set, K, is the disjoint union of two recursively isomorphic, r.e., sets, L_0 and L_1, which are not* 1-1 *semi-cylinders.*

Discussion. Since all creative sets are recursively isomorphic, we may assume that $K = \{x \mid \phi_x(x) \text{ is defined}\}$. Since all creative sets are cylinders, there exists a 1-1 recursive permutation t such that

(i) for all x, $t(x) \neq x$,

(ii) for all x, $t^2(x) = x$, and

(iii) for all x, $x \in K$ if and only if $t(x) \in K$.

Let h be a 1-1 recursive function whose range is K. Define $g(2n) = h(y)$, where y is the least z such that

$$h(z) \notin \{g(0), g(1), \cdots, g(2n-1)\},$$

and $g(2n+1) = t(g(2n))$.

Since $h(z) \in \{g(0), g(1), \cdots, g(2n-1)\}$ if and only if

$$t(h(z)) \in \{g(0), g(1), \cdots g(2n-1)\},$$

g is a 1-1 recursive function whose range is K.

In proving the theorem, we will proceed in stages, constructing the sets L_0 and L_1 in such a way that the permutation t will be the required isomorphism between L_0 and L_1. A priority argument will be used to assure that L_0 has no function of the kind described in the definition of 1-1 semi-cylinders. The idea is to spoil each total recursive function f by finding some index r such that $\phi_r = f$ and either

(i) $\phi_r(r) = r$,

(ii) ϕ_r is not a 1-1 function,

(iii) $r \in L_1$ and $\phi_r(r) \in L_0$, or

(iv) $r \in L_0$ and $\phi_r(r) \notin L_0$.

It is easily seen that any one of these conditions is sufficient to assure that f will not be the function required by the definition if L_0 is to be a 1-1 semi-cylinder.

In constructing the sets L_0 and L_1, we imagine the non-negative integers to occur in a list, and as the construction proceeds we associate certain markers with some of the integers in the list. We will use two infinite collections of markers: $\{\Lambda_i\}$ and $\{\Lambda_i'\}$, $0 \leqq i < \infty$.

At any stage in the construction, for any i, at most one of the markers Λ_i and Λ_i' will be associated with some number. At any stage, any marker is beside at most one number and no number has more than one marker beside it. At Stage n, only markers Λ_i and Λ_i' for $i \leqq 2n+1$ can be in use. If at the end of Stage n, neither Λ_i nor Λ_i' for $i \leqq 2n+1$ is in use, then the function $\phi_{g(i)}$ has been spoiled.

Each stage of the construction will have two steps, Step 1 and Step 2. The first of these will be used to assure that exactly one of $g(2n)$ and $g(2n+1)$ $(=t(g(2n)))$ has a marker beside it. The second step will be used to assure that L_0 is not a 1-1 semi-cylinder. The action taken at the second step will depend on the marker which is associated with either $g(2n)$ or $g(2n+1)$ at the end of the first step.

Construction and proof. $L_i(a)$ will be the *set of integers* placed in L_i by the *beginning* of Stage a $(i=0,1)$. We now give the construction:

Stage n $(n \geqq 0)$.

Step 1. Compute $g(2n)$ and $g(2n+1)$ $(=t(g(2n)))$.

If neither $g(2n)$ nor $g(2n+1)$ has a marker beside it, place the marker Λ_{2n} beside $g(2n)$. Then find an r such that $\phi_r = \phi_{g(2n+1)}$ and (i) neither r nor $t(r)$ has a marker beside it, and (ii)

$$r \notin L_0(n) \cup L_1(n) \cup \{g(2n), g(2n+1)\}.$$

Place Λ_{2n+1} beside r and go on to Step 2.

If either $g(2n)$ or $g(2n+1)$ has a marker beside it, find r_0 and r_1 such that $\phi_{r_0} = \phi_{g(2n)}$ and $\phi_{r_1} = \phi_{g(2n+1)}$ $(r_0 \neq r_1)$ and (i) $t(r_0) \neq r_1$ and neither r_0, r_1, $t(r_0)$, nor $t(r_1)$ has a marker beside it, and (ii) neither r_0 nor r_1 belongs to $L_0 \cup L_1 \cup \{g(2n), g(2n+1)\}$. Place the marker Λ_{2n} beside r_0 and Λ_{2n+1} beside r_1 and go on to Step 2.

Step 2. After completing Step 1, we find that exactly one of $g(2n)$ and $g(2n+1)$ has a marker beside it. We shall assume the marker is beside $g(2n)$. The case where it is beside $g(2n+1)$ is treated similarly.

Case A. The marker beside $g(2n)$ is Λ_i. (So $\phi_{g(2n)} = \phi_{g(i)}$.) In this case we first compute $\phi_{g(2n)}(g(2n))$.

Subcase I. The marker Λ_i has at some stage in the construction been beside a number $m \neq g(2n)$, and $\phi_m(m)$ was computed and $\phi_m(m) = \phi_{g(2n)}(g(2n))$.

In this case $\phi_{g(i)}$ is not a 1-1 function which, according to (ii) of our preliminary discussion, spoils $\phi_{g(i)}$. Consequently we erase the marker Λ_i and place $g(2n)$ in L_0 and $g(2n+1)$ in L_1 and go on to Stage $n+1$.

Subcase II. $\phi_{g(2n)}(g(2n)) = g(2n)$ or $g(2n+1)$.

In the first of these cases, applying (i) of our preliminary discussion, $\phi_{g(i)}$ is automatically spoiled, so place $g(2n)$ in L_0 and $g(2n+1)$ in L_1. In the second case, applying (iv) of our preliminary discussion, $\phi_{g(i)}$ can be spoiled by placing $g(2n)$ in L_0 and $g(2n+1)$ in L_1. In either case, erase Λ_i, and go on to Stage $n+1$.

Subcase III. $\phi_{g(2n)}(g(2n)) \in L_q(n)$ $(q = 0$ or $1)$.

In this case, applying (iii) and (iv) of our preliminary discussion, spoil $\phi_{g(i)}$ by placing $g(2n)$ in L_{1-q}. Since $\phi_{g(i)}$ is spoiled, erase Λ_i. To preserve the isomorphism t between L_0 and L_1, place $g(2n+1)$ in L_q, and go on to Stage $n+1$.

Subcase IV. None of the preceding subcases applies, and neither $\phi_{g(2n)}(g(2n))$ nor $t(\phi_{g(2n)}(g(2n)))$ has a marker beside it.

In this case, erase Λ_i and place Λ_i' beside $\phi_{g(2n)}(g(2n))$. Then put $g(2n)$ in L_0 and $g(2n+1)$ in L_1, and go on to Stage $n+1$.

Subcase V. None of the preceding subcases applies, but either $\phi_{g(2n)}(g(2n))$ or $t(\phi_{g(2n)}(g(2n))$ has beside it a marker Λ_x or Λ_x' with $x > i$.

In this case find r such that $\phi_r = \phi_{g(x)}$, and (i) neither r nor $t(r)$ has a marker beside it, and (ii) $r \notin L_0(n) \cup L_1(n) \cup \{g(2n), g(2n+1)\}$. Erase Λ_x or Λ_x' and place Λ_x beside r. Then erase Λ_i and place Λ_i' beside $\phi_{g(2n)}(g(2n))$. Finally, put $g(2n)$ in L_0 and $g(2n+1)$ in L_1, and go on to Stage $n+1$.

Subcase VI. None of the preceding subcases applies. Thus either $\phi_{g(2n)}(g(2n))$ or $t(\phi_{g(2n)}(g(2n)))$ has beside it a marker Λ_x or Λ_x' with $x \leqq i$. (When the description of the construction is completed, it will be clear that $x \neq i$; hence $x < i$.)

In this case, find r such that $\phi_r = \phi_{g(i)}$ and (i) neither r nor $t(r)$ has a marker beside it, and (ii) $r \notin L_0(n) \cup L_1(n) \cup \{g(2n), g(2n+1)\}$. Remove Λ_i from $g(2n)$ and place it next to r. Then put $g(2n)$ in L_1 and $g(2n+1)$ in L_0 and go on to Stage $n+1$.

Case B. The marker beside $g(2n)$ is Λ_i'. In this case, by examining the construction we see that there exists r such that $\phi_r = \phi_{g(i)}$, $r \in L_0$, and $\phi_r(r) = g(2n)$. We spoil $\phi_{g(i)}$ by placing $g(2n)$ in L_1. Then we put $g(2n+1)$ in L_0, erase the marker Λ_i', and go on to Stage $n+1$.

This completes the description of the construction.

Since $x \in K$ if and only if $x = g(2n)$ or $g(2n+1)$ for some n, and since $g(2n) \in L_i$ $(i = 0$ or $1)$ if and only if $t(g(2n))$ $(= g(2n+1)) \in L_{1-i}$, we easily see that $L_0 \cap L_1 = \emptyset$, $L_0 \cup L_1 = K$, and $t(L_0) = L_1$ (so L_0 and L_1 are recursively isomorphic). Since L_0 and L_1 are clearly r.e., we can complete the proof of the theorem by showing that L_0 is not a 1-1 semi-cylinder.

In the following discussion, when we speak of *replacing* Λ_i we mean that Λ_i is erased *and* Λ_i' is introduced beside some number. Similarly, when we say that Λ_i' is *replaced* we mean that Λ_i' is erased *and* Λ_i is introduced beside some number. When in the following discussion we say that a marker is *erased* we mean that it is erased *and* not replaced.

At Stage 0, the markers Λ_0 and Λ_1 are introduced. At Step 2 of this stage, the marker Λ_0 is either erased or replaced by Λ_0', but Λ_0', although it can be erased, can never be replaced or moved. This establishes the basis for an inductive argument showing that no marker gets moved infinitely often.

Suppose all markers Λ_j and Λ_j' with $j < i$ have either been permanently erased or have reached a final resting spot by Stage n_0. Suppose also that the marker Λ_i or Λ_i' is moved infinitely often. Λ_i' can be moved only if it is first replaced by Λ_i, so we may suppose that at some Stage n_1 $(n_1 \geqq n_0)$ Λ_i

is next to some number r. Λ_i can then be moved or replaced only if $\phi_r(r)$ is computed and fails to belong to $L_0(n_1) \cup L_1(n_1)$ and either

(a) $\phi_r(r)$ or $t(\phi_r(r))$ has beside it a marker Λ_j or Λ'_j with $j < i$, in which case Λ_i is moved immediately, or

(b) $\phi_r(r)$ and $t(\phi_r(r))$ have no marker or a marker Λ_j or Λ'_j with $j > i$ beside them, in which case Λ'_i is placed beside $\phi_r(r)$, replacing Λ_i.

Since Case A, Subcase I of Step 2 of the construction is used to spoil functions which are not 1-1, case (a) of the preceding paragraph can occur at most finitely often after Stage n_1. Consequently, at some Stage n_2 ($n_2 \geqq n_0$), case (b) of the preceding paragraph must occur if Λ_i or Λ'_i is to be moved infinitely often. But no marker Λ_j or Λ'_j with $j < i$ ever gets moved after Stage n_0, so Λ'_i can never be replaced by Λ_i. Thus Λ'_i, although it may be erased, can never be replaced by Λ_i. Thus Λ'_i will never appear in another position, and Λ_i will never appear again. This shows that all markers get moved at most finitely often.

Furthermore, we see that the only way in which Λ_i can come permanently to rest beside some number r is for $\phi_r(r)$ never to be computed. Since ϕ_r must be $\phi_{g(i)}$, this cannot happen if $\phi_{g(i)}$ is a total function. Thus if $\phi_{g(i)}$ is defined everywhere, Λ_i cannot come permanently to rest beside some number.

Now if L_0 were a 1-1 semi-cylinder, there would be a 1-1 total recursive function ϕ_e such that $x \in L_0$ implies $\phi_e(x) \in L_0 - \{x\}$ and $x \in L'_0$ implies $\phi_e(x) \in L'_0 - \{x\}$. If ϕ_e is defined everywhere, every index for ϕ_e must belong to K ($= \{x \mid \phi_x(x)$ is defined$\}$). Therefore for some number n, $e = g(2n)$ or $g(2n+1)$. Suppose $e = g(2n)$. (The case where $e = g(2n+1)$ is treated similarly.) Then at Stage n, the marker Λ_{2n} is introduced. Either at some succeeding stage the symbols Λ_{2n} and Λ'_{2n} are both erased (and the function ϕ_e spoiled) or, since ϕ_e is total, the marker Λ'_{2n} must come to rest permanently beside a number m. But this implies that m fails to belong to K, and that there exists a number r such that $\phi_r(r) = m$, and $r \in L_0$, and $\phi_r = \phi_e$. In view of (iv) of our preliminary discussion, ϕ_e cannot have the properties described at the beginning of this paragraph. Thus L_0 is not a 1-1 semi-cylinder.

COROLLARY 1. *The sets L_0 and L_1 described in the statement of the theorem are bounded-truth-table-complete noncreative sets.*

Proof. Since a creative set is necessarily a 1-1 semi-cylinder, it suffices to show that the set K of the theorem is bounded-truth-table reducible to L_0. Let f be a recursive function such that $x \in L_1$ if and only if $f(x) \in L_0$. Then $x \in K$ if and only if $x \in L_0$ or $f(x) \in L_0$, which is an example of bounded-truth-table reducibility.

COROLLARY 2 (FISCHER). *Bounded-truth-table reducibility differs from many-one reducibility on the r.e. sets.*

Proof. This is now immediate since all many-one-complete sets are creative.

A noncreative r.e. set A has been called pseudo-creative if for every r.e. set $B \subset A'$ there is an infinite r.e. set $C \subset A'$ such that $B \cap C = \emptyset$. A standard method for showing that the class of pseudo-creative sets is nonempty is to consider the cylinder of any nonrecursive, noncreative, r.e. set. (See [3].) In another paper [7], we show that there is a pseudo-creative set which is not a cylinder but which is bounded-truth-table reducible to a simple set. Corollary 1 can be used to obtain additional information about pseudo-creative sets which are not cylinders.

COROLLARY 3. *There is a pseudo-creative set which is not a cylinder and which is bounded-truth-table reducible, among all the r.e. sets, only to pseudo-creative sets and to creative sets.*

Proof. From the following lemma and from the transitivity of bounded-truth-table reducibility, it follows that any bounded-truth-table-complete noncreative set satisfies the conditions of the corollary.

LEMMA (POST-SHOENFIELD). *Every bounded-truth-table-complete set is either creative or pseudo-creative.*

Proof. In [4] it is proved that no bounded-truth-table-complete set is either recursive or simple. In [6] it is proved that if A is r.e. but not recursive and if there is an infinite r.e. subset $B \subset A'$ such that $A \cup B$ is simple (i.e., if A is pseudo-simple), then A is not bounded-truth-table-complete. It is easily shown that any r.e. set which is neither recursive, simple, nor pseudo-simple is either creative or pseudo-creative. (See [3].)

REMARK. The proofs in [4] and [6] on which this lemma depends are rather involved. It has been pointed out to the author by T. G. McLaughlin that an easier proof of Corollary 3 is obtained by observing that the class of recursive, simple, and pseudo-simple sets forms a lattice under intersection and union. Hence the sets L_0 and L_1 must be either pseudo-creative or creative.

II. AN INFINITE SPLINTER WHICH IS NOT A CYLINDER

Introduction. Ullian has defined a splinter to be a set of the form $\{f^i(n) \mid i \in N\}$ where n is some fixed integer and f is a total recursive function. A 1-1 splinter is a splinter of a 1-1 recursive function. It can be shown that an infinite and coinfinite r.e. set is a cylinder if and only if it is an infinite and coinfinite 1-1 splinter. Myhill has raised the question of whether every infinite and coinfinite splinter is a cylinder [3].

In this part of the paper we construct an infinite and coinfinite splinter which is not a cylinder.

THEOREM. *There exists an infinite and coinfinite splinter which is not a semi-cylinder.*

Proof. Let $K = \{x \mid \phi_x(x) \text{ converges}\}$. Let g be any 1-1 recursive function whose range is K.

We will proceed in stages, constructing a total recursive function h such that $S = \{h^i(1) \mid i \in N\}$ will be the desired splinter. $\{h^i(0) \mid i \in N\}$ will be a subset of S'.

As we proceed with our construction, we will use an infinite collection of sets, $\{S_k^{(n)}\}$ $(k, n \in N)$. For all k, $S_k^{(0)} = \emptyset$; if $S_k^{(n)} \neq \emptyset$, $S_k^{(n)}$ will be linearly ordered under the relation $a \prec b$ if and only if there is some $i > 0$ such that $h^i(a) = b$. If $S_k^{(n)} \neq \emptyset$, we define $r_{k,0}^{(n)}$ to be the first member of $S_k^{(n)}$ under the order $\prec$, and we define $r_{k,1}^{(n)}$ to be the last member of $S_k^{(n)}$ under the order $\prec$.

We define o_n to be that unique y for which there exists $j \geqq 0$ such that $h^j(0) = y$ and such that $h(y)$ is not defined by the end of Stage n. We define i_n to be that unique z for which there exists $j \geqq 0$ such that $h^j(1) = z$ and such that $h(z)$ is not defined by the end of Stage n.

Construction.

Stage 0. Define $i_0 = 1$ and $o_0 = 0$.

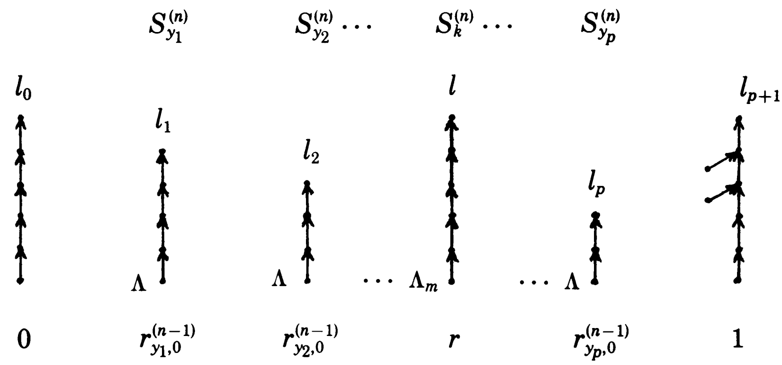
Stage n $(n > 0)$. Let $F_n = \{y \mid S_y^{(n-1)} \neq \emptyset$ and during Stage $n-1$ it was not specified that $S_y^{(n)} = \emptyset\}$. This is a finite set, so we may suppose it has p members, $y_1, y_2, \cdots, y_p$. Let $l_0, l_1, \cdots, l_p, l_{p+1}$ be the smallest $p+2$ integers which have not appeared in our construction of h. Define $h(r_{y_k,1}^{(n-1)}) = l_k$ $(1 \leqq k \leqq p)$, $h(o_{n-1}) = l_0$, and $h(i_{n-1}) = l_{p+1}$. For $1 \leqq k \leqq p$, define $S_{y_k}^{(n)} = S_{y_k}^{(n-1)} \cup \{l_k\}$. If $y \notin F_n$, we do not yet define $S_y^{(n)}$, but instead wait until the end of Stage n, at which time we define $S_y^{(n)} = \emptyset$ unless $S_y^{(n)}$ is defined to be a unit set during Stage n, as described below.

Notation. When in the following we say "(re)introduce the marker Λ_q," we mean: find a z such that z has not yet been used as an index, i, for some $S_i^{(w)}$ with $S_i^{(w)} \neq \emptyset$; then find a q' such that q' has not appeared in our construction of h and such that $\phi_{q'} = \phi_{g(q)}$; define $S_z^{(n)} = \{q'\}$ and place the marker Λ_q beside q'.

Now compute $g(n-1)$ and introduce the marker Λ_{n-1}, introducing $g(n-1)$ and placing Λ_{n-1} beside $g(n-1)$ if $g(n-1)$ has not yet appeared in our construction of h. Then if $g(n-1)$ fails to have beside it a marker, Λ_m, go on to Stage $n+1$. Otherwise, as can be seen as the construction is fully described, $g(n-1) = r_{k,0}^{(n)}$ for some k. For the duration of Stage n, we abbreviate $r_{k,0}^{(n)}$ to r. Since $g(n-1) = r \in K$, $\phi_r(r)$ converges; we compute $\phi_r(r)$. (As the construction is fully described, it will be seen that $\phi_r = \phi_{g(m)}$. The remainder of Stage n is an attempt to prevent $\phi_{g(m)}$ from being a function which would make S a semi-cylinder.)

FIGURE 1.

The graph of h during the examination of cases in Stage n.



Case A. $\phi_r(r) = r$. In this case, $\phi_{g(m)}$ cannot be the function required by the definition if S is to be a semi-cylinder. Erase the marker Λ_m, set $h(l_0) = r$, and define $S_k^{(n+1)} = \emptyset$. Then go on to Stage $n+1$.

Case B. $\phi_r(r) \in S_j^{(n)}$ for $j \neq k$. Let the marker beside $r^{(n)}$ be Δ_x.

Subcase I. $x < m$. In this case define $h(l_{p+1}) = r$. Erase Λ_m, making its removal contingent upon S_j remaining in S'. Now for those markers, $\Lambda_{q_0}, \cdots, \Lambda_{q_d}$, whose removal was contingent upon S_k remaining in S', we successively reintroduce the markers $\Lambda_{q_0}, \cdots, \Lambda_{q_d}$. We then define $S_k^{(n+1)} = \emptyset$ and go on to Stage $n+1$.

Subcase II. $m \leqq x$. (As the construction is fully described, it can be seen that in fact m must be less than x.) In this case define $h(l_{p+1}) = r$ and $h(l_0) = r_{j,0}^{(n)}$. Erase marker Λ_x from its position beside $r_{j,0}^{(n)}$, and then reintroduce the marker Λ_x. For those markers, $\Lambda_{q_0}, \cdots, \Lambda_{q_d}$, whose removal was contingent upon S_k remaining in S', successively reintroduce the markers $\Lambda_{q_0}, \cdots, \Lambda_{q_d}$. Erase marker Λ_m, define $S_k^{(n+1)} = \emptyset = S_j^{(n+1)}$, and go on to Stage $n+1$.

Case C. $\phi_r(r) \in S_k^{(n)}$ but $\phi_r(r) \neq r$. In this case, define $h(l_{p+1}) = h(r)$. (This puts r in S' and $\phi_r(r)$ in S.) Erase the marker Λ_m and set $S_k^{(n+1)} = \emptyset$. Now for those markers, $\Lambda_{q_0}, \cdots, \Lambda_{q_d}$, whose removal was contingent upon S_k remaining in S', successively reintroduce the markers $\Lambda_{q_0}, \cdots, \Lambda_{q_d}$. Then go on to Stage $n+1$. (As the construction is fully described, one can see that for any such $q_j, q_j > m$.)

Case D. $\phi_r(r) = h^i(0)$ for some $i \geqq 0$ *or* $\phi_r(r) \in h^{-1}h^i(1)$ but $\phi_r(r) \neq h^{i-1}(1)$ for some $i > 0$. (In this case $\phi_r(r) \in S'$.) Define $h(l_{p+1}) = r$, erase marker Λ_m, and define $S_k^{(n+1)} = \emptyset$. For those markers, $\Lambda_{q_0}, \cdots, \Lambda_{q_d}$, whose removal was contingent upon S_k remaining in S', successively reintroduce the markers $\Lambda_{q_0}, \cdots, \Lambda_{q_d}$. Then go on to Stage $n+1$.

Case E. $\phi_r(r) = h^i(1)$ for some $i \geqq 0$. In this case define $h(l_0) = r$, erase the marker Λ_m, set $S_k^{(n+1)} = \emptyset$, and go on to Stage $n+1$.

Case F. $\phi_r(r)$ has not yet appeared in our construction of h. In this case, define $h(l_{p+1}) = \phi_r(r)$ and $h(l_0) = r$. Then erase Λ_m, set $S_k^{(n+1)} = \emptyset$, and go on to Stage $n+1$.

This completes the description of the construction of h.

It is clear that h is a total recursive function and that $S = \{h^i(1): i \in N\}$ is an infinite splinter. In order to show that S is not a semi-cylinder, we first show, by induction, that every marker comes permanently to rest beside some number or is permanently erased.

The marker Λ_0 is introduced at Stage 1, and can never be moved or reintroduced (though it can be erased).

Now suppose all of the markers $\Lambda_0, \cdots, \Lambda_{n-1}$ have been either permanently erased or have come permanently to rest by Stage m_0. If the marker Λ_n were not to satisfy these conditions, it would at some Stage m_1, with $m_1 > m_0$, be beside a number, r, from which it would be moved or removed at Stage m_1. Basically there are only two ways this could occur: either $g(m_1 - 1) = r$ or $g(m_1 - 1) = l$ where l has beside it a marker Λ_q with $q < n$. The latter case cannot occur, for an examination of cases shows that Λ_q would then be either moved or removed, contradicting our induction hypothesis. In the former case, Λ_n must be erased and its erasure is permanent unless Subcase I of Case B has obtained. But in this latter event, the removal of Λ_n is contingent upon S_j remaining in S', where the first element of $S_j^{(m_1)}$ has beside it a marker Λ_q with $q < n$. Thus reintroducing the marker Λ_n requires moving or removing the marker Λ_q, contrary to our induction hypothesis.

Now suppose S were a semi-cylinder with a total recursive function ϕ_e such that $x \in S$ implies $\phi_e(x) \in S - \{x\}$ and $x \in S'$ implies $\phi_e(x) \in S' - \{x\}$. Since ϕ_e is total, $\phi_e(e)$ is defined, so $e = g(n)$ for some n. At Stage $n+1$ the marker Λ_n is introduced. If Λ_n were ever permanently erased, by examining the various cases, we see that ϕ_e could not be as described. Therefore we may assume that Λ_n comes permanently to rest beside some number s. But by our construction, this implies that $\phi_e = \phi_s$ and hence, since ϕ_e is total, $s \in K$. Since s cannot have appeared in our construction before Λ_n is placed beside it, $s = g(n_0)$ for some $n_0 > n$, and so at Stage $n_0 + 1$ we compute $\phi_s(s)$. Examining our construction, we see that Λ_n is then either moved or erased. This contradiction shows that ϕ_e cannot be the desired function, and hence S is not a semi-cylinder. Since both S and S' have two elements by the end of Stage 1, neither is a unit set, and so both must be infinite.

COROLLARY. *There is an infinite and coinfinite splinter which is not a cylinder.*

Proof. Every cylinder is a semi-cylinder.

REMARKS. 1. The set S is "almost" a 1-1 splinter in the following sense: If $x \in S'$, $h^{-1}(x)$ has at most one element. If $x \in S$, $h^{-1}(x)$ has at most two elements and $h^{-2}(x)$ has at most one element.

2. In Part I and in [7], we have constructed pseudo-creative sets which are not cylinders. Since Ullian has shown that all splinters are recursive, creative, or pseudo-creative (see [3], and see Part I for a definition of pseudo-creative), the set S of the Theorem must also be a pseudo-creative noncylinder. The question of whether all pseudo-creative sets are splinters remains open.

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