SOME TYPES OF BANACH SPACES, HERMITIAN OPERATORS, AND BADE FUNCTIONALS(1)

BY

EARL BERKSON

Introduction. In [6] and [11] a general notion of hermitian operator has been developed for arbitrary complex Banach spaces (see §1 below). In terms of this notion, a family of operators on a Banach space is said to be hermitian-equivalent if the operators of this family can be made simultaneously hermitian by equivalent renorming of the underlying space [7]. Let X be a complex Banach space with norm $\| \quad \|$, and let F be a commutative hermitian-equivalent family of operators on X. A norm for X equivalent to $\| \quad \|$, and relative to which the operators of F are hermitian will be called an F-norm. Such families have been studied in [7], where it is shown that if X is a Hilbert space, then there is an F-norm which is also a Hilbert space norm. It is natural to seek other properties which, if enjoyed by X, can be preserved by choosing an F-norm appropriately. Such an investigation is conducted in this paper. Specifically, we show in §§3 and 4 that if the Banach space X has uniformly Fréchet differentiable norm (resp., is uniformly convex), then there is an F-norm which preserves uniform Fréchet differentiability (resp., uniform convexity). Moreover, we show in §6 that if $X = L^p(\mu)$, $\infty > p > 1$, μ a measure, then there is an F-norm which preserves both uniform Fréchet differentiability and uniform convexity.

Our result for the case where X is uniformly convex enables us to establish in Theorem (5.4) a strong link between the notions of semi-inner-product (see §1) and Bade functional. This link adds to the analogy with Hilbert space inherent in these notions.

Throughout this paper all spaces are over the complex field, and an operator will be a bounded linear transformation with range contained in its domain. In some cases it will be convenient to employ a notation for Banach spaces which explicitly exhibits the norm. Thus, if Y is a linear space, and $| \quad |$ is a Banach space norm for Y, we shall sometimes designate the resulting Banach space by $(Y, | \quad |)$.

1. **Semi-inner-products and Hermitian operators.** In this section and the next we reproduce some machinery from other papers which will be needed

---

Received by the editors March 17, 1964.

(1) The research contained in this paper was done under NSF Grant GP 1846.

in the sequel. The definitions and results stated in §1 are taken from [6] except where otherwise specified.

DEFINITION. Let X be a vector space. A semi-inner-product (abbreviated s.i.p.) on X is a mapping $[\ ,\ ]$ of $X \times X$ into the field of complex numbers such that:

(i) $[x+y,z] = [x,z] + [y,z]$ for $x,\ y,\ z \in X$.

(ii) $[\lambda x,y] = \lambda[x,y]$ for $x,\ y \in X,\ \lambda$ complex.

(iii) $[x,x] > 0$ for $x \neq 0$.

(iv) $|[x,y]|^2 \leqq [x,x][y,y]$ for $x,\ y \in X$.

When a s.i.p. is defined on X, we call X a semi-inner-product space (abbreviated s.i.p.s.).

If X is a s.i.p.s., then $[x,x]^{1/2}$ is a norm on X. On the other hand every normed linear space can be made into a s.i.p.s. (in general, in infinitely many ways) so that the s.i.p. is consistent with the norm, i.e., $[x,x]^{1/2} = \|x\|$, for each $x \in X$. By virtue of the Hahn-Banach theorem this can be accomplished by choosing for each $x \in X$ exactly one bounded linear functional f_x such that $\|f_x\| = \|x\|$ and $f_x(x) = \|x\|^2$, and then setting $[x,y] = f_y(x)$, for arbitrary $x,\ y \in X$.

DEFINITION. Given a linear transformation T mapping a s.i.p.s. into itself, we denote by $W(T)$ the set, $\{[Tx,x] \mid [x,x] = 1\}$, and call this set the numerical range of T.

Let T be an operator on the Banach space $(X, \|\ \ \|)$. Although in principle there may be many different semi-inner-products consistent with $\|\ \ \|$, nonetheless if the numerical range of T relative to one such s.i.p. is real, then the numerical range relative to any such s.i.p. is real. If this is the case, T is said to be a hermitian operator.

In [11] I. Vidav introduces the following notion of hermiticity:

DEFINITION. An element h of a Banach algebra with identity of norm 1 will be called hermitian if and only if for α real, $\|1 + i\alpha h\| = 1 + o(\alpha)$ as $\alpha \to 0$.

It is shown in [6, §9] that an operator T on the Banach space $(X, \|\ \ \|)$ is a hermitian operator (in the first sense described above) if and only if it is hermitian in the sense of Vidav's definition, i.e., if and only if for α real, $\|I + i\alpha T\| = 1 + o(\alpha)$, where I is the identity operator. Thus we have at our disposal two equivalent formulations of the notion of hermitian operator.

2. **Hermitian equivalence.** The next two definitions are taken from [7].

DEFINITION. Let F be a commutative set of operators on the Banach space X, and denote by $L(F)$ the real linear span of F in the space of operators on X. The exponential group of F is the set, $\{e^{iT} \mid T \in L(F)\}$. We denote this set by $G(F)$.

Definition. A set S of operators on the Banach space X is said to be hermitian-equivalent (resp., hermitian) if and only if there is an equivalent renorming of X which makes the operators of S hermitian (resp., S consists of hermitian operators).

Very important for our purposes is the following [7, Theorem 6]:

(2.1) Let X be a Banach space and F a commutative family of operators on X. Then F is hermitian-equivalent if and only if $G(F)$ is uniformly bounded.

Moreover, it is easy to see from the proof of [7, Theorem 6] the following fact:

(2.2) Let F be a commutative family of operators on the Banach space $(X, \| \ \|)$. Then F is a hermitian family if and only if $G(F)$ is a group of isometries on $(X, \| \ \|)$.

Definition. Let F be a commutative hermitian-equivalent family of operators on the Banach space $(X, \| \ \|)$. A norm on X equivalent to $\| \ \|$ and relative to which F is a hermitian family will be called an F-norm.

Throughout what follows $(X, \| \ \|)$ will be a Banach space, and G will be a commutative, uniformly bounded, multiplicative group of operators on $(X, \| \ \|)$, containing I. Under various hypotheses on $(X, \| \ \|)$, we shall give procedures for obtaining a norm equivalent to $\| \ \|$, having specified properties, and making the operators of G isometries. By (2.1) these procedures will apply to the exponential group of a commutative hermitian-equivalent family F, and will supply, by (2.2), an F-norm possessing the specified properties. This fact will be used without further mention.

3. The p-norms, uniform Fréchet differentiability.

Let $B(G)$ denote the set of bounded complex-valued functions on G. We now choose an invariant mean J on $B(G)$, i.e., a positive linear functional invariant under translation with respect to G in the usual sense, and such that $J(1) = 1$, 1 denoting the function identically 1 on G. The existence of such a functional for any abstract commutative group can be seen, as in [7, §2], by temporarily imposing the discrete topology on the group and applying [3, p. 115]. The functional J will be fixed throughout this paper.

Suppose $1 \leqq p < \infty$. For fixed $x \in X$, the function which assigns to each $T \in G$ the number $\|Tx\|^p$ is clearly in $B(G)$. We define $| \ |_p$ on X as follows:

$$|x|_p = [J(\|Tx\|^p)]^{1/p}, \quad \text{for } x \in X.$$

(3.1) Theorem. *For $1 \leqq p < \infty$, $| \ |_p$ is a norm on X equivalent to $\| \ \|$. Relative to $| \ |_p$, each $T \in G$ is an isometry.*

Proof. (For the case $p = 1$ see also the proof of [7, Theorem 6].) For each subset M of G with characteristic function C_M, let $\nu(M) = J(C_M)$.

Then ν is a finitely additive function on the power class of G. For each $f \in B(G)$, there is a sequence of functions of finite range tending to f uniformly on G. From this it is easy to see that in terms of the integration theory for finitely additive set functions given in [5, III. 1-III. 3], each $f \in B(G)$ is ν-integrable and $J(f) = \int_G f d\nu$. By [5, III. 3.3] Minkowski's Inequality is valid for ν. Hence for $1 \leqq p < \infty$ and $f, g \in B(G)$,

$$(3.2) \qquad [J(|f+g|^p)]^{1/p} \leqq [J(|f|^p)]^{1/p} + [J(|g|^p)]^{1/p}.$$

It is easy to see from (3.2) that $| \quad |_p$ is a seminorm. From now on, let K be an upper bound for $\{\|T\| \mid T \in G\}$. For $x \in X$, $T \in G$, we have

$$(3.3) \qquad K^{-1}\|x\| = K^{-1}\|T^{-1}Tx\| \leqq \|Tx\| \leqq K\|x\|.$$

From (3.3) it follows that

$$(3.4) \qquad K^{-1}\|x\| \leqq |x|_p \leqq K\|x\|.$$

Hence $| \quad |_p$ is a norm equivalent to $\| \quad \|$. The fact that each $T \in G$ is an isometry relative to $| \quad |_p$ follows from the invariance of J.

Definition. $\| \quad \|$ is said to be uniformly Fréchet differentiable (abbreviated (UF)) if and only if for t real, $t^{-1}(\|x + th\| - 1)$ tends uniformly to a limit as $t \to 0$ for $\|x\| = \|h\| = 1$. For $x \neq 0$, let $x' = \|x\|^{-1}x$. By observing that

$$t^{-1}(\|x + th\| - \|x\|) = \|h\| \left(\frac{t\|h\|}{\|x\|}\right)^{-1}\left(\left\|x' + t\frac{\|h\|}{\|x\|}h'\right\| - 1\right),$$

it is straightforward to verify that $\| \quad \|$ is (UF) if and only if for each pair (r_1, r_2) of positive numbers, $t^{-1}(\|x + th\| - \|x\|)$ tends uniformly to a limit as $t \to 0$, for $r_1 \leqq \|x\|$ and $\|h\| \leqq r_2$.

(3.5) Theorem. *If $\| \quad \|$ is (UF), then so is each $| \quad |_p$.*

Proof. Set $G(u, v) = \lim_{t \to 0} t^{-1}(\|u + tv\| - \|u\|)$, t real, $u, v \in X$, $u \neq 0$. If $|x|_p = |h|_p = 1$ and $T \in G$, we have by (3.4) and the fact that T is a $| \quad |_p$-isometry that $K^{-1} \leqq \|Tx\| \leqq K$ and $K^{-1} \leqq \|Th\| \leqq K$. Thus for $|x|_p = |h|_p = 1$, $T \in G$, and t real, we have:

$$(3.6) \qquad \begin{aligned} &\|Tx + tTh\| = \|Tx\| + tG(Tx, Th) + t\alpha(T, x, h, t), \\ &\qquad \text{where } \alpha(T, x, h, t) \to 0 \text{ as } t \to 0, \quad \text{uniformly in } T, x, h. \end{aligned}$$

By (3.6) and the law of the mean

$$\begin{aligned} \|Tx + tTh\|^p - \|Tx\|^p = p[&tG(Tx, Th) + t\alpha(T, x, h, t)] \\ &\cdot [\|Tx\| + \theta\{tG(Tx, Th) + t\alpha(T, x, h, t)\}]^{p-1}, \end{aligned}$$

where θ depends on t, x, h, t and satisfies $0 < \theta < 1$.

From this last equation, it is easy to see that

$$\|Tx + tTh\|^p = \|Tx\|^p + ptG(Tx, Th)\|Tx\|^{p-1} + t\beta(T, x, h, t), \tag{3.7}$$
$$\text{where } \beta(T, x, h, t) \to 0 \text{ as } t \to 0, \quad \text{uniformly in } T, x, h.$$

Applying J to (3.7) and taking $1/p$ powers, we get (since $|x|_p = 1$):

$$|x + th|_p = \{1 + ptJ[G(Tx, Th)\|Tx\|^{p-1}] + tJ[\beta(T, x, h, t)]\}^{1/p}. \tag{3.8}$$

It should be noted that since (see the proof of (3.1)) $J(f) = \int_G f d\nu$ for $f \in B(G)$, one has $|J(f)| \leqq J(|f|)$. Hence

$$J[\beta(T, x, h, t)] \to 0 \quad \text{as } t \to 0 \text{ uniformly in } x, h.$$

With the aid of this fact and the differentiability of $(1 + c)^{1/p}$ at $c = 0$, we get from (3.8) that

$$|x + th|_p = 1 + tJ[G(Tx, Th)\|Tx\|^{p-1}] + t\gamma(x, h, t),$$
$$\text{where } \gamma(x, h, t) \to 0 \text{ as } t \to 0, \quad \text{uniformly in } x, h.$$

Hence $t^{-1}(|x + th|_p - 1)$ tends to $J[G(Tx, Th)\|Tx\|^{p-1}]$ as $t \to 0$ uniformly for $|x|_p = |h|_p = 1$, and so $|\ \ |_p$ is (UF).

(3.9) COROLLARY. *If F is a commutative, hermitian-equivalent family of operators on $(X, \|\ \ \|)$, and $\|\ \ \|$ is (UF), then there is an F-norm which is (UF).*

4. The uniformly convex case.

DEFINITION ([4]). $(X, \|\ \ \|)$ is said to be uniformly convex (abbreviated (UC)) if and only if to each ϵ, $0 < \epsilon \leqq 2$, there corresponds a $\delta(\epsilon) > 0$ such that the conditions $\|x\| = \|y\| = 1$, $\|x - y\| \geqq \epsilon$ imply $\|x + y\| \leqq 2(1 - \delta(\epsilon))$.

Equivalently, $(X, \|\ \ \|)$ is (UC) if and only if whenever $\{x_n\}$, $\{y_n\}$ are sequences of vectors with $\|x_n\|$, $\|y_n\| \leqq 1$, for each n, and $\|x_n + y_n\| \to 2$, then $\|x_n - y_n\| \to 0$.

The key fact for our considerations in this section is the following ([10]):

(4.1) A Banach space is (UC) (resp., has its norm (UF)) if and only if the norm in the dual space is (UF) (resp., the dual space is (UC)).

(4.2) THEOREM. *If $(X, \|\ \ \|)$ is (UC), then there is an equivalent renorming of X which makes each $T \in G$ an isometry and gives again a (UC) space.*

Proof. We denote the dual space of $(X, \|\ \ \|)$ by $(X^*, \|\ \ \|)$. By (4.1) and application of (3.1) and (3.5) to the group G^* of adjoints of operators of G, we get an equivalent renorming of X^* which yields a (UF) norm and makes each T^*, for $T \in G$, an isometry. This, in turn, gives by (4.1) an equivalent renorming of X^{**} which yields a (UC) space and makes each

T^{**}, for $T \in G$, have norm 1. A (UC) Banach space is reflexive ([8],[9]). In view of this fact (or merely by recalling that the second adjoint of an operator "extends" the operator), we can apply the inverse of the natural mapping of X into X^{**} to get a norm $|\ \ |$ on X equivalent to $\|\ \ \|$, with $(X,|\ \ |)$ (UC) and $|T| \leqq 1$, for $T \in G$. Thus for each $x \in X$ and $T \in G$, $|x| = |T^{-1}Tx| \leqq |Tx| \leqq |x|$. Hence each $T \in G$ is an isometry relative to $|\ \ |$.

(4.3) COROLLARY. *If F is a commutative, hermitian-equivalent family of operators on $(X,\|\ \ \|)$, and $(X,\|\ \ \|)$ is (UC), then an F-norm can be chosen so as to obtain again a (UC) space.*

5. Bade functionals and semi-inner-products.

DEFINITION (see [1]). Let $x \in X$ and let $\mathscr{B}$ be a bounded Boolean algebra (abbreviated B.A.) of projection operators with domain X. A continuous linear functional $x^* \in X^*$ such that

(i) $x^*Ex \geqq 0$, for $E \in \mathscr{B}$,

(ii) $x^*Ex = 0$, with $E \in \mathscr{B}$, implies $Ex = 0$,

will be called a Bade functional for x with respect to $\mathscr{B}$.

By [1, 2.9 and 3.1] if X is weakly complete, then there is a Bade functional with respect to $\mathscr{B}$ for each $x \in X$. In particular, if X is (UC), then it is reflexive, and the existence of a Bade functional for each $x \in X$ is assured. We show in Theorem (5.4) that in the case of uniform convexity, Bade functionals with respect to $\mathscr{B}$ can be obtained from any s.i.p. consistent with an appropriate norm equivalent to $\|\ \ \|$ and preserving uniform convexity.

DEFINITION. $(X,\|\ \ \|)$ is said to be strictly convex (abbreviated (SC)) if and only if each point of the unit surface is an extreme point of the unit ball.

As is well known, uniform convexity implies strict convexity.

We remark that it is possible to characterize strict convexity in terms of the notion of semi-inner-product as follows:

(5.1) THEOREM. *Let $[\ \ ,\ \ ]$ be a s.i.p. for X consistent with $\|\ \ \|$. Then $(X,\|\ \ \|)$ is (SC) if and only if whenever $[x,y] = \|x\|\ \|y\|$, with $x \neq 0$, then $y = cx$, for some $c \geqq 0$.*

Since we do not need to make use of (5.1), we omit its proof for expository reasons.

(5.2) LEMMA. *Let u be a nonzero hermitian idempotent in a Banach algebra $(A,|\ \ |)$, A having an identity of norm 1. Then $|u| = 1$.*

Proof. Clearly the spectrum of u has a maximum of 1. By [11, Lemma 3], $|e^{tu}| = e^t$, for $t \geqq 0$. It is easy to see from the series expansion for the exponential function and from the idempotence of u that $e^{tu} = 1 + (e^t - 1)u$. Thus $|1 + (e^t - 1)u| = e^t$. Multiplying by e^{-t} gives $|e^{-t} + (1 - e^{-t})u| = 1$. Letting $t \to +\infty$, we get $|u| = 1$.

(5.3) LEMMA. *If, relative to* $\|\ \|$, *E is a hermitian projection with domain X, and* $[\ ,\ ]$ *is a s.i.p. consistent with* $\|\ \|$, *then:*

(i) $[Ex, x] \geqq 0$, *for* $x \in X$.

(ii) *If* $(X, \|\ \|)$ *is (SC), then* $[Ex, x] = 0$ *implies* $Ex = 0$.

Proof. By [7, Proof of Lemma 14], the infimum of the numerical range of E and the minimum of the spectrum of E are identical. Hence the numerical range of E is non-negative. By [6, Theorem 11], if the numerical range of an operator T is non-negative, then $[Tx, x] \geqq \mathbf{0}$, for each $x \in X$. This proves (i). To prove (ii), suppose x is a nonzero vector with $[Ex, x] = 0$. Then for $0 \leqq t \leqq 1$, let $y_t = tx + (1 - t)(I - E)x = x - (1 - t)Ex$. Thus $[y_t, x] = [x, x] = \|x\|^2$. Hence $\|y_t\| \geqq \|x\|$. Since $I - E$ is hermitian, we have by (5.2) that $\|y_0\| = \|(I - E)x\| \leqq \|x\|$. It follows that $\|(I - E)x\|$ $\|x\| = \|y_t\|$ for each t. By strict convexity $x = (I - E)x$, and hence $Ex = 0$.

REMARK. Conclusion (ii) above is no longer valid if the restriction of strict convexity is removed. This is shown by the following simple example. Let $X = 1^\infty(2)$ (i.e., the space of all ordered pairs of complex numbers, with $\|(\alpha, \beta)\| = \max\{|\alpha|, |\beta|\}$). The operator E defined by $E\{(\alpha, \beta)\} = (\alpha, 0)$ is a projection, and it is easy to verify that for real t, $\|I + itE\| = |1 + it|$. Hence E is hermitian. Let $x = (1/2, 1)$. Then $\|x\| = 1$, and a simple calculation shows that the only linear functional of norm 1 assuming the value 1 at x is the functional x^* given by $x^*\{(\alpha, \beta)\} = \beta$. Hence for any s.i.p. $[\ ,\ ]$ consistent with $\|\ \|$, the linear functional $[\ , x]$ must be x^*, and thus $[Ex, x] = x^*\{(1/2, 0)\} = 0$, while $Ex = (1/2, 0) \neq 0$.

(5.4) THEOREM. *If* $(X, \|\ \|)$ *is (UC), and* $\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n$, *are commuting bounded B.A.'s of projections with domain X, then there is a norm* $|\ |$ *on X equivalent to* $\|\ \|$ *such that* $(X, |\ |)$ *is (UC), and such that* $\mathscr{B} = \bigcup_{j=1}^n \mathscr{B}_j$ *is a hermitian family. Moreover, if* $[\ ,\ ]$ *is any s.i.p. consistent with* $|\ |$, *then for each* $x \in X$, $[\ , x]$ *is a Bade functional with respect to each* $\mathscr{B}_j$, $j = 1, 2, \cdots, n$.

Proof. The proof of [2, Lemma 2.3], though stated for the resolution of the identity of a scalar type operator, easily adapts to an arbitrary bounded Boolean algebra of projections and gives the result that an arbitrary bounded B.A. of projections is hermitian-equivalent. It now follows from (2.1) (see [7, Corollary 7]) that $\mathscr{B}$ is hermitian-equivalent. By (4.3) there is a $\mathscr{B}$-norm $|\ \ |$ such that $(X, |\ \ |)$ is (UC). The final assertion follows by applying (5.3) to the Banach space $(X, |\ \ |)$.

6. **The space** $L^p(\mu)$. In this section, we take $X = L^p(\mu)$, where $\infty > p > 1$, and μ is a measure. $\|\ \ \|$ will be the standard $L^p(\mu)$ norm, and q will be the index conjugate to p. We shall need the following inequalities due to James A. Clarkson [4, Theorem 2]:

(6.1) If $p \geqq 2$, then

$$2(\|x\|^p + \|y\|^p) \leqq \|x+y\|^p + \|x-y\|^p \leqq 2^{p-1}(\|x\|^p + \|y\|^p).$$

(6.2) If $1 < p \leqq 2$, then

$$\|x+y\|^q + \|x-y\|^q \leqq 2(\|x\|^p + \|y\|^p)^{q-1}.$$

We now return to our consideration of the group G in this setting.

(6.3) Theorem. (i) *If* $p \geqq 2$, *then* (6.1) *is also valid for* $|\ \ |_p$.
(ii) *If* $1 < p < 2$, *then* (6.2) *is also valid for* $|\ \ |_q$.

Proof. We first consider (i). In this case by (6.1), we have for $x, y \in X$, $T \in G$:

$$2(\|Tx\|^p + \|Ty\|^p) \leqq \|T(x+y)\|^p + \|T(x-y)\|^p \leqq 2^{p-1}(\|Tx\|^p + \|Ty\|^p).$$

The desired conclusion now follows upon application of J.

To prove (ii) we first observe that by (6.2) we have for $x, y \in X$, $T \in G$:

$$\|T(x+y)\|^q + \|T(x-y)\|^q \leqq 2(\|Tx\|^p + \|Ty\|^p)^{q-1}.$$

Applying J, we get:

$$|x+y|_q^q + |x-y|_q^q \leqq 2J[(\|Tx\|^p + \|Ty\|^p)^{q-1}]. \tag{6.4}$$

Since $q - 1 > 1$ here, we can apply (3.2) to $q-1$ and the functions f, g given by $f(T) = \|Tx\|^p$, $g(T) = \|Ty\|^p$ to obtain:

$$\{J[(\|Tx\|^p + \|Ty\|^p)^{q-1}]\}^{1/(q-1)} \leqq [J(\|Tx\|^{p(q-1)})]^{1/(q-1)} + [J(\|Ty\|^{p(q-1)})]^{1/(q-1)}. \tag{6.5}$$

Since $p(q-1) = q$, we get from (6.4) and (6.5)

$$|x+y|_q^q + |x-y|_q^q \leqq 2(|x|_q^p + |y|_q^p)^{q-1}.$$

This completes the proof.

(6.6) THEOREM. *If $p \geqq 2$, then $(X, |\ \ |_p)$ is (UC), while if $1 < p < 2$, $(X, |\ \ |_q)$ is (UC).*

Proof. Consider first the case $p \geqq 2$. If $\{x_n\}$, $\{y_n\}$ are sequences of vectors with $|x_n|_p$, $|y_n|_p \leqq 1$ and $|x_n + y_n|_p \to 2$, then applying conclusion (i) of (6.3) we get:

$$|x_n + y_n|_p^p + |x_n - y_n|_p^p \leqq 2^p.$$

Thus $0 \leqq |x_n - y_n|_p \leqq (2^p - |x_n + y_n|_p^p)^{1/p}$. Hence $|x_n - y_n|_p \to 0$. The proof for the case $1 < p < 2$ proceeds similarly from conclusion (ii) of (6.3). Since $L^p(\mu)$, with the norm $\|\ \ \|$, is the dual space of $L^q(\mu)$, and the latter space is (UC) (see [4]), it follows from (4.1) that $\|\ \ \|$ is (UF). Thus with the aid of (3.5) we have:

(6.7) THEOREM. *For $X = L^p(\mu)$, with $\infty > p > 1$, and the group G, we have:*

(i) If $p \geqq 2$, then $|\ \ |_p$ is a (UF) norm equivalent to $\|\ \ \|$, relative to which each $T \in G$ is an isometry, and $(X, |\ \ |_p)$ is (UC).

(ii) If $1 < p < 2$, then $|\ \ |_q$ is a (UF) norm equivalent to $\|\ \ \|$, relative to which each $T \in G$ is an isometry, and $(X, |\ \ |_q)$ is (UC).

(6.8) COROLLARY. *If F is a commutative, hermitian-equivalent family of operators on $L^p(\mu)$, $\infty > p > 1$, then there is an F-norm which preserves uniform Fréchet differentiability and uniform convexity.*

By (6.8) and the reasoning used in the proof of (5.4), we have:

(6.9) COROLLARY. *If $\mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n$, are commuting bounded B.A.'s of projections with domain $L^p(\mu)$, $\infty > p > 1$, and $\mathscr{B} = \bigcup_{j=1}^n \mathscr{B}_j$, then there is a $\mathscr{B}$-norm $|\ \ |$ which preserves uniform Fréchet differentiability and uniform convexity. Moreover, if $[\ \ ,\ \ ]$ is any s.i.p. consistent with $|\ \ |$, then for each $x \in L^p(\mu)$, $[\ \ , x]$ is a Bade functional with respect to each $\mathscr{B}_j$, $j = 1, 2, \cdots, n$.*

REFERENCES

1. W. G. Bade, *On Boolean algebras of projections and algebras of operators,* Trans. Amer. Math. Soc. **80** (1955), 345-360.

2. E. Berkson, *A characterization of scalar type operators on reflexive Banach spaces,* Pacific J. Math. **13** (1963), 365-373.

3. N. Bourbaki, *Éléments de mathématique,* Livre V, *Espaces vectoriels topologiques,* Hermann, Paris, 1953.

4. J. A. Clarkson, *Uniformly convex spaces,* Trans. Amer. Math. Soc. **40** (1936), 396-414.

5. N. Dunford and J. T. Schwartz, *Linear operators,* Part I, *General theory,* Interscience, New York, 1958.

6. G. Lumer, *Semi-inner-product spaces,* Trans. Amer. Math. Soc. **100** (1961), 29-43.

7. ———, *Spectral operators, Hermitian operators, and bounded groups,* Acta Sci. Math. (Szeged) (to appear).

8. D. Milman, *On some criteria for the regularity of spaces of the type* (*B*), C. R. (Doklady) Acad. Sci. URSS (N.S.) **20** (1938), 243-246.

9. B. J. Pettis, *A proof that every uniformly convex space is reflexive*, Duke Math. J. **5** (1939), 249-253.

10. V. Smulian, *Sur la dérivabilité de la norme dans l'espace de Banach*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **27** (1940), 643-648.

11. I. Vidav, *Eine metrische Kennzeichnung der selbstadjungierten Operatoren*, Math. Z. **66** (1956), 121-128.

UNIVERSITY OF CALIFORNIA,
LOS ANGELES, CALIFORNIA