UNIFORMIZATION OF SYMMETRIC RIEMANN SURFACES BY SCHOTTKY GROUPS

BY

ROBERT J. SIBNER

1. Introduction. A Riemann surface $S$ is called symmetric if there exists an anti-conformal map $\phi$ of $S$ onto itself such that $\phi^2 = \text{identity}$. We say that $\phi$ is a symmetry on $S$.

The classical "retrospection theorem" asserts the existence of representations of closed Riemann surfaces of genus $g$ by "Schottky groups," groups generated by Möbius transformations $A_1, \cdots, A_g$ such that $A_i$ maps the exterior of $\Gamma_i$ into the interior of $\Gamma'_i$, where $\Gamma_1, \Gamma'_1, \cdots, \Gamma_g, \Gamma'_g$ are disjoint Jordan curves bounding a $2g$-times connected domain, a standard fundamental domain for the group.

We will show that a closed symmetric Riemann surface of genus $g$ can be represented by a Schottky group which has a standard fundamental domain which exhibits the symmetry. This result is contained in Theorems I, II and III of §§4-6. The proof does not use the classical theorem.

As a corollary, in §7, we obtain a new proof of the Koebe theorem: every $n$-times connected planar domain can be conformally mapped onto a plane domain exterior to $n$ disjoint circles.

Techniques from the theory of quasiconformal mappings are used to obtain these results.

I would like to thank Professor Lipman Bers for his invaluable advice and assistance.

2. Quasiconformal mappings. We recall [1], [3] that a homeomorphism $w(z)$ of a plane domain $\Delta$ onto another plane domain $\Delta'$ is said to be quasiconformal if it has generalized derivatives satisfying, at each point $z \in \Delta$, a Beltrami equation $w_z = \mu(z)w_z$ with $\mu(z) \in M_\Delta$, where $\mu(z) \in M_\Delta$ if it is defined and measurable in $\Delta$ and $\text{ess. sup} |\mu(z)| \leq k < 1$ for $z \in \Delta$.

For a given $\mu(z) \in M_C$ ($C$ the complex plane) there exists a unique quasiconformal mapping $w^*(z)$ of $C$ onto itself satisfying $w^*_z = \mu(z)w_z$ and normalized by the conditions $w(0) = 0$ and $w(1) = 1$.

If $\mu(z) \in M_C$ is compatible with the Möbius transformation $A(z)$:

$$\mu \circ A = (A_z/\overline{A_z})\mu$$

Received by the editors May 13, 1964.

(1) This work represents some of the results contained in the author's Ph. D. thesis at New York University. It was supported, in part, by the Army Research Office under Contract No. DA-ARO-(D)-31-124-6156 and by the Office of Naval Research under Contract No. ONR(G)-00010-64.
or with the anti-Möbius transformation $B(z) = (az + b)/(cz + d)$:

$$
\mu \circ B = (B_{\overline{\mu}} \overline{B})\overline{\mu}
$$

then $A^* = \omega^* \circ A \circ (\omega)^{-1}$ and $B^* = \omega^* \circ B \circ (\omega)^{-1}$ are Möbius and anti-Möbius transformations respectively. If $G$ is a Schottky group generated by the transformations \{\$Aj\}, then the transformations \{\$Aj^*\} generate a Schottky group $G^*$.

We need, also, the following lemmas:

**Lemma 1.** Let $M(z)$ and $N(z)$ be two anti-Möbius transformations. If $\mu(z)$ is compatible with $M(z)$ and $M \circ N(z)$ then it is compatible with $N(z)$.

The proof is by calculation.

**Lemma 2.** If $\mu(z) \in M_G$ and is compatible with the anti-Möbius transformation $R(z)$, reflection in $C_{a,\rho}$ (the circle with radius $\rho$ and center at $z = a$), that is

$$
R(z) = a + \rho^2/z - a
$$

then $\omega^r(C_{a,\rho})$ is a circle (with center at $\omega^r(a)$) and

$$
R^r(\omega^r(z)) = \omega^r(a) + \lambda^2/\omega^r(z) - \omega^r(a),
$$

i.e., reflection in $w(C_{a,\rho})$.

**Proof.** Let $\psi(z) = \omega^r(z) + \rho^2/\omega^r(R(z)) - \omega^r(a)$. Then it is easily shown that $\psi(z) - \omega^r(a)$ and $\omega^r(z) - \omega^r(a)$ both satisfy the same Beltrami equation. They are both 0 when $z = a$ and $\infty$ when $z = \infty$. It follows by uniqueness that one is a multiple of the other. Then $[\omega^r(z) - \omega^r(a)]$$/[\omega^r(R(z) - \omega^r(a))] = \text{constant}.

For $z'$ on $C_{a,\rho}, R(z') = z'$ and $|\omega^r(z') - \omega^r(a)| = \lambda$ where $\lambda$ is a positive constant. Hence $\omega^r(C_{a,\rho})$ is a circle with center $\omega^r(a)$ and radius $\lambda$.

**Remark.** If, in Lemma 2, $C_{a,\rho}$ is the unit circle, then $R^* = \omega^r \circ R \circ (\omega)^{-1}$ is again reflection in the unit circle. If, in addition, $\mu$ is compatible with the Möbius transformations $A$ and $B$ and $A = R \circ B \circ R$, then $A^* = R^* \circ B^* \circ R^*$. This follows by a simple calculation.

**Lemma 3.** If $\mu(z) \in M_G$ and is compatible with the anti-Möbius transformation

$$
Q(z) = a - \rho^2/z - a
$$

which is reflection in the circle $C_{a,\rho}$ of radius $\rho$ and center $a$, followed by a rotation about $a$ by the angle $\pi$, then $\omega^r(C_{a,\rho})$ is a "quasicircle" (i.e., if $w_1$ is on $\omega^r(C_{a,\rho})$ then the line through $w_1$ and the "center" $b = \omega^r(a)$ intersects $\omega^r(C_{a,\rho})$ in a point $w_2$ such that $(w_2 - b)(w_1 - b) = \text{negative constant}$). The anti-Möbius transformation $Q(z)$ maps the exterior of $\omega^r(C_{a,\rho})$ into its interior in such a way that a point on $\omega^r(C_{a,\rho})$ is mapped into its "diametrically opposed" point.
Proof. As in Lemma 2, one can show that

\[ w'(z) - w'(a) \left[ w'(Q(z)) - w'(a) \right] = \text{constant} = c. \]

Setting \( z = a + \rho \) and then \( z = a - \rho \) we see that \( c \) is real. Suppose \( c > 0 \). Then for \( z_0 \) such that \( w'(z_0) = w'(a) + \sqrt{c} \), \( w'(Q(z_0)) = w'(a) + \sqrt{c} \) also. Hence \( c = -\lambda^2 \) and

\[ Q'(w'(z)) = w'(Q(z)) = w'(a) - \lambda^2 w'(z) - w'(a). \]

Suppose now that \( w_0 \) is on \( w'(C_{a,\rho}) \). Let \( w_0 = w'(a) + \rho e^i \). Then \( Q'(w_0) = w'(a) - \lambda^2/w_0 - w'(a) = w'(a) - (\lambda^2/\rho)e^i \). But then \( Q'(w_0) \), which is on \( w'(C_{a,\rho}) \), is diametrically opposed to \( w_0 \). Hence \( w'(C_{a,\rho}) \) is a quasicircle.

We recall also [3] that if a homeomorphic map \( f \) of a compact Riemann surface \( S \) onto another compact Riemann surface \( S' \) is given, there exists a quasiconformal map \( \tilde{f} \) of \( S \) onto \( S' \) (i.e., a map which is quasiconformal in terms of local parameters) which is homotopic to \( f \). If, in addition, \( S \) and \( S' \) admit anti-conformal involutions \( \phi \) and \( \phi' \) and if \( f \circ \phi = \phi' \circ f \), then \( \tilde{f} \) may be chosen so as to satisfy the relation \( \tilde{f} \circ \phi = \phi' \circ \tilde{f} \). If \( \mu(z) \in \mathcal{M} \) and \( \nu(z) \) is compatible with the generators \( \{A_i\} \) of a Schottky group \( G \), then, denoting by \( L \) and \( L' \) the set of limit points of \( G \) and \( G' \) respectively, \( w^*: C \to C \) induces a quasiconformal map of \( (C - L)/G \) onto \( (C - L')/G' \). Furthermore, if \( S \), \( S' \), and \( S'' \) are three Riemann surfaces and \( f^* \) and \( h^* \) quasiconformal maps; \( f^*: S \to S' \) and \( h^*: S \to S'' \) both satisfying (in terms of local coordinates) the same Beltrami equation on \( S \), then \( h^* \circ (f^*)^{-1} \) is a conformal map of \( S' \) onto \( S'' \).

3. Symmetric surfaces. If a symmetry \( \phi \) on \( S \) leaves fixed a point of \( S \), then it leaves fixed a closed, analytic, Jordan curve through the point, which we call a transition curve. If the \( \tau \geq 0 \) transition curves separate \( S \), a symmetric Riemann surface of genus \( g \), into two disjoint surfaces (orthosymmetry), we say that \( S \) is symmetric of type \( (g, + \tau) \) with respect to the symmetry \( \phi \); otherwise (diASYMMETRY) it is of type \( (g, - \tau) \). In the former case \( S/\phi \) is an orientable surface with \( \tau \) holes and \( (g - \tau + 1)/2 \) handles. In the latter case \( S/\phi \) is a nonorientable surface with \( \tau \) holes. Since, topologically, on a nonorientable surface a handle can be replaced by two cross caps, \( S/\phi \) is homeomorphic to a surface with \( \tau \) holes and, say, \( k \) cross caps. It is easily seen that \( k = g - \tau + 1 \). From these remarks we observe that if \( S \) is symmetric of type \( (g, \epsilon \tau) \), \( \epsilon = \pm 1 \), then:

\[
\begin{align*}
\text{if } \epsilon = +1, \text{ then } g - \tau + 1 & \text{ is even and } 0 \leq g - \tau + 1 \leq g, \\
\text{if } \epsilon = -1, \text{ then } 0 & \leq \tau \leq g.
\end{align*}
\]

4. Orthosymmetric surfaces. Given \( \epsilon = \pm 1 \) and integers \( g > 0 \) and \( \tau \geq 0 \) satisfying (1), we construct a “standard model of type \( (g, \epsilon \tau) \).” We as-
sume at first that \( \epsilon = +1 \) and hence \( \tau > 0 \). Let \( C_s \) \((1 \leq s \leq \tau - 1)\) and \( H_r \) \((1 < r < g - \tau + 1)\) be \( g \) disjoint circles exterior to the unit circle \( C_0 \), and with centers \((a_s, a)\) respectively on the real axis. Denote by \( R_s(z) \) reflection in the circle \( C_s \). Let \( A_s(z) \) be a Möbius transformation which maps the exterior of \( H_r \) onto the interior of \( H_{r+1} \), \( r = 1, 3, \ldots, g - \tau \).

Reflect the circles \( C_1, \ldots, C_{\tau - 1} \) and \( H_1, \ldots, H_{g - r + 1} \) in \( C_0 \), obtaining circles \( C'_1, \ldots, C'_{\tau - 1} \) and \( H'_1, \ldots, H'_{g - r + 1} \). The exterior of these \( 2g \) circles we denote by \( F \) and note that \( F \) is a standard fundamental domain of the Schottky group \( G \) generated by the \( g \) Möbius transformations

\[ A_1, A_3, \ldots, A_{g-\tau}, A'_1, A'_3, \ldots, A'_{\tau-1}, R_0 \circ R_1, \ldots, R_0 \circ R_{\tau-1} \]

where \( A'_s(z) = R_0 \circ A_s \circ R_0(z) \). We observe that \( R_0 \circ R_s(z) \) (reflection in \( C_s \) followed by reflection in \( C_0 \)) maps \( C_s \) onto \( C'_s \) in such a way that points on \( C_s \) and \( C'_s \) which are symmetrically situated with respect to \( C_0 \), are identified by \( R_0 \circ R_s(z) \) and hence by the group \( G \).

Denoting by \( \pi \) the canonical mapping of \((C - L)\) onto \((C - L)/G\), we see that the surface \( F/G = (C - L)/G \) is symmetric of type \((g, + \tau)\) with respect to the symmetry \( R, R \circ \pi = \pi \circ R_0 \). We call it the standard model of type \((g, + \tau)\). If it is identified under \( R \), the resulting surface \((F/G)/R = F/\{G, R_0\}\) has, by computing the Euler characteristic, \( \tau \) holes and \((g - \tau + 1)/2 \) handles.

Given a symmetric surface \( S \) of type \((g, + \tau)\), there exists, therefore, a homeomorphism \( f: (F/G)/R = ((C - L)/G)/R \to S/\phi \). We extend \( f \) to a map of \( F/G \) onto \( S \) by the requirement \( f \circ R = \phi \circ f \). The homeomorphism \( f \) can be deformed into a quasiconformal map, which we again denote by \( f \), of \( F/G \) onto \( S \) satisfying the same requirement.

The map \( f \) defines in \( F \) a function \( \mu(z) = f^* f^* \) where \( f^*(z) = \xi \circ f \circ z^{-1} \) \((\xi \text{ and } \xi \text{ being local coordinates near } p_0 \text{ on } F/G \text{ and near } f(p_0) \text{ on } S \text{ respectively})\). Due to the above requirement, \( \mu(z) \) is compatible with \( R_0(z) \). We extend \( \mu \) to \( C \) by requiring that it be compatible with \( G \) and observe that \( \mu \in M_C \). Let \( w^r(z) \) be the (unique) quasiconformal map of \( C \) onto itself satisfying \( w^r_0 = \mu(z)w_z \) with \( w(0) = 0 \) and \( w(1) = 1 \). Denote by \( \tilde{w}^r \) the induced map of \((C - L)/G \) onto \((C - L^r)/G^r \). It is easily seen that \( F^r = w^r(F) \) is a standard fundamental domain of the Schottky group \( G^r \). But then \( h = f \circ (\tilde{w}^r)^{-1} \) is a conformal map of \((C - L^r)/G^r \) onto \( S \). The Schottky group \( G^r \), which has the fundamental domain \( F^r \), therefore represents the symmetric surface \( S \).

We examine now the fundamental domain \( F^r \). By the remark following Lemma 2,

(a) \( F^r \) is symmetric with respect to reflection in the unit circle and

\[ h \circ R^r = f \circ (\tilde{w}^r)^{-1} \circ R^r = f \circ R \circ (\tilde{w}^r)^{-1} \# f \circ (\tilde{w}^r)^{-1} = f \circ h \]

so that the symmetry \( \phi \) in \( S \) is represented by reflection in the unit circle.
(b) $A_r^*(w)$ and $A_r'^*(w)$ map the exterior of the symmetrically situated Jordan curves $H_r^*$ and $H_r'^*$ onto the interior of the symmetrically situated Jordan curves $H_{r+1}^*$ and $H_{r+1}'^*$ respectively ($r = 1, 3, \ldots, g - r$). Here, we denote by $\Gamma^*$, the image of a curve $\Gamma$ under $w^*$. Furthermore

$$A_r^*(w) = R_0^0 \circ A_r^* \circ R_0^0(w).$$

(c) $C_r^*$ and $C_r'^*$ are circles (by Lemmas 1 and 2) and the Möbius transformation $(R_0 \circ R_s)^* = R_0^0 \circ R_s^*$ maps the exterior of $C_r^*$ onto the interior of $C_r'^*$ in such a way that two symmetrically situated points on $C_r^*$ and $C_r'^*$ are identified under the group $G^*$ (specifically, by the element $R_0^0 \circ R_s^*$). As a result, the points on $F^*$ which lie on these circles are left fixed (as is the unit circle $C_0^*$) by the symmetry: reflection in $C_0^*$. We summarize these results in

**Theorem I.** A symmetric surface $S$, of type $(g, + \tau)$ with respect to a symmetry $\phi$, can be represented by a Schottky group which has a fundamental domain symmetric with respect to reflection in the unit circle $C$, and bounded by (i) $\tau - 1$ identified pairs of symmetrically situated circles $\Gamma_1, \Gamma_1', \ldots, \Gamma_{\tau-1}, \Gamma_{\tau-1}'$ and (ii) $(g - \tau + 1)/2$ identified pairs of Jordan curves in the exterior of $C$ and $(g - \tau + 1)/2$ symmetrically situated identified pairs of Jordan curves in the interior of $C$. The symmetry $\phi$ on $S$ is represented by reflection in $C$ and the $\tau$ transition curves on $S$ by $C$ and the $\tau - 1$ pairs of circles $\Gamma_1, \Gamma_1', \ldots, \Gamma_{\tau-1}, \Gamma_{\tau-1}'$.

5. Diasymmetric surfaces with fixed points. We now extend the results of §4 to symmetric surfaces for which $\tau \neq 0$ but $\epsilon = -1$.

To obtain the standard model of type $(g, - \tau)$ we construct $g$ pairs of circles:

$$C_1, C'_1, \ldots, C_{g-1}, C'_{g-1}, K_1, K_1', \ldots, K_{\tau-1}, K'_{\tau-1}$$

symmetrically situated with respect to the unit circle $C_0$ as in §4. If we let $Q_i(z)$ be reflection in $K_i$ followed by rotation about the center $b_i$ of $K_i$ by the angle $\pi$, and define $R_s(z)$ as in §4 we find that $F$, the exterior of the $2g$ circles, is a fundamental domain of the Schottky group

$$G = \{R_0^0 \circ Q_1, \ldots, R_0^0 \circ Q_{\tau-1}, R_0^0 \circ R_1, \ldots, R_0^0 \circ R_{\tau-1}\}.$$

Again, under $R_0^0 \circ R_s$, symmetrical points on $C_s$ and $C'_s$ are identified and $R_0^0 \circ Q_i$ maps each point $P$ of $K_i$ onto the point of $K'_i$ which is diametrically opposed to $R_0(P)$, the reflection of $P$ in $C_0$.

$F/G$, the standard model of type $(g, - \tau)$ has genus $g$, is symmetric with respect to $R(z)$, and has $\tau$ transition curves. $(F/G)/R$ has $\tau$ holes and $g - \tau + 1$ holes with diametrically opposed points identified (i.e., cross caps). Then, if $S$ is a symmetric surface of type $(g, - \tau)$, $\tau \neq 0$, there exists a homeomorphism $f: F/G \to S$ satisfying $f \circ R = \phi \circ f$. 
The procedure of §4 can be repeated to obtain a Schottky group $G'$ (with a fundamental domain $F'$) which represents $S$. $F'$ has the properties (a) and (c) of §4 and, in addition, by Lemmas 1 and 3, (b') $K_i'$ and $K_i''$ are quasicircles and the Möbius transformation $(R_0 \circ Q)(z)$ maps the exterior of $K_i'$ onto the interior of $K_i''$ in such a way that each point $P$ on one quasicircle is identified with the point $P'$ on the other quasicircle which is diametrically opposed to the reflection $R_0(P)$ of $P$. As a result, a point on $F'$ which lies on one of the quasicircles is identified, under the group $\{G', R_0\}$, with its diametrically opposed point on the quasicircle.

We state

Theorem II. A symmetric Riemann surface $S$ of type $(g, -\tau), \tau \neq 0$, with respect to a symmetry $\phi$, can be represented by a Schottky group which has a standard fundamental domain symmetric with respect to reflection in the unit circle $C$, and bounded by (i) $\tau - 1$ identified pairs of symmetrically situated circles $\Gamma_1, \Gamma_1', \ldots, \Gamma_{1-1}, \Gamma_{1-1}'$ and (ii) $g - \tau + 1$ identified pairs of symmetrically situated quasicircles $A_1, A_1', \ldots, A_{g-\tau+1}, A_{g-\tau+1}'$. The symmetry $\phi$ on $S$ is represented by reflection in $C$; the $\tau$ transition curves on $S$ by $C$ and the $\tau - 1$ pairs of circles $\Gamma_0, \Gamma_0'$. The pairs of quasicircles $A_1, A_1'$ represent (when identified under reflection) $g - \tau + 1$ cross caps on $S/\phi$.

6. Fixed point free diasymmetric surfaces. To extend the results of §§4 and 5 to symmetric surfaces of type $(g, 0)$ we use a different representation of the symmetry. This is clearly necessary, since the reflection $R_0(z)$ always leaves the points on the unit circle.

Let $K_0, \ldots, K_g$ be disjoint circles and let $Q_j(z), 0 \leq j \leq g$, be reflection in $K_j$ followed by rotation about the center of $K_j$ by the angle $\pi$. Denote by $K_j'$ the circle $Q_0(K_j), 1 \leq j \leq g$. Denoting by $F$ the exterior of the $2g$ circles $K_0, K_1', \ldots, K_g, K_g'$ we see that $F$ is a standard fundamental domain of the Schottky group $G = \{Q_0 \circ Q_1, \ldots, Q_0 \circ Q_g\}$. $F/G$ is a symmetric surface (with respect to the symmetry $Q$, $Q \circ \pi = \pi \circ Q_0$). It has genus $g$ and no transition curves. Since, for a point $P$ on $K_0, Q_0(K_0(P))$ and $Q_0 \circ Q_0(P)$ are diametrically opposed points on $K_0'$. $(F/G)/Q = F/\{G, Q_0\}$ is a sphere with $g + 1$ cross caps. If $S$ is a symmetric surface of type $(g, 0)$ there exists a homeomorphism $f$ of $F/G$ onto $S$ such that $\phi \circ f = f \circ Q$.

As in §§4 and 5 we obtain a fundamental domain $F'$ of a Schottky group $G'$ which represents $S$. Furthermore,

(a) The symmetry $\phi$ on $S$ is represented by a symmetry $Q_0'(w)$ which is of the form (see Lemma 3)

$$Q_0'(w) = b - \lambda^2/w - b.$$

(b) Again by Lemma 3, $K_0'$ and $K_0''$ are quasicircles and the Möbius transformation $Q_0'(Q_0)(w)$ maps the exterior of $K_0'$ onto the interior of $K_0''$ in such a way that each point $P$ on one quasicircle is identified with
that point $P''$ on the other quasicircle which is diametrically opposed to the point $Q_0(P)$. Therefore, points on $F^*$ which lie on the quasicircles are identified, under the group $\{G^*, Q_0^*\}$ with their diametrically opposed points.

We assume without loss of generality that the “center” of $K_0^*$ is at the origin and that $\lambda = 1$, so that $Q_0^*(w) = -1/\bar{w}$. We can then state

**Theorem III.** A symmetric Riemann surface $S$ of type $(g,0)$ with respect to a symmetry $\phi$ can be represented by a Schottky group $G$ having a fundamental domain bounded by $g$ identified pairs of disjoint quasicircles $A_1, A'_1, \cdots, A_g, A'_g$ which are symmetrically situated with respect to the symmetry $\tilde{Q}(w) = -1/\bar{w}$. The symmetry $\phi$ on $S$ is represented by the transformation $\tilde{Q}(w)$. There is also a quasicircle $\Lambda$ which is a closed Jordan curve and which contains in its interior the quasicircles $A_1, A_2, \cdots, A_g$. $\tilde{Q}(w)$ transforms $\Lambda$ into itself in such a way that diametrically opposed points are identified, and transforms $A_i$ into $A'_i$ in such a way that a point $P$ on $A_i$ is identified with the point on $A'_i$ which is diametrically opposed to the point on $A_i$ which is identified with $P$ under the group $G$. As a result, the quasicircle $\Lambda$, together with the $g$ pairs of quasicircles $A_i, A'_i$ represent, when identified under $\tilde{Q}$, $g+1$ cross caps on $S/\phi$.

7. **Mappings of multiply connected domains.** We recall that a multiply connected plane domain $D$, bounded by $n$ nondegenerate continua, can be mapped conformally onto a plane domain bounded by $n$ closed analytic Jordan curves. This follows at once from the Riemann mapping theorem.

Given a domain $D$ bounded by $n$ closed analytic Jordan curves $\gamma_1, \cdots, \gamma_n$, let $S$ be the closed surface obtained by doubling $D$ [2, pp. 118-119]. We observe that $S$ is a Riemann surface of genus $n-1$, orthosymmetric with respect to the symmetry $\phi$ defined by the doubling process. Furthermore $S/\phi = D$. Then by Theorem I, $D$ is conformally equivalent to a region bounded by $n$ disjoint circles. The above arguments give a new proof of the

**Koebe Theorem.** A multiply connected plane domain, bounded by $n$ nondegenerate continua can be mapped conformally onto a plane domain bounded by $n$ circles.

**Remark.** This theorem, which has been obtained as a corollary of Theorem I, can also be obtained directly from Lemma 2.

**References**


Stanford University,
Stanford, California